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POSITIVE LINEAR MAPPINGS BETWEEN C^*-ALGEBRAS

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ABSTRACT. We prove that a positive unital linear mapping from a von Neumann algebra to a unital C^*-algebra is a Jordan homomorphism if it maps invertible self-adjoint elements to invertible elements, and that for any compact Hausdorff space X, all positive unital linear mappings from $C(X)$ into a unital C^*-algebra that preserve the invertibility for self-adjoint elements are $*$-homomorphisms if and only if X is totally disconnected.

Let $\mathcal{A}$ be a unital C^*-algebra. An element of $\mathcal{A}$ is *positive* if it is self-adjoint with non-negative spectrum. Suppose ϕ is a linear mapping from $\mathcal{A}$ to another unital C^*-algebra $\mathcal{B}$. Consider the following conditions on ϕ:

(1) ϕ maps the unit element of $\mathcal{A}$ to the unit element of $\mathcal{B}$,

(2) ϕ maps self-adjoint elements of $\mathcal{A}$ to self-adjoint elements of $\mathcal{B}$, or equivalently, $\phi(A^*) = \phi(A)^*$ for all $A \in \mathcal{A}$,

(2′) ϕ maps positive elements to positive elements,

(3) ϕ maps invertible elements to invertible elements,

(3′) ϕ maps invertible self-adjoint elements to invertible elements.

Depending on which condition ϕ satisfies, we call it (1) *unital*, (2) *self-adjoint*, (2′) *positive*, (3) *invertibility preserving*, and (3′) *invertibility preserving for self-adjoint elements*, respectively. It is obvious that if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\phi: \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (1) and (3′), then ϕ is self-adjoint if and only if it is positive.

Positive linear mappings have been widely studied. It is known that if $\phi: \mathcal{A} \longrightarrow \mathcal{B}$ is a positive linear mapping, then ϕ is bounded and $\|\phi\| \leq 2\|\phi(1)\|$ [10, Proposition 2.1], and that if $\phi: \mathcal{A} \longrightarrow \mathcal{B}$ is a unital linear mapping, then ϕ is positive if and only if $\|\phi\| = 1$ [12, Corollary 1].

A *Jordan homomorphism* of a C^*-algebra $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$ is a linear self-adjoint mapping ϕ with the property that $\phi(A^2) = \phi(A)^2$ for every self-adjoint element $A \in \mathcal{A}$. The concept of Jordan homomorphism is from Kaplansky [9]. It has been shown ([6], [7], [13]) that any Jordan homomorphism is the sum of a $*$-homomorphism and a $*$-anti-homomorphism, and therefore, a $*$-homomorphism if the range is commutative, and that any Jordan homomorphism is a $*$-homomorphism if the domain is commutative.

A number of sufficient conditions for a linear mapping to be a Jordan homomorphism have been obtained. Let $\phi: \mathcal{A} \longrightarrow \mathcal{B}$ be a linear mapping between unital C^*-algebras $\mathcal{A}$

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and $\mathcal{B}$. It has been proven that ϕ is a Jordan homomorphism if it is unital and maps unitary elements of $\mathcal{A}$ to unitary elements of $\mathcal{B}$ [12, Corollary 2], and that ϕ is a Jordan homomorphism if $\mathcal{A}$ is a von Neumann algebra and ϕ satisfies conditions (1), (2) and (3) [11, Theorem 2]. Similar results can also be found in [1].

After proving [11, Theorem 2] Russo asked the following question: Does the result remain true if $\mathcal{A}$ is only a unital C^*-algebra? It is Russo's observation that there is no loss of generality in assuming $\mathcal{A}$ to be commutative since only self-adjoint elements are involved. When the range of the linear mapping is also commutative, the question has been answered positively by Gleason [5] and Kahane and Zelazko [8].

