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THE MINIMAL PRIMAL IDEAL SPACE OF A C^*-ALGEBRA AND LOCAL COMPACTNESS

Dedicated to my teacher Prof. G. Maltese on the occasion of his 60th birthday

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ABSTRACT. This paper is concerned with local compactness of the minimal primal ideal space of a C^*-algebra, a sufficient condition is given. The property in question has bad hereditary properties as is shown by examples.

1. **Introduction.** In [1] R. G. Archibold started the investigation of the minimal primal ideal space of a C^*-algebra $\mathcal{A}$ (definitions in the next chapter) and studied representation theory on Min-Primal($\mathcal{A}$) under the assumption that this space is closed. Local compactness is enforced by this condition.

In general Min-Primal($\mathcal{A}$) is not locally compact, there is an example where $\mathcal{A}$ is unital liminal separable and $\dim(\pi) \leq 2$ for all irreducible representations of $\mathcal{A}$ (see §4).

If Prim($\mathcal{A}$) is Hausdorff, then Prim($\mathcal{A}$) = Min-Primal($\mathcal{A}$) is clearly locally compact. It will be proved here that Min-Primal($\mathcal{A}$) is locally compact provided Prim($\mathcal{A}$) is nearly Hausdorff, i.e. Prim($\mathcal{A}$) is a T_1-space such that all limit sets are finite and each limit set L possesses a neighbourhood U so that $(U \setminus L) \cup \{P\}$ is Hausdorff for all points P in L.

Local compactness of Min-Primal($\mathcal{A}$) has very bad hereditary properties. Let I be a closed two-sided ideal in $\mathcal{A}$.

If Min-Primal($\mathcal{A}$) is locally compact, Min-Primal(I) or Min-Primal($\mathcal{A}/I$) are not in general.

If Min-Primal(I) and Min-Primal($\mathcal{A}/I$) are locally compact, Min-Primal($\mathcal{A}$) is not in general.

2. **Preliminaries.** For any C^*-algebra $\mathcal{A}$ let Prim($\mathcal{A}$) be the primitive ideal space equipped with the Jacobson topology. Id($\mathcal{A}$) is the set of all closed two-sided ideals of $\mathcal{A}$. $I \in$ Id($\mathcal{A}$) is said to be primal iff the following holds true: if $I_1, \ldots, I_n, \in$ Id($\mathcal{A}$) and $I_1 \cap \cdots \cap I_n = 0$ then I contains at least one of the I_j. These ideals have been introduced in [2]. Another description is the following: I is primal iff Prim($\mathcal{A}/I$) $\subset$ Prim($\mathcal{A}$) is a Lié set in the sense of [3]. Define Primal($\mathcal{A}$) to be the space of these ideals. By Zorn's lemma each primal ideal contains a minimal primal ideal, and the set of the latter is denoted by Min-Primal($\mathcal{A}$).

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$\mathrm{Id}(\mathcal{A})$ carries at least two important topologies.

(i) The Fell topology or strong topology τ. An open base is given by the sets $U(C : I_1, \ldots, I_n) := \{I \in \mathrm{Id}(\mathcal{A}) \mid I^c \cap C = \emptyset \text{ and } I_1 \not\subset I, \ldots, I_n \not\subset I\}$, where I^c is the set of ideals containing I, C is a quasi-compact set in $\mathrm{Prim}(\mathcal{A})$, $n \in \mathbb{N}$, and $I_1, \ldots, I_n$ are ideals in $\mathcal{A}$. A net I_α is τ-convergent to I iff $\|x+I_\alpha\| \longrightarrow \|x+I\|$ for all x in $\mathcal{A}$. $\big(\mathrm{Id}(\mathcal{A}), \tau\big)$ is a compact Hausdorff space (see [5] and [6] for this).

(ii) The weak topology ω. An open base is given by the sets $U(I_1, \ldots, I_n) = \{I \in \mathrm{Id}(\mathcal{A}) \mid I_1 \not\subset I, \ldots, I_n \not\subset I\}$, where $n \in \mathbb{N}$, $I_1, \ldots, I_n \in \mathrm{Id}(\mathcal{A})$. ω restricted to $\mathrm{Prim}(\mathcal{A})$ coincides with the Jacobson topology.

