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Generalized Torsion in Knot Groups

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Abstract. In a group, a nonidentity element is called a generalized torsion element if some product of its conjugates equals the identity. We show that for many classical knots one can find generalized torsion in the fundamental group of its complement, commonly called the knot group. It follows that such a group is not bi-orderable. Examples include all torus knots, the (hyperbolic) knot 5_2, and algebraic knots in the sense of Milnor.

1 Introduction

The purpose of this note is to initiate a study of generalized torsion in classical knot groups. For two elements x and y of a group, we use the notation $x^y := y^{-1}xy$ for the conjugate of x by y and $[x, y] := x^{-1}y^{-1}xy$ for their commutator. The identity of a group is denoted by 1. A *generalized torsion* element of a group is an element $x \neq 1$ for which some (nonempty finite) product of conjugates of that element is the identity: $x^{y_1}x^{y_2}\cdots x^{y_k} = 1$.

Example 1.1 An example of generalized torsion is the fundamental group of the Klein bottle $\langle x, y \mid y^{-1}xy = x^{-1}\rangle$ in which x is a generalized torsion element: $x^y x = 1$.

2 Knot Groups and Bi-ordering

A classical knot is a subset K of euclidean space $\mathbb{R}^3$ that is abstractly homeomorphic to the circle S^1. We assume that K is smooth or piecewise linear. The *knot group* of K is the fundamental group of its complement: $\pi_1(\mathbb{R}^3 \setminus K)$. It has long been known that knot groups do not contain torsion, that is, elements of finite order. In fact, knot groups have the stronger property of being locally indicable; every nontrivial finitely-generated subgroup surjects to $\mathbb{Z}$, the integers (see [2, 6]). It follows that knot groups support left-invariant orderings, meaning that the elements of the group can be given a total ordering $<$ such that for elements x, y, z of the group one has $y < z \iff xy < xz$. The groups of some knots, for example 4_1, are bi-orderable; there is a strict total ordering of the elements that is invariant under multiplication on both sides (see[10]). But not all knot groups are bi-orderable; generalized torsion is a well-known obstruction to bi-orderability. The following proposition is clear, for if we assume

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without loss of generality that $x > 1$ in a bi-ordering, then all its conjugates are also greater than the identity, so $x^{y_1}x^{y_2}\cdots x^{y_k} > 1$.

Proposition 2.1 *Bi-orderable groups do not have generalized torsion elements.*

Although left-orderable groups cannot have torsion, they can have generalized torsion; the Klein bottle group of Example 1.1 is an instance. In fact, it has exactly four left-orderings. As we will see, many knot groups are also examples.

Among the various reasons bi-orderability of a knot group is of interest is the following result from [5].

Theorem 2.2 *If the group of the knot K is bi-orderable, then surgery on K cannot produce a* 3*-manifold with finite fundamental group, or more generally any L-space in the sense of Ozsváth and Szabó* [9].

3 Commutators as Generalized Torsion

Example 3.1 Consider the group of the trefoil knot $G_{2,3} = \langle x, y \mid x^2 = y^3 \rangle$. The commutator $[x, y]$ is a nontrivial element, because G is non-abelian. But

$$x^{-1}[x,y]x[x,y] = x^{-2}y^{-1}xyxx^{-1}y^{-1}xy = x^{-2}y^{-1}x^2y = y^{-3}y^{-1}y^3y = 1.$$

So $[x, y]$ is a generalized torsion element in the knot group of the trefoil. This is expanded in Theorem 3.4 below for other torus knots, that is, knots that can be inscribed on the surface of an unknotted solid torus in $\mathbb{R}^3$.

The following lemma is a standard fact of group theory, easily proved by induction using the identities $[x^n, y] = [x^{n-1}, y]^x[x, y]$ and $[x, y^n] = [x, y][x, y^{n-1}]^y$. Its relation to orderability is pointed out in [3, Examples 2.1 and 2.2], which also inspired the remaining results of this section.