A topological space X is called *totally disconnected* if every component in X is a singleton (Dugundji [3]). Through a careful examination of the proof of [11, Theorem 2], we can see that the assumption of $\mathcal{A}$ being a von Neumann algebra is needed only to ensure that any given self-adjoint element A can be approximated by real linear combinations of commutative orthogonal projections in $\mathcal{A}$ commuting with A. Suppose X is a compact, totally disconnected, Hausdorff topological space. Then X has a topological base consisting of open-and-closed sets [4, Theorem 6.2.9]. Therefore, every real continuous function on X can be approximated by real linear combinations of characteristic functions of mutually disjoint open-and-closed sets in X. Consequently, any linear mapping ϕ from $C(X)$ into a unital C^*-algebra $\mathcal{B}$ satisfying conditions (1), (2) and (3) is a $*$-homomorphism.

For any bounded Borel function h on the unit circle, the restriction T_h of the multiplication operator M_h to the Hardy space $\mathcal{H}^2$ (consisting of all $\mathcal{L}^2$-functions whose negative Fourier coefficients are 0) is called the *Toeplitz operator* induced by h. M-D. Choi et al proved [1, Theorem 2] that if X is a compact Hausdorff space containing a continuous injective image of [0, 1], then there exists a linear mapping ϕ from $C(X)$ to $\mathcal{B}(\mathcal{H}^2)$ that satisfies the conditions (1), (2) and (3) but is not a Jordan homomorphism. The proof involves Toeplitz operators and is based on the fact (Douglas [2, Corollary 7.28]) that the spectrum $\sigma(T_h)$ for a continuous function h is the range of h together with those points not in the range with respect to which h has non-zero winding number. At the end of [1], M-D. Choi et al asked the following question: What is the necessary and sufficient condition on X that forces all linear mappings from $C(X)$ into a unital C^*-algebra satisfying (1), (2) and (3) to be $*$-homomorphisms? The main result of this paper is a theorem (Theorem 6) that answers this question with condition (3) replaced by (3$'$).

It is clear that condition (3$'$) is significantly weaker than condition (3). Moreover, when we study Jordan homomorphisms, condition (3$'$) is more natural than condition (3) since in most cases only self-adjoint elements are involved. Later it will be shown that condition (3$'$) is the right one for characterizing the total disconnectedness of a compact Hausdorff space. Thus any generalization of known results obtained by only replacing condition (3) with (3$'$) is meaningful. The following two results are just such generalizations. Their proofs are essentially the same as those of [11, Lemma 3] and [11, Theorem 2] respectively.

LEMMA 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Suppose $\phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a linear mapping and satisfies conditions (1), (2) and (3′). Then*

(i) ϕ maps projections into projections.

(ii) ϕ maps every pair of orthogonal projections into a pair of orthogonal projections.

THEOREM 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Suppose any self-adjoint element $A \in \mathcal{A}$ can be approximated by real linear combinations of commutative orthogonal projections in $\mathcal{A}$ commuting with A. Then every linear mapping $\phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ that satisfies conditions (1), (2) and (3′) is a Jordan homomorphism.*

COROLLARY 3. *Suppose $\mathcal{A}$ is a von Neumann algebra and $\mathcal{B}$ is a unital C^*-algebra. Then every linear mapping $\phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ that satisfies conditions (1), (2) and (3′) is a Jordan homomorphism.*

PROOF. It follows immediately from Theorem 2. ∎

COROLLARY 4. *Let $\mathcal{B}$ be a unital C^*-algebra. If X is a compact, totally disconnected Hausdorff space, then every linear mapping $\phi\colon C(X) \longrightarrow \mathcal{B}$ that satisfies conditions (1), (2) and (3′) is a Jordan homomorphism, and therefore, a $*$-homomorphism.*

PROOF. Since the total disconnectedness of X implies that the conditions of Theorem 2 are satisfied with $\mathcal{A} = C(X)$, the result follows immediately. ∎

The following theorem is the converse of Corollary 4. It shows that condition (3′) is the right one for characterizing the total disconnectedess of a compact Hausdorff space.