Obviously ω is weaker than τ. The ω-closure of $\mathrm{Prim}(\mathcal{A})$ is $\mathrm{Primal}(\mathcal{A})$, $\mathrm{Min\text{-}Primal}(\mathcal{A})$ is always contained in the τ-closure of $\mathrm{Prim}(\mathcal{A})$, τ and ω coincide on $\mathrm{Min\text{-}Primal}(\mathcal{A})$. All this can be found in [1].

3. **C^*-algebras with a nearly Hausdorff primitive ideal space.** The aim of this section is the following theorem:

THEOREM 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra having a nearly Hausdorff primitive ideal space. Then Min-Primal($\mathcal{A}$) is locally compact.*

Let $X = \mathrm{Min\text{-}Primal}(\mathcal{A})$, $\mathcal{A}^v = \mathrm{Prim}(\mathcal{A})$. The first step is

LEMMA 3.2. *Let $\mathcal{A}^v$ be T_1 and $K \in X$. Assume that $L = \{P \in \mathcal{A}^v \mid K \subset P\}$ is finite and has a closed neighbourhood F, such that $(F \setminus L) \cup \{P\}$ is Hausdorff for all $P \in L$. Then K has a compact neighbourhood in X.*

PROOF. Say $L = \{P_1, \ldots, P_n\}$. Since F is a neighbourhood of each P_i there is an ideal J_i such that $P_i \subset \{P \in \mathcal{A}^v \mid J_i \not\subset P\} \subset F$. Now $\mathcal{A}^v$ is a T_1-space and so $P_j \not\subset P_i$ for $j \neq i$. Since $\mathcal{A}^v$ is locally compact, there are compact neighbourhoods V_i of P_i satisfying $P_i \in V_i \subset \{P \in \mathcal{A}^v \mid J_i \cap \bigcap_{j=1, j\neq i}^n P_j \not\subset P\} \subset F$. In fact V_i is a compact Hausdorff space, since $V_i \cap L$ is the singleton P_i. It will be shown that $W := \bigcap_{i=1}^n (V_i \setminus L) \cup \{K\}$ is the required neighbourhood.

1.) If (P_α) is a net in F, such that its limit set L_0 contains elements of L, then it is already contained in L. In order to prove this let $P \in L \cap L_0$ and $Q \in L_0 \setminus L$. Since F is closed, we have $Q \in F \setminus L$ and so P_α is in $F \setminus L$ for large α, w.l.o.g. for all α. Since P and Q are in the Hausdorff space $(F \setminus L) \cup \{P\}$ we arrive at the contradiction $P = Q$.

2.) $V_i \setminus L \subset X$. If $P \in V_i \setminus L$ and $J \in X$ with $J \subset P$, there is a net (P_α) in $\mathcal{A}^v$ converging to J with respect to ω. By ([1], 3.2), $P_\alpha \longrightarrow P$ which is in $\mathrm{int}(F)$. So w.l.o.g. all P_α are in F, and by 1.) we conclude that the limit set is contained in $F \setminus L$. But this space is Hausdorff, and so the limit set must be a singleton, which in turn must be $\{P\}$. This shows $J = P$, and so P is in X.

3.) W is a neighbourhood of K in X. By 2.) we have $W \subset X$. There are ideals I_i satisfying $P_i \in U(I_i) \subset \mathrm{int}(V_i)$. Obviously, $K \in U(I_1, \ldots, I_n) \cap X$ so let us show, that this set is contained in W. Let $J \in U(I_1, \ldots, I_n) \cap X$. Then there are primitive

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ideals Q_i satisfying $J \subset Q_i$ and $I_i \not\subset Q_i$. If $Q_i = P_i$ for all i, then clearly J is in K and then $J = K$ by minimality. So assume $Q_j \neq P_j$ for one j. Since Q_j is in $U(I_j)\backslash L \subset V_j\backslash L$, we conclude by 2.) $Q_j = J \subset Q_i$ for all I, and so $Q_j = Q_i$ for all i, because $\mathcal{A}^v$ is T_1. But then J is in $V_1\backslash L$ for all I, and so in W.