Lemma 3.2 *In any group, for all positive integers p, q the commutator $[x^p, y^q]$ is a product of conjugates of $[x, y]$.*

Proposition 3.3 *Suppose G is any group containing elements x and y that do not commute, but for which some positive powers x^p and y^q do commute. Then the commutator $[x, y]$ is a generalized torsion element of G.*

Proof By Lemma 3.2, $[x^p, y^q]$ is a product of conjugates of $[x, y]$. But $[x, y] \neq 1$ and $[x^p, y^q] = 1$. ∎

Theorem 3.4 *For any integers p and q with $|p| > 1$ and $|q| > 1$, the group*

$$G = G_{p,q} = \langle x, y \mid x^p = y^q \rangle$$

contains a generalized torsion element, namely the commutator $[x, y]$.

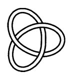
Proof The conditions on p and q imply that G is nonabelian, so $[x, y] \neq 1$. Proposition 3.3 completes the proof. ∎

If p and q are relatively prime integers, $G_{p,q}$ is the group of the p, q-torus knot (two examples pictured below).

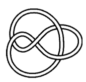
Corollary 3.5 *The group of a nontrivial torus knot has generalized torsion.*

4 The First Few Prime Knots

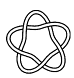
Let us consider the nontrivial prime knots up to six crossings in their usual tabulated order:



3_1: The trefoil, or $2, 3$-torus knot. Its group has generalized torsion, as observed in Example 3.1.

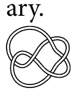


4_1: The figure-eight knot. It has group $\langle a, b \mid ab^3a = ba^2b \rangle$. Its Alexander polynomial, $1 - 3t + t^2$, has real positive roots. It is shown in [10] that its group is bi-orderable using that fact, and that 4_1 is a fibred knot (see below for definition), hence it has no generalized torsion.



5_1: The $2, 5$-torus knot or cinquefoil. It has generalized torsion by Theorem 3.4.

All the above knots are fibred knots; that is, their complements in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ fibre over S^1 with fibres being open surfaces whose closure has the knot as its boundary.



5_2: We will see that this knot's group also has generalized torsion.

The knot 5_2 is the first knot in the standard table that is not fibred. It has Alexander polynomial $2t^2 - 3t + 2$, whereas fibred knots have monic polynomials. Its knot group G_{5_2} has presentation

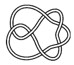
$$G_{5_2} = \langle a, b \mid b^2a^{-2}b^2 = a^{-1}b^3a^{-1} \rangle.$$

It is shown in [3] that G_{5_2} is not bi-orderable using the fact that the Alexander polynomial has no real roots. In this note we strengthen this result by noting there is generalized torsion in G_{5_2}.

Theorem 4.1 *In the group G_{5_2}, $a^{-1}bab^{-1}$ is a generalized torsion element.*

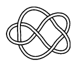
This will be proved in the next section.

The knot in the tables that follows 5_2 is sometimes known as the stevedore knot.



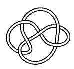
6_1: This knot has Alexander polynomial $2 - 5t + 2t^2$, whose roots are $1/2$ and 2. According to [4], using the results of [3], the group of this knot is bi-orderable. Therefore, its group does not have generalized torsion.

The next knot in the tables is still a mystery.



6_2: This is the first knot in the tables for which bi-orderability, as well as the existence of generalized torsion, is unknown at this writing. Its Alexander polynomial is $1 - 3t + 3t^2 - 3t^3 + t^4$, which has two positive real roots, $2.15372\ldots$ and $0.446431\ldots$, and two complex roots, approximately $0.19098 \pm 0.98159i$.

The last prime knot with at most six crossings is the following.



6_3: Its Alexander polynomial is $1 - 3t + 5t^2 - 3t^3 + t^4$, which has no real roots. Using this fact, it is shown in [4], again using the results of [3], that the group of this knot is not bi-orderable. We do not know if its group contains generalized torsion.

5 Proof of Theorem 4.1

We adopt the convention of using capital letters to denote the inverse of a group element, so that the defining relation of the group G_{5_2} becomes $b^2A^2b^2 = Ab^3A$. The existence of generalized torsion in this group was discovered with the help of a Python program written by the first author to test for bi-orderability. The first proof we offer is adapted from the non-bi-orderability argument discovered by computer, and seems rather unmotivated. We will also present a second proof, suggested by Andrew Glass, which the reader may find somewhat more mathematically satisfying.

Proof First we need to argue that $AbaB$ is not the identity. But this follows, since the group, like all groups of nontrivial knots, is nonabelian.