THEOREM 5. *Let X be a compact Hausdorff space. Suppose all linear mappings from $C(X)$ into unital C^*-algebras that satisfy conditions (1), (2) and (3′) are Jordan homomorphisms. Then X is totally disconnected.*

PROOF. Suppose, on the contrary, there exist two distinct points u and v in the same component X_0 of X. Form the disjoint union $X \cup_d [0,1]$, and let $\tilde{X}$ be the topological space derived from $X \cup_d [0,1]$ by identifying u with 0 and v with 1. Then $\tilde{X}$ is compact and Hausdorff.

Define θ to be the mapping $\theta\colon C(X) \longrightarrow C(\tilde{X})$ given by

$$(\theta f)(x) = \begin{cases} f(x) & \text{if } x \in \tilde{X} \setminus [0,1] \\ (1-x)f(u) + xf(v) & \text{if } x \in [0,1] \end{cases} \qquad f \in C(X).$$

It is easy to check that θ is well-defined, and that θ is linear, unital and positive. For any real $f \in C(X)$, since X_0 is connected, $f(X_0)$ is a connected subset of the real line, and hence, an interval. Therefore, $u, v \in X_0$ implies that

$$(1-x)f(u) + xf(v) \in f(X_0)$$

for all $x \in [0,1]$. Thus, θ satisfies (3′) because

$$\sigma\big(\theta(f)\big) = (\theta f)(\tilde{X}) = f(X) = \sigma(f).$$

Fix a continuous surjective mapping h from the unit circle to the unit interval $[0, 1]$, and let $\tau: [0, 1] \longrightarrow \tilde{X}$ be the embedding of $[0, 1]$ into $\tilde{X}$, *i.e.*,

$$\tau(x) = x \quad x \in [0, 1].$$

Define ψ to be the linear mapping $\psi: C(\tilde{X}) \longrightarrow \mathcal{B}(\mathcal{H}^2)$ given by

$$\psi(f) = T_{f\circ\tau\circ h},$$

where $\mathcal{H}^2$ is the Hardy space and $T_{f\circ\tau\circ h}$ is the Toeplitz operator induced by the continuous function $f\circ\tau \circ h$ on the unit circle. Straightforward verifications show that ψ is linear and unital, and that $\psi(1) = 1 = \|\psi\|$. It follows from [12, Corollary 1] that ψ is positive.

Let $\phi = \psi \circ \theta$. Then $\phi: C(X) \longrightarrow \mathcal{B}(\mathcal{H}^2)$ is linear and satisfies (1), (2) and (3′). We complete the proof by showing that the linear mapping ϕ is not a Jordan homomorphism. Choose any real function $f \in C(X)$ with $f(u) = 0$ and $f(v) = 1$. Such functions do exist by the Tietze Extension Theorem. By the definition, $\phi(f) = T_{(\theta f)\circ\tau\circ h}$. Since, for any $x \in [0, 1]$,

$$\theta(f^2)(x) = (1 - x)f^2(u) + xf^2(v) = \theta(f)(x),$$

we have that $\phi(f^2) = \phi(f)$. Therefore, it is impossible that $\phi(f^2) = [\phi(f)]^2$, for otherwise we have that $\phi(f) = [\phi(f)]^2$ is a self-adjoint idempotent. It follows that $\phi(f)$ is a projection, and consequently $\sigma\big(\phi(f)\big) \subseteq \{0, 1\}$. However, $\sigma\big(\phi(f)\big) = \big(\theta(f)\big)([0, 1]) = [0, 1]$, and we have a contradiction. ∎

We finish this paper with a theorem that answers one of the questions raised by Choi et al at the end of [1].

THEOREM 6. *Let X be a compact Hausdorff space. Then all linear mappings from $C(X)$ to a unital C^*-algebras satisfying conditions (1), (2) and (3′) are $*$-homomorphisms if and only if X is totally disconnected.*

PROOF. It follows immediately from Corollary 4 and Theorem 5. ∎

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