4.) W is compact. To this end let (P_α) be a net in W, and let us show that there is a convergent subnet. Clearly we may assume $P_\alpha \in \bigcap_{i=1}^n V_i$ for all α. By successive choice of convergent subnets in V_i we can produce a subnet (P_β) such that $P_\beta \longrightarrow Q_i \in V_i$ for all $i = 1, \dots, n$. If $Q_i = P_i$ for all i then clearly $P_\beta \longrightarrow \bigcap_{i=1}^n P_i = K$, so we may assume $Q_j \neq P_j$ for one j, and like in 3.) we conclude $Q_1 = \cdots = Q_n$, and so $P_\beta \longrightarrow Q_1 \in W$.

This finishes the proof.

COROLLARY 3.3. *If $\mathcal{A}^v$ is a T_1-space and if each maximal limit set is finite and possesses a closed neighbourhood F such that $(F\backslash L) \cup \{P\}$ is Hausdorff for all $P \in L$, then $X = $ Min-Primal($\mathcal{A}$) is locally compact.*

PROOF. By Lemma 2 each minimal primal ideal (which corresponds to a maximal limit set) has a compact neighbourhood. As X is Hausdorff, this proves the claim.

EXAMPLE 3.4. Let M_2 be the 2×2 matrices, and let D_2 be the diagonal matrices in M_2. Define

$$\mathcal{A} = \{x\colon [0,2] \longrightarrow M_2 \mid x \text{ is continuous and } x(t) \in D_2 \text{ for all } t \geq 1\}.$$

Then $\mathcal{A}^v$ is homeomorphic to the quotient space of $[0,2] \times \{0,1\}$ which one gets by identifying $(t,0)$ and $(t,1)$ for all $t \in [0,1)$. It is easily seen, that $\mathcal{A}$ satisfies the conditions of the above corollary, indeed Min-Primal($\mathcal{A}$) is homeomorphic to $[0,1] \cup \big((1,2] \times \{-1,1\}\big) \subset \mathbb{R}^2$. In this example each $P \in \mathcal{A}^v$ contains a unique minimal primal ideal I_p, and in view of [1], 5.1 one might ask whether the map $P \longrightarrow I_p$ is open continuous or whether the topology on Min-Primal($\mathcal{A}$) is given by the hull-kernel-process. Such maps are always open by the proof of [1], 5.1. Here $P \longrightarrow I_p$ is obviously discontinuous at $(1,0)$ and $(1,1)$. The last property also fails to hold.

In order to prove Theorem 1, i.e., to delete the closedness condition in Corollary 3.3, a few lemmas will be helpful.

LEMMA 3.5. *Let $\mathcal{A}$ be an arbitrary C^*-algebra, $I \in$ Id($\mathcal{A}$).*

(i) If $J \in$ Primal($\mathcal{A}$) then $I \cap J \in$ Primal(I).

(ii) For each $J \in$ Primal(I) there is a $\tilde{J} \in$ Primal($\mathcal{A}$) with $J = \tilde{J} \cap I$.

(iii) For each $K \in$ Min-Primal(I) there is a $\tilde{K} \in$ Min-Primal($\mathcal{A}$) with $K = \tilde{K} \cap I$.

PROOF. (i) is trivial since Id(I) $\subset$ Id($\mathcal{A}$).

(ii) $\tilde{J} := \{x \in \mathcal{A} \mid xI \subset J\}$ does the job.

(iii) By (ii) there exists $L \in$ Primal($\mathcal{A}$) such that $L \cap I = K$. If $\tilde{K} \in$ Min-Primal($\mathcal{A}$) is contained in L we know by (i) that $\tilde{K} \cap I$ is primal in I and hence $\tilde{K} \cap I = K$ by minimality.

The following lemma is the key step in the proof of Theorem 1.

LEMMA 3.6. *Let $\mathcal{A}^\vee$ be a compact space, $I \in \mathrm{Id}(\mathcal{A})$, $K \in X = \text{Min-Primal}(\mathcal{A})$ such that $K + I = \mathcal{A}$. Then there is an open neighbourhood U of K in X such that $\varphi_1 \colon U \longrightarrow \text{Min-Primal}(I)$. $\varphi_1(I) = I \cap L$, is a homeomorphism onto an open set in Min-Primal(I).*

PROOF.