Let X denote the set of elements of G_{5_2} that are (nonempty) products of conjugates of $AbaB$. Our goal is to show that $1 \in X$. It is clear that X is closed under multiplication and conjugation, and of course

(5.1) $$AbaB \in X.$$

Now, $B^2(AbaB)b^2 = B^2Abab$, so

(5.2) $$B^2Abab \in X.$$

Conjugating $AbaB$ by B shows that

(5.3) $$bAbaB^2 \in X.$$

Now consider the product of (5.1) and (5.3): $(AbaB)(bAbaB^2) = Ab^2aB^2$, which is equal to aB^2Ab by the following calculation:

$$\begin{aligned}(Ab^2aB^2)(aB^2Ab)^{-1} &= (Ab^2aB^2)(Bab^2A) = Ab^2(aB^3a)b^2A \\ &= Ab^2(B^2a^2B^2)b^2A = 1.\end{aligned}$$

Here we used the relation $aB^3a = B^2a^2B^2$, which follows from the defining relation. So we have shown that

$$aB^2Ab \in X. \tag{5.4}$$

Conjugating (5.4) gives

$$(BaB^2)(aB^2Ab)(b^2Ab) = BaB^2aB^2(Ab^3A)b = BaB^2aB^2(b^2A^2b^2)b = BaB^2Ab^3.$$

This last expression equals BAb^2a from the calculation

$$(BaB^2Ab^3)(AB^2ab) = BaB^2(Ab^3A)B^2ab = BaB^2(b^2A^2b^2)B^2ab = 1,$$

so we conclude

$$BAb^2a \in X. \tag{5.5}$$

Another conjugate of (5.4) gives $b(aB^2Ab)B = baB^2A$ so $baB^2A \in X$. Next we conjugate (5.1): $ba(AbaB)AB = b^2aBAB$ so that $b^2aBAB \in X$. Multiplying those last two elements of X yields $(b^2aBAB)(baB^2A) = b^2aB^3A$:

$$b^2aB^3A \in X. \tag{5.6}$$

Similarly, the product of (5.2) and (5.5) gives $(B^2Abab)(BAb^2a) = B^2Ab^3a$,

$$B^2Ab^3a \in X \tag{5.7}$$

Finally, we conjugate (5.7) by A^4 and multiply by (5.6) to conclude

$$\begin{aligned} a^4(B^2Ab^3a)A^4(b^2aB^3A) &= a^4B^2(Ab^3A)A^2b^2aB^3A \\ &= a^4B^2(b^2A^2b^2)A^2b^2aB^3A \\ &= a^2(b^2A^2b^2)aB^3A \\ &= a^2(Ab^3A)aB^3A = 1 \in X \end{aligned}$$

■

Second proof We are interested in showing that $c := [a, b^{-1}]$ is a generalized torsion element in the 5_2 knot group. Obviously, if we are to have a hope, we had better look at products of conjugates of c by elements of the form a^mb^n where $|m|, |n|$ are small. Note that $c^{a^{-1}} = [b^{-1}, a^{-1}]$ and $c^b = [b, a]$. Since $a^{-2}b^2a = b^{-2}a^{-1}b^3$ (one form of the defining relation), we obtain $c^{a^{-1}}cc^{b^{-1}} = bab^{-3}a^{-1}b$. Hence, $(c^{a^{-1}}cc^{b^{-1}})^{b^{-1}a^2} = b^{-2}$. Now $c^{b^2}c^b = [b^2, a]$, so

$$c^{b^2}(c^{b^2}c^b)^{a^{-1}b} = b^{-2}a^{-1}ba^2b^{-2}a^{-1}b^3 = b^{-2}a^{-1}b^3a$$

(using $a^{-1}b^3 = b^2a^{-2}b^2a$). So we can conclude that $(c^{b^2}(c^{b^2}c^b)^{a^{-1}b})^{a^{-2}} = b^2$. Thus, one product of conjugates of c is the inverse of another product of conjugates of c, so c is a generalized torsion element. ■

6 Satellites and Sums

Satellites of knots are constructed as follows. Let K be a knot in the interior of a solid torus V which in turn is standardly embedded in $\mathbb{R}^3$; that is, V is a regular neighbourhood of the trivial knot. We assume that K is essential in V, in the sense that the inclusion induces an *injective* homomorphism $\pi_1(\partial V) \to \pi_1(V \setminus K)$. Now let K_1 be some other knot in $\mathbb{R}^3$ with tubular neighbourhood $N(K_1)$. Since $N(K_1)$ is a solid torus, there is a homeomorphism $h : V \to N(K_1)$; in fact, there are infinitely many isotopy classes of such homeomorphisms. Finally, we let $K_2 = h(K)$; it is itself a knot in $\mathbb{R}^3$. In this situation, we say that K_2 is a *satellite* of K_1 with *pattern* knot K. By a van Kampen argument, one can see that the group of K_1 is isomorphic with a subgroup of the group of K_2. This implies the following proposition.

Proposition 6.1 *If K_2 is a* satellite *of K_1 and the knot group of K_1 has generalized torsion, then the same is true of the group of K_2.*

Corollary 6.2 *If one of the knots in a connected sum of knots has generalized torsion in its group, then the same is true of the sum.*

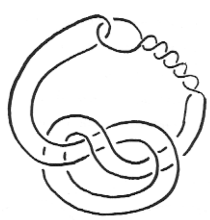
This follows, since a connected sum of two knots can be viewed as a satellite of either of the summands.

Corollary 6.3 *Algebraic knots in the sense of Milnor* [7] *have generalized torsion in their groups.*

This is true, because they are iterated cables of the unknot, and therefore a satellite of a torus knot.

Corollary 6.4 *There exist knots with trivial Alexander polynomial whose group contains generalized torsion.*

For example, consider Whitehead doubles of knots with generalized torsion (see [11, p. 167] or [12]). If they are "untwisted", they have trivial polynomials, but being satellites they inherit generalized torsion in their groups.



Untwisted double of the trefoil. It has trivial Alexander polynomial, but there is generalized torsion in its group.

The following corollary shows that the Alexander polynomial cannot detect the nonexistence of generalized torsion.

Corollary 6.5 *For any knot K, there exists another knot with the same Alexander polynomial as K, but whose group contains generalized torsion.*

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Proof Just take the connected sum of K with, say, the doubled trefoil above. ■

It is known that for nontrivial knots that are fibred [5] or have one-relator presentations of a particular form [3], if the Alexander polynomial has no positive real roots, then the knot group is not bi-orderable. This raises the following question.

Question 6.6 *If K is a knot whose Alexander polynomial is nontrivial and has no real positive roots, does it follow that the group of K contains generalized torsion?*

7 Conclusion

In summary, we have seen that all nontrivial torus knots, as well as the hyperbolic knot 5_2, have generalized torsion in their groups. Moreover, possessing generalized torsion in the group is preserved under taking satellites.

In all the cases we have discussed, the generalized torsion element identified in the knot group is a commutator in appropriate generators. Since for any knot group G the abelianization is infinite cyclic, any generalized torsion element must be in the kernel of the abelianization map $G \to \mathbb{Z}$, in other words, the commutator subgroup. That it is a *single* commutator in all these cases is interesting.

It is also interesting to compare the two knot groups, for 4_1 and 5_2. The latter can be rewritten, exchanging a with its inverse, so that we have

$$G_{4_1} \cong \langle a, b \mid ab^3a = ba^2b \rangle \quad G_{5_2} \cong \langle a, b \mid ab^3a = b^2a^2b^2 \rangle.$$

Curiously, the latter has generalized torsion while G_{4_1} does not.

From [10] we have infinitely many knots whose groups are bi-orderable, namely fibred knots whose Alexander polynomial has all roots real and positive. See [4] for further examples. These knot groups cannot contain generalized torsion by Proposition 2.1. Although this is an infinite class, it does not seem to be proportionately large. There are certainly many (most) knots for which bi-orderability of its group is an open question. The same is true of generalized torsion.

The question of whether the absence of generalized torsion in a group is actually equivalent to bi-orderability was answered in the negative by Bludov [1] (see also [8, p. 89]) by construction of examples of non-bi-orderable groups without generalized torsion. We ask whether that question (also raised in [3]) has an affirmative answer when restricted to classical knot groups.

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