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SEIBERG–WITTEN FLOW IN HIGHER DIMENSIONS

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Abstract

We show that for manifolds of dimension $m \geq 5$, the flow of a Seiberg–Witten-type functional admits a global smooth solution.

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1. Introduction

The Seiberg–Witten invariant has proven a very effective tool in four-dimensional geometry. Its computation involves finding nontrivial solutions to the system of first-order Seiberg–Witten equations, called monopoles. Monopoles represent the zeros of the Seiberg–Witten functional (1.1) (see [10]). In [5], the flow for the Seiberg–Witten functional on a 4-manifold was studied. It was shown that the flow admits a global solution which converges in C^∞ to a critical point of the functional.

The Seiberg–Witten equations and functional do not generalize immediately to higher dimensions, since they depend on the notion of self-duality on four-dimensional manifolds. Nonetheless, a number of generalizations of Seiberg–Witten theory have been suggested for higher-dimensional manifolds (see, for example, [1, 2, 4]). In this paper, we extend the global existence result obtained for the Seiberg–Witten functional in [5] for dimension four to a functional of similar form in higher dimensions.

Let M be a compact oriented Riemannian m-manifold which admits a Spin^c structure $\mathfrak{s}$. Denote by $\mathcal{S} = W \otimes \mathcal{L}$ the corresponding spinor bundle, and by $\mathcal{L}^2$ the corresponding determinant line bundle. Let A be a unitary connection on $\mathcal{L}^2$. Note that we can write $A = A_0 + a$, where A_0 is some fixed connection and $a \in i\Lambda^1 M$ with $i = \sqrt{-1}$. Denote by $F_A = dA \in i\Lambda^2 M$ the curvature of the line bundle connection A.

Let $\{e_j\}$ be a local orthonormal basis of the tangent bundle TM. A Spin(4)^c-connection on the spinor bundle $\mathcal{S}$ is locally defined by

$$\nabla_A = d + \tfrac{1}{2}(\omega + A),$$

where $\omega = \sum_{i<j} \omega_{ij} e_i e_j$ is induced by the Levi-Civita connection matrix ω_{jk} and $e_j e_k$ acts by Clifford multiplication (see [7]). We denote the curvature of ∇_A by Ω_A. We define the configuration space $\Gamma(\mathcal{S}) \times \mathscr{A}$, where $\mathscr{A}$ is the space of unitary connections on $\mathcal{L}^2$, and let $(\varphi, A) \in \Gamma(\mathcal{S}) \times \mathscr{A}$. Note that in [5] we took $\varphi \in \Gamma(\mathcal{S}^+)$. However, the splitting $\mathcal{S} = \mathcal{S}^+ \bigoplus \mathcal{S}^-$ is available only if m is even. The exact nature of the bundle to which φ belongs does not affect our results, and we may assume $\varphi \in \Gamma(\mathcal{S})$ for simplicity.

We first recall the definition of the Seiberg–Witten functional on 4-manifolds. The Seiberg–Witten functional $\mathcal{SW} : \Gamma(\mathcal{S}^+) \times \mathscr{A} \to \mathbb{R}$ is given by

$$\mathcal{SW}(\varphi, A) = \int_M |\nabla_A \varphi|^2 + |F_A^+|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4 \, dV, \tag{1.1}$$

where S is the scalar curvature of M. The Seiberg–Witten functional (1.1) is invariant under the action of a gauge group. The group of gauge transformations is

$$\mathscr{G} = \{g : M \to U(1)\}.$$

The group $\mathscr{G}$ acts on elements of the configuration space via

$$g^*(\varphi, A) = (g^{-1}\varphi, A + 2g^{-1} dg).$$

Using the relation

$$\|F_A\|_{L^2} = 2\|F_A^+\|_{L^2} - 4\pi^2 c_1(\mathcal{L}^2)^2,$$

where $c_1(\mathcal{L}^2)$ is the first Chern class of $\mathcal{L}^2$ (see [11]), one can also write the functional in the form

$$\mathcal{SW}(\varphi, A) = \int_M |\nabla_A \varphi|^2 + \frac{1}{2}|F_A|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4 \, dV + \pi^2 c_1(\mathcal{L}^2)^2. \tag{1.2}$$