1.) There is an open neighbourhood U of K such that for all $L \in U$ we have $I \not\subset Q$ whenever $L \subset Q \in \mathcal{A}^\vee$. To see this let $C := I^c \cap \mathcal{A}^\vee$. Then C is a closed and therefore compact subset of $\mathcal{A}^\vee$. By assumption $K \in U := \{L \in X \mid L^c \cap C = \emptyset\}$, and U is such a neighbourhood. Note that $U = \{L \in X \mid L + I = \mathcal{A}\}$.
2.) $L \cap I$ is in Min-Primal(I) for all $L \in U$. To see this let $L \in U$ and $J \in \text{Primal}(I)$ such that $J \subset L \cap I$ (by 3.5(i) $L \cap I$ is primal in I), and let us prove that $J = L \cap I$. By 3.5(ii) there is a $\tilde{J} \in \text{Primal}(\mathcal{A})$ with $J = I \cap \tilde{J}$. Suppose that $P \in \mathcal{A}^\vee$ and $L \subset P$. Then $I \not\subset P$ and $I \cap \tilde{J} = I \cap L \subset P$. Since P is prime we have $\tilde{J} \subset P$. This proves that $\tilde{J}$ is contained in L and hence $\tilde{J} = L$ by minimality. From this we see that φ_I is a well defined map $U \longrightarrow \text{Min-Primal}(I)$.
3.) φ_I is injective. Let $L_1, L_2 \in U$ and suppose $\varphi_I(L_1) = \varphi_I(L_2)$. If P is a primitive ideal containing L_1, then $I \not\subset P$ and $L_2 \cap I = L_1 \cap I \subset P$ and so $L_2 \subset P$ since P is prime. This shows $L_2 \subset L_1$ and therefore $L_1 = L_2$ by minimality or symmetry.
4.) If $U(I_1, \ldots, I_q) \cap X \subset U$ then $\varphi_I\big(U(I_1, \ldots, I_q) \cap X\big) = U(I_1 \cap I, \ldots, I_q \cap I) \cap \text{Min-Primal}(I)$. In particular φ_I is open and $\varphi_1(U)$ is contained in Min-Primal(I). To show this let $L \in U(I_1, \ldots, I_q) \cap X$ and assume $I_j \cap I \subset L \cap I$. Then $I_j \cap I \subset P$ for all $P \in \mathcal{A}^\vee$ such that $L \subset P$. Since $I \not\subset P$ we conclude $I_j \subset P$. Thus we arrive at the contradiction $I_j \subset L$. This gives $\varphi_1(L) \in U(I_1 \cap I, \ldots, I_q \cap I)$. If conversely $J \in U(I_1 \cap I, \ldots, I_q \cap I) \cap \text{Min-Primal}(I)$, then by 3.5(iii) there is an $L \in \text{Min-Primal}(\mathcal{A})$ with $\varphi_I(L) = J$. This proves 4.).
5.) Let J be an ideal in I, and let $U_I(J) := \{L \in \mathrm{Id}(I) \mid J \not\subset L\}$. Then we can check easily that $\varphi_I^{-1}\big(U_I(J) \cap \varphi_I(U)\big) = U(J) \cap U$, especially φ_I is continuous.

This finishes the proof of the lemma.

COROLLARY 3.7. *Let $\mathcal{A}^\vee$ be a compact space and assume that each maximal limit set has an open neighbourhood V such that the ideal corresponding to $V \subset \mathcal{A}^\vee$ has a locally compact minimal primal ideal space. Then Min-Primal($\mathcal{A}$) is locally compact.*

PROOF. Let $K \in X = \text{Min-Primal}(\mathcal{A})$. Then $K^c = \{Q \in \mathcal{A}^\vee \mid K \subset Q\}$ is a maximal limit set. Let V be a neighbourhood of the kind guaranteed in the assumptions, and let I be the corresponding ideal. Then by Lemma 3.6 K has a neighbourhood U in X, which is homeomorphic to an open subset of Min-Primal(I). Since this space is locally compact we are done.

Now let us attack the proof of the theorem. Let L be a maximal limit set in $\mathcal{A}^\vee$. By assumption there is an open neighbourhood U of L such that $(U \setminus L) \cup \{P\}$ is relatively Hausdorff for all $P \in L$. Let I be the ideal corresponding to U. By 3.7 it is enough to show that Min-Primal(I) is locally compact, and this will be an application of 3.3.

Since Prim(I) $\cong U$, Prim(I) is a T_1-space and all limit sets are finite. Let L_0 be a maximal limit set in U, say $L_0 = \lim P_\alpha$ where $P_\alpha \in U$, in the relative topology on U.

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Assume first $L_0 \cap L = \emptyset$. Then $P_\alpha \in U \setminus L$ finally and so L_0 is a singleton $\{Q\}$ since $U \setminus L$ is Hausdorff. As $Q \not\subset L$, there is a relatively closed neighbourhood V of Q disjoint from L, so $V = (V \setminus L_0) \cup \{Q\}$ is Hausdorff.

Assume next, that $L \cap L_0 \neq \emptyset$, say $P_\alpha \longrightarrow P \in L \cap L_0$. If we had $Q \in L_0 \setminus L$, then $P_\alpha \in (U \setminus L) \cup \{P\}$ finally, and since this is a Hausdorff space, we conclude $Q = P$, which is impossible. So we have $L_0 = L$, and clearly $V := U$ is a relatively closed neighbourhood of L_0 such that $(V \setminus L_0) \cup \{P\}$ is Hausdorff for all $P \in L_0$.

By Corollary 3.3 Min-Primal(I) is locally compact, and we are done.

EXAMPLE 3.8. Consider Example 4.12 of [1], and the notation used there. A neighbourhood base of Q_1, Q_2 and Q_3 is given by $\{Q_1\} \cup M_1 \cup M_3$, $\{Q_2\} \cup M_1 \cup M_2$ and $\{Q_3\} \cup M_2 \cup M_3$ respectively, where M_i runs through the cofinal subsets of $\{P_{3n+i-1} \mid n \in \mathbb{N}\}$. So a closed neighbourhood F of the maximal limit set $L = \{Q_1, Q_2\}$ also contains Q_3, and so $(F \setminus L) \cup \{Q_i\}$ contains two elements of $\{Q_1, Q_2, Q_3\}$, i.e., it isn't Hausdorff. In this example Corollary 3.3 is not applicable, but Theorem 3.1 obviously is.

4. **Negative examples.**

EXAMPLE 4.1. Let M_2 and D_2 like in 3.4. Define $B = \bigcup_{n \in \mathbb{N}} \left[\frac{1}{2n+1}, \frac{1}{2n}\right]$ and

$$\mathcal{A} := C([0,1], M_2) = C[0,1] \otimes M_2,$$
$$\mathcal{A}_1 := \{x \in \mathcal{A} \mid x(t) \in D_2 \text{ for all } t \text{ in } B\},$$
$$\mathcal{A}_2 := \{x \in \mathcal{A}_1 \mid x(0)_{1,1} = x(0)_{2,2}\},$$
$$\mathcal{A}_3 := \{x \in \mathcal{A}_1 \mid x(0)_{2,2} = 0\}.$$

Let $X_i :=$ Min-Primal($\mathcal{A}_i$). $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are liminal separable C^*-algebras which are continuous fields of finite dimensional C^*-algebras. $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital, $\mathcal{A}_3$ is an ideal of $\mathcal{A}_1$. The following will be shown.

(i) X_2 and X_3 are not locally compact. So $\mathcal{A}_2$ is a liminal separable unital C^*-algebra with $\dim(\pi) \leq 2$ for all irreducible representations π of $\mathcal{A}_2$, and Min-Primal($\mathcal{A}_2$) is not locally compact.

(ii) X_1 is locally compact, and by (i) the ideal $\mathcal{A}_3$ of $\mathcal{A}_1$ doesn't share this property.

a) First of all let us describe the spectra $\mathcal{A}_i^\vee$. We will identify irreducible representations and primitive ideals. For t in $(0,1] \setminus B$ define $\pi_t(x) = x(t)$, for $t \in B \cup \{0\}$ let $\lambda_t(x) := x(t)_{1,1}$ and $\mu_t(x) := x(t)_{2,2}$. Let P be the set of all the π_t, λ_t and μ_t where $0 < t \leq 1$. An application of [4], 10.4.3 tells us $\mathcal{A}_2^\vee = \mathcal{A}_3^\vee = P \cup \{\lambda_0\}$ (equality as sets, not as topological spaces) and $\mathcal{A}_1^\vee = P \cup \{\lambda_0, \mu_0\}$. The Jacobson topology on $\left\{\pi_t \mid t \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right)\right\}$ coincides with the Euclidean topology on the interval, the same holds true for $\left\{\lambda_t \mid t \in \left[\frac{1}{2n+1}, \frac{1}{2n}\right]\right\}$ and $\left\{\mu_t \mid t \in \left[\frac{1}{2n+1}, \frac{1}{2n}\right]\right\}$. A neighbourhood base of λ_s, $s = \frac{1}{2n+1}$, is given by the sets $\{\pi_t \mid t \in (s-\varepsilon, s)\} \cup \{\lambda_t \mid t \in [s, s+\varepsilon)\}$ where $\varepsilon > 0$. There are similar neighbourhood bases for μ_s and for $s = \frac{1}{2n}$. From this we can conclude that $(\pi_t)_t$ converges for $t \longrightarrow s = \frac{1}{2n+1}$ to $\{\lambda_s, \mu_s\}$ from below and similarly $(\pi_t)_t$ converges to $\{\lambda_s, \mu_s\}$ as $t \longrightarrow s = \frac{1}{2n}$ from above.

b) LEMMA. *Let $((A_t)_{t\in T}, \mathcal{A})$ be a continuous field of C^*-algebras on a locally compact Hausdorff space T. For an ideal $I \subset \mathcal{A}$ let $I(t) := \{x(t) \mid x \in I\}$. For each primal ideal I there is a unique $t \in T$ such that $I(t) \neq \mathcal{A}(t)$. In this situation we say that I belongs to t.*

(The proof of [4], 10.4.3. applies here).

c) By [1], 3.1 and 3.2 for each primal ideal I of $\mathcal{A}_i$ there is a limit set in $\mathcal{A}_i^\vee$ containing all primitive ideals in I^c. This fact together with a) and b) gives us the following list of minimal primal ideals: $\pi_t, t \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right)$; λ_t and μ_t, $t \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right)$; and $J(s) := \lambda_s \cap \mu_s$, $s = \frac{1}{n}$. By a) we conclude that $K_n := \left\{J\left(\frac{1}{2n}\right), J\left(\frac{1}{2n-1}\right)\right\} \cup \left\{\pi_t \mid t \in \left(\frac{1}{2n}, \frac{1}{2n-1}\right)\right\}$, $L_n := \left\{\lambda_t \mid t \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right)\right\}$ and $M_n := \left\{\mu_t \mid t \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right)\right\}$ are clopen, and K_n is compact. A minimal primal ideal which is not in $X = \bigcup_n (K_n \cup L_n \cup M_n)$ must belong to 0.

d) Now let $i \in \{2, 3\}$. By b) we know $X_i = X \cup \{\lambda_0\}$. It is routine to verify that a closed neighbourhood base for λ_0 is given by the sets

$$W_N := \{\lambda_0\} \cup \bigcup_{N \leq n} (K_n \cup L_n \cup M_n) \quad \text{if } i = 2$$

$$W_N := \{\lambda_0\} \cup \bigcup_{N \leq n} (K_n \cup L_n) \qquad \text{if } i = 3.$$

So each neighbourhood U of λ_0 contains L_n as a closed subset for large n, and so no neighbourhood of λ_0 can be compact.

e) In the case $i = 1$ we have $X_1 = X \cup \{\lambda_0 \cap \mu_0\}$, and a neigbourhood base of $\lambda_0 \cap \mu_0$ is given by the sets $W_N = \{\lambda_0 \cap \mu_0\} \cup \bigcup_{N \leq n} K_n$. These sets are clearly compact, and we are done.

f) It can be shown, that $\mathcal{A}_3^\vee$ is nearly Hausdorff. This shows that in Theorem 3.1 you cannot remove the assumption, that the C^*-algebra is unital.

EXAMPLE 4.2. There is a C^*-algebra $\mathcal{A}$ with an ideal I such that Prim(I) and Prim($\mathcal{A}/I$) are Hausdorff, but Min-Primal($\mathcal{A}$) is not locally compact. Let $\mathbb{N}_\infty$ be the one point compactification of $\mathbb{N}$, and let $\mathcal{K}$ be the compact operators on a separable Hilbert space with a given orthonormal basis.

$$\mathcal{A} := \left\{x \in C(\mathbb{N}_\infty^2, \mathcal{K}) \mid x(\infty, n) = x(n, \infty) = \operatorname{diag}\left(\lambda_1(x), \ldots, \lambda_n(x), 0, \ldots\right)\right\}$$

$\mathcal{A}$ is a separable liminal C^*-algebra which is a continuous field of C^*-algebras on the base space $\mathbb{N}_\infty^2$. We have $\mathcal{A}_{(n,m)} = \mathcal{K}$ for $n, m \in \mathbb{N}$, $\mathcal{A}_{(n,\infty)} = \mathcal{A}_{(\infty,n)} = \mathbb{C}^n \subset \mathcal{K}$ and $\mathcal{A}_{(\infty,\infty)} = \mathcal{D} \cap \mathcal{K}$ (diagonal operators in $\mathcal{K}$). From [4], 10.4.3 we conclude that the following list of irreducible representations is complete.

$$\pi_{(n,m)}(x) := x(n, m) \text{ for } n, m \in \mathbb{N} \text{ and } \lambda_1, \lambda_2, \lambda_3, \ldots$$

Let $\rho_n : \mathcal{A} \longrightarrow \mathbb{C}^n$ be the $*$-homomorphism $\rho_n(x) = \left(\lambda_1(x), \ldots, \lambda_n(x)\right)$. Define $I_n = \ker \rho_n$. Then for any x in $\mathcal{A}$ we have $\|x + \ker \pi_{n,m}\| = \|x(n, m)\| \underset{m}{\longrightarrow} \max\{\,|\lambda_1(x)|, \ldots,$

$|\lambda_n(x)|\} = \|\rho_n(x)\| = \|x+I_n\|$. It is easy to see that all the $\ker \pi_{n,m}$ are minimal primal, and so I_n is in the τ-closure of Min-Primal($\mathcal{A}$). Now let $I_\infty := \bigcap_{n\in\mathbb{N}} I_n$. Then I_∞ is primal as all the I_n are primal. Since any proper ideal of I_∞ must be a proper ideal of some $\ker \pi_{(n,m)}$, we may conclude by that I_∞ in fact is minimal primal.

Now if U were a compact neighbourhood of I_∞ in X then U would be compact in the Hausdorff space $\big(\mathrm{Id}(\mathcal{A}), \tau\big)$ and hence closed. Since there is $N \in \mathbb{N}$ such that $\ker \pi_{n,m} \in U$ for all $n, m \geq N$, we arrive at the contradiction $I_N \in U$, since I_N is not in X.

(Another possible argument is: by the above we have shown that X is not open in its τ-closure which is a locally compact space, and so it cannot be locally compact by ([7], 18.4).

We have $\mathrm{Prim}(\mathcal{A}/I_\infty) = \{\lambda_n \mid n \in \mathbb{N}\}$ and $\mathrm{Prim}(I_\infty) \cong \mathbb{N}^2$ equipped with the discrete topology. This establishes the promised example.

EXAMPLE 4.3. There is a C^*-algebra $\mathcal{B}$ such that Min-Primal($\mathcal{B}$) is locally compact and a quotient of $\mathcal{B}$ which does not have this property. Let $\mathcal{A}_2$ be the C^*-algebra of Example 4.1 and define

$$\mathcal{B} := \{x \in C([0,1] \times \mathbb{N}_\infty, M_2) \mid x(\cdot,\infty) \in \mathcal{A}_2\}.$$

The irreducible representations of $\mathcal{B}$ are $\pi_{(t,n)}(x) = x(t,n)$ where $(t,n) \in [0,1] \times \mathbb{N}$, $\pi_t(x) = x(t,\infty)$, $\lambda_t(x) = x(t,\infty)_{1,1}$ and $\mu_t(x) = x(t,\infty)_{2,2}$ like in Example 4.1. Let $I := \{x \in \mathcal{B} \mid x(\cdot,\infty) = 0\}$, then $\mathcal{B}/I \cong \mathcal{A}_2$ and so by 4.1 Min-Primal($\mathcal{B}/I$) is not locally compact, let us show that Min-Primal($\mathcal{B}$) is. Obviously we have $\pi_{(t,n)} \underset{n}{\longrightarrow} \lambda_t \cap \mu_t$ which shows that $\lambda_t \cap \mu_t$ is primal hence minimal primal. This yields Min-Primal($\mathcal{B}$) $\cong [0,1] \times \mathbb{N}_\infty$ which clearly is locally compact.

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