Now, consider again the case of an m-manifold M. The functional (1.1) is not defined here, since self-duality is a phenomenon that occurs only in dimension four. However, we may use (1.2) to extend the Seiberg–Witten functional to higher dimensions. As mentioned, we can allow $\varphi \in \Gamma(\mathcal{S})$ in the case where m is odd. Note that the constant term $\pi^2 c_1(\mathcal{L}^2)^2$ does not affect the Euler–Lagrange equations and so is irrelevant for the results in this paper. The Euler–Lagrange equations for the Seiberg–Witten functional are

$$-\nabla_A^* \nabla_A \varphi - \tfrac{1}{4}[S + |\varphi|^2]\varphi = 0,$$
$$-d^* F_A - i \,\mathrm{Im}\langle \nabla_A \varphi, \varphi \rangle = 0.$$

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As in [5], we define the flow of the Seiberg–Witten functional by

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= -\nabla_A^* \nabla_A \varphi - \frac{1}{4}[S + |\varphi|^2]\varphi, \\ \frac{\partial A}{\partial t} &= -d^* F_A - i\mathrm{Im}\langle \nabla_A \varphi, \varphi \rangle, \end{aligned} \tag{1.3}$$

with initial data

$$(\varphi(0), A(0)) = (\varphi_0, A_0).$$

Regarding the existence of solutions to the flow (1.3), we prove the following theorem.

THEOREM 1.1. *For any given smooth initial data* (φ_0, A_0) *and* m*-dimensional Riemannian manifold* M *for* $m \geq 5$*, equations (1.3) admit a unique global smooth solution on* $M \times [0, \infty)$.

In proving global existence in dimension four, a blow-up or rescaling argument was used in order to obtain a contradiction with the assumption of singularity formation. Importantly, the boundedness of $\int_M |F_A|^2\, dV$ under the flow was used to imply the boundedness of the corresponding energy $\int_{\mathbb{R}^4} |F_{\tilde{A}}|^2\, dy$ of the limiting curvature $F_{\tilde{A}}$ on the rescaled space. In higher dimensions, however, this observation is not sufficient to ensure a bound on the rescaled energy. This necessitates a stronger result through which to obtain the desired contradiction, along with some modifications to the blow-up argument.

The main additional estimate needed in establishing global existence in higher dimensions is a so-called monotonicity formula. This idea was used by Struwe for the heat flow of harmonic maps in higher dimensions [13], and has also been used to study the Yang–Mills and Yang–Mills–Higgs flows in higher dimensions [6, 12]. See also [8, 9] for the harmonic map flow, and [14] for sequences of weakly converging Yang–Mills connections.

2. Estimates

As in the four-dimensional case [5], we have an energy inequality

$$\frac{d}{dt}\mathcal{SW}(\varphi(t), A(t)) = -\int_M \left[2\left|\frac{\partial \varphi}{\partial t}\right|^2 + \left|\frac{\partial A}{\partial t}\right|^2\right] \leq 0,$$

or

$$\int_0^T \left[2\left\|\frac{\partial \varphi}{\partial t}\right\|_{L^2}^2 + \left\|\frac{\partial A}{\partial t}\right\|_{L^2}^2\right] = \mathcal{SW}(\varphi_0, A_0) - \mathcal{SW}(\varphi(T), A(T)). \tag{2.1}$$

The proof of the energy inequality and many other results from [5] do not contain dimensional considerations, and are also valid in the m-dimensional case. For the proof of the energy inequality and of the following lemmas, we direct the reader to that paper.

The first step is to establish the existence of a local solution to the flow.

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LEMMA 2.1. *For any given smooth initial data* (φ_0, A_0)*, the system (1.3) admits a unique local smooth solution on* $M \times [0, T)$ *for some* $T > 0$.

In this paper, (φ, A) will typically represent a local solution to (1.3) on $M \times [0, T)$ for some initial value (φ_0, A_0). Next, Lemma 2.2 gives us a uniform bound on φ under the flow.

LEMMA 2.2. *Write* $S_0 = \min\{S(x) : x \in M\}$ *and* $k_0 = \sup_{x\in M} |\varphi_0|$. *Then for all* $t \in [0, T)$,

$$\sup_{x\in M} |\varphi(x, t)| \le \max\{k_0, \sqrt{|S_0|}\}.$$

The following Bochner formula gives us a constraint on the evolution of the first derivatives of φ and A.

LEMMA 2.3. *There exist positive constants* c, c' *such that the following estimate holds:*

$$\begin{aligned}&\frac{\partial}{\partial t}(|\nabla_A\varphi|^2 + |F_A|^2) + \Delta(|\nabla_A\varphi|^2 + |F_A|^2)\\ &\quad\le -c'(|\nabla_A^2\varphi|^2 + |\nabla F_A|^2) + c(|F_A| + 1)(|\nabla_A\varphi|^2 + |F_A|^2 + 1).\end{aligned}$$

Finally, the following lemma and corollary show that a bound on the first derivatives of φ and A implies a bound on derivatives of all orders.

LEMMA 2.4. *Suppose that* $|\nabla_A\varphi| \le K_1$ *and* $|F_A| \le K_1$ *in* $M \times [0, T)$ *for some constant* $K_1 > 0$. *Then for any positive integer* $n \ge 1$*, there is a constant* K_{n+1} *independent of* T *such that*

$$|\nabla_A^{(n+1)}\varphi| \le K_{n+1}, \quad |\nabla_M^{(n)}F_A| \le K_{n+1} \quad \text{in } M \times [0, T),$$

where (n) *denotes* n *iterations of the derivative.*

COROLLARY 2.5. *Suppose that* $|\nabla_A^{(j)}\varphi| \le K_n$ *and* $|\nabla_M^{(j-1)}F_A| \le K_n$ *in* $P_1(x_0, t_0)$ *for each* $1 \le j \le n$ *and some constant* K_n. *Then there is a positive constant* K_{n+1} *such that*

$$|\nabla_A^{(n+1)}\varphi| \le K_{n+1}, \quad |\nabla_M^{(n)}F_A| \le K_{n+1} \quad \text{in } P_{1/2}(x_0, t_0).$$

In order to extend the global existence result in four dimensions to higher dimensions, we begin be deriving a monotonicity inequality for the flow (1.3). We define

$$e(\varphi, A)(x, t) = |\nabla_A\varphi|^2 + \frac{1}{2}|F_A|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4.$$

Let $z = (x, t)$ denote a point of $M \times \mathbb{R}$, with $z_0 = (x_0, t_0) \in M \times [0, T]$. We define

$$T_R(z_0) = M \times [t_0 - 4R^2, t_0 - R^2]$$

and

$$P_R(z_0) = B_R(x_0) \times [t_0 - R^2, t_0],$$

where $B_R(x_0) \subset M$ denotes a ball centered at x_0 with radius R. Note that in constructing $T_R(z_0)$ we require that $t_0 - 4R^2 \ge 0$ or $R \le \sqrt{t_0}/2$. We abbreviate $T_R(0, 0) = T_R$

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and $P_R(0,0) = P_R$. The fundamental solution to the backward heat equation with singularity at z_0 is

$$G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\Big(-\frac{(x - x_0)^2}{4(t_0 - t)}\Big),$$

where $t < t_0$. We also write $G = G_{(0,0)}$. Let $i(M)$ be the injectivity radius of M, and suppose that (φ, A) is a solution to the flow (1.3) on $M \times [0, T)$. Let ϕ_x be a smooth cut-off function with $|\phi_x| \le 1$, $\phi_x \equiv 1$ on $B_{i(M)/2}(x)$, $\phi_x \equiv 0$ outside $B_{i(M)}(x)$ and $|\nabla\phi_x| \le c/i(M)$ for some constant c. We also abbreviate $\phi = \phi_{x_0}$. We define

$$\Phi(R;\varphi, A) = R^2 \int_{T_R(z_0)} e(\varphi, A)(z)\phi^2 G\, dV\, dt$$

and

$$\mathscr{F}(R;\varphi, A) = \int_{T_R(z_0)} Rt\Big(\Big|\frac{\partial A}{\partial t} + \frac{x_k}{2t}\frac{\partial}{\partial x_k}\lrcorner F_A\Big|^2 + 2\Big|\frac{\partial\varphi}{\partial t} + \frac{x_k}{2t}\nabla^k_A\varphi\Big|^2\Big)\phi^2 G\sqrt{g}\, dz,$$

where

$$\frac{\partial}{\partial x_k}\lrcorner F_A = F_A\Big(\frac{\partial}{\partial x_k}, \cdot\Big) = F^{kj}\, dx^j$$

defines a 1-form.

Lemma 2.6. *Let (φ, A) be a smooth solution of (1.3) on $M \times [0, T)$ with initial data (φ_0, A_0). Then for $z_0 \in M \times [0, T]$ and any R_a and R_b satisfying $0 < R_a \le R_b \le R_0$ for some $R_0 \le \min\{i(M), \sqrt{t_0}/2\}$,*

$$\Phi(R_a;\varphi, A) + \int_{R_a}^{R_b} e^{aR}\mathscr{F}(R)\, dR \le e^{c(R_b - R_a)}\Phi(R_b;\varphi, A) + c(R_b^2 - R_a^2)\mathcal{SW}(\varphi_0, A_0),$$

where c depends only on the geometry of M.

Proof. We show that

$$\frac{d}{dR}\Phi(R;\varphi, A) + \mathscr{F}(R;\varphi, A) \ge -c\Phi(R;\varphi, A) - cR\mathcal{SW}(\varphi_0, A_0). \tag{2.2}$$

The required result then follows by multiplying (2.2) by e^{aR} for some sufficiently large $a > 0$, and integrating from R_a to R_b. To show (2.2), we may assume that $z_0 = (0, 0)$, which implies that $t < 0$ on T_R. We rescale the coordinates to $x = R\tilde{x}$, $t = R^2\tilde{t}$. In these coordinates,

$$\Phi(R;\varphi, A) = \int_{T_1} R^4 e(\varphi, A)(R\tilde{x}, R^2\tilde{t})\phi^2(R\tilde{x})G(\tilde{z})\sqrt{g(R\tilde{x})}\, d\tilde{z},$$

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where $d\tilde{z} = d\tilde{x}\, d\tilde{t}$. For some $R \le R_0$, we compute

$$\begin{aligned}
\frac{d}{dR}\Phi(R;\varphi,A) &= \int_{T_1} \frac{d}{dR}[R^4 e(\varphi,A)(R\tilde{x},R^2\tilde{t})\phi^2(R\tilde{x})\sqrt{g(R\tilde{x})}]G(\tilde{z})\,d\tilde{z} \\
&= \int_{T_1} 4R^3 e(\varphi,A)(R\tilde{x},R^2\tilde{t})\phi^2(R\tilde{x})G(\tilde{z})\sqrt{g(R\tilde{x})}\,d\tilde{z} \\
&\quad + \int_{T_1} R^4\tilde{x}_k\frac{\partial}{\partial x_k}e(\varphi,A)(R\tilde{x},R^2\tilde{t})\phi^2(R\tilde{x})G(\tilde{z})\sqrt{g(R\tilde{x})}\,d\tilde{z} \\
&\quad + \int_{T_1} 2R^5\tilde{t}\frac{\partial}{\partial t}e(\varphi,A)(R\tilde{x},R^2\tilde{t})\phi^2(R\tilde{x})G(\tilde{z})\sqrt{g(R\tilde{x})}\,d\tilde{z} \\
&\quad + \int_{T_1} R^4 e(\varphi,A)(R\tilde{x},R^2\tilde{t})\tilde{x}_k\frac{\partial}{\partial x_k}(\phi^2\sqrt{g})(R\tilde{x})G(\tilde{z})\,d\tilde{z} \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Rescaling coordinates back to (x,t),

$$I_1 = \int_{T_R} 4Re(\varphi,A)\phi^2 G\sqrt{g}\,dz$$

and

$$I_4 = \int_{T_R} Re(\varphi,A)x_k\frac{\partial}{\partial x_k}(\phi^2\sqrt{g})G\,dz.$$

For the second term,

$$I_2 = \int_{T_R} Rx_k\frac{\partial}{\partial x_k}e(\varphi,A)\phi^2 G\sqrt{g}\,dz.$$

This simplifies as follows:

$$\begin{aligned}
&\frac{\partial}{\partial x_k}\Big[|\nabla_A\varphi|^2 + \frac{1}{2}|F_A|^2 + \Big(\frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4\Big)\Big] \\
&\quad = \langle\nabla_M^k F_A, F_A\rangle + 2\,\mathrm{Re}\langle\nabla_A^k\nabla_A^j\varphi, \nabla_A^j\varphi\rangle + \frac{1}{2}(S + |\varphi|^2)\,\mathrm{Re}\langle\nabla_A^k\varphi,\varphi\rangle.
\end{aligned}$$

Note that

$$2\,\mathrm{Re}\langle\nabla_A^k\nabla_A^j\varphi,\nabla_A^j\varphi\rangle = 2\,\mathrm{Re}\langle\nabla_A\nabla_A^k\varphi,\nabla_A\varphi\rangle - 2\,\mathrm{Re}\langle\Omega_A^{jk}\varphi,\nabla_A^j\varphi\rangle.$$

Using the fact that

$$\frac{\partial G}{\partial x_j} = \frac{x_j}{2t}G,$$

we have

$$-2\int_{T_R} Rx_k\,\mathrm{Re}\langle d(G)\wedge\nabla_A^k\varphi,\nabla_A\varphi\rangle\phi^2\sqrt{g}\,dz = -4\int_{T_R} Rt\Big|\frac{x_k}{2t}\nabla_A^k\varphi\Big|^2\phi^2 G\sqrt{g}\,dz.$$

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For the curvature term, we recall the Bianchi identity $dF_A = 0$ which implies that

$$\partial_k F^{ij} = \partial_i F^{kj} - \partial_j F^{ki},$$

and we compute in local coordinates

$$\begin{aligned} x_k\partial_k \sum_{i<j} (F^{ij})^2 G\phi^2 &= 2x_k \sum_{i<j} F^{ij}\partial_k F^{ij} G\phi^2 \\ &= 2x_k \sum_{i<j} F^{ij}(\partial_i F^{kj} - \partial_j F^{ki}) G\phi^2 \\ &= x_k F^{ij}\partial_i F^{kj} G\phi^2 \\ &= \partial_i(x_k F^{ij} F^{kj} G\phi^2) - x_k F^{kj}\partial_i F^{ij} G\phi^2 - F^{kj}F^{ij}\partial_i(x_k G\phi^2). \end{aligned}$$

Observe that the first term will integrate to zero by Stoke's theorem, and for the second term we have $(d^*F_A)_j = \partial_i F^{ij}$. To deal with the third term, we see that

$$-x_k F^{kj} x_i F^{ij} \frac{1}{2t} G\phi^2 = -2t\left|\frac{x_k}{2t}\frac{\partial}{\partial x_k}\rfloor F_A\right|^2 G\phi^2.$$

Note that $|d\phi| G \leq c$ since $|d\phi| = 0$ on $B_{i(M)/2}(x_0)$. Then

$$\begin{aligned} I_2 &= \int_{T_R} R x_k\Big[\frac{1}{2}\partial_k |F_A|^2 + 2\,\mathrm{Re}\langle \nabla_A \nabla_A^k \varphi, \nabla_A\varphi\rangle - 2\,\mathrm{Re}\langle \Omega_A^{jk}\varphi, \nabla_A^j \varphi\rangle \\ &\quad + \frac{1}{2}(S + |\varphi|^2)\,\mathrm{Re}\langle \nabla_A^k\varphi, \varphi\rangle\Big]\phi^2 G\sqrt{g}\,dz \\ &= \int_{T_R} R x_k\Big[\frac{1}{2}\partial_k |F_A|^2 + 2\,\mathrm{Re}\langle \nabla_A^k\varphi, \nabla_A^*\nabla_A\varphi\rangle \\ &\quad - 2\,\mathrm{Re}\langle \Omega_A^{jk}\varphi, \nabla_A^j\varphi\rangle + \frac{1}{2}(S + |\varphi|^2)\,\mathrm{Re}\langle \nabla_A^k\varphi, \varphi\rangle\Big]\phi^2 G\sqrt{g}\,dz \\ &\quad - 2\int_{T_R} R x_k\,\mathrm{Re}\langle d(\phi^2 G x_k)\nabla_A^k\varphi, \nabla_A\varphi\rangle\sqrt{g}\,dz \\ &\geq -\int_{T_R} R|x|\Big(\Big|\frac{\partial A}{\partial t}\Big||F_A| + 2\Big|\frac{\partial\varphi}{\partial t}\Big||\nabla_A\varphi|\Big)\phi^2 G\sqrt{g}\,dz \\ &\quad - 4\int_{T_R} Rt\Big|\frac{x_k}{2t}\nabla_A^k\varphi\Big|^2\phi^2 G\sqrt{g}\,dz - \int_{T_R} Rt\Big|\frac{x_k}{2t}\frac{\partial}{\partial x_k}\rfloor F_A\Big|^2 G\phi^2\sqrt{g}\,dz \\ &\quad - c\Phi(R;\varphi, A) - cR\mathcal{SW}(\varphi_0, A_0), \end{aligned}$$

where we note that $\Omega_A = \Omega_{A_0} + \frac{1}{2}F_A$, and we also recall from [5] that

$$\mathrm{Re}\Big\langle \frac{\partial A}{\partial t}\varphi, \nabla_A\varphi\Big\rangle = \Big\langle \frac{\partial A}{\partial t}, i\,\mathrm{Im}\langle \nabla_A\varphi, \varphi\rangle\Big\rangle;$$

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note that $\partial A/\partial t$ can be replaced by any 1-form. For the third term,

$$\begin{aligned}
I_3 &= \int_{T_R} 2Rt\Big[\Big\langle d\frac{\partial A}{\partial t}, F_A\Big\rangle + 2\,\mathrm{Re}\Big\langle \nabla_A \frac{\partial \varphi}{\partial t}, \nabla_A \varphi\Big\rangle + \mathrm{Re}\Big\langle \frac{\partial A}{\partial t}\varphi, \nabla_A\varphi\Big\rangle \\
&\qquad + \Big(\frac{1}{2}[S + |\varphi|^2]\,\mathrm{Re}\Big\langle \frac{\partial \varphi}{\partial t}, \varphi\Big\rangle\Big)\Big]\phi^2 G\sqrt{g}\,dz \\
&= -\int_{T_R} 2Rt\Big[\Big|\frac{\partial A}{\partial t}\Big|^2 + 2\Big|\frac{\partial \varphi}{\partial t}\Big|^2\Big]\phi^2 G\sqrt{g}\,dz \\
&\qquad - \int_{T_R} 2Rt\Big\langle d(\phi^2 G)\wedge \frac{\partial A}{\partial t}, F_A\Big\rangle\sqrt{g}\,dz \\
&\qquad - \int_{T_R} 4Rt\,\mathrm{Re}\Big\langle d(\phi^2 G)\wedge \frac{\partial \varphi}{\partial t}, \nabla_A\varphi\Big\rangle\sqrt{g}\,dz.
\end{aligned}$$

Next we obtain

$$\Big\langle dG\wedge \frac{\partial A}{\partial t}, F_A\Big\rangle = \Big\langle \frac{x_k}{2t}\,dx^k\wedge \frac{\partial A}{\partial t}, F_A\Big\rangle G = \Big\langle \frac{\partial A}{\partial t}, \frac{x_k}{2t}\frac{\partial}{\partial x_k}\lrcorner F_A\Big\rangle G$$

and

$$\Big\langle dG\wedge \frac{\partial \varphi}{\partial t}, \nabla_A\varphi\Big\rangle = \Big\langle \frac{x_k}{2t} dx^k\wedge \frac{\partial \varphi}{\partial t}, \nabla_A\varphi\Big\rangle G = \Big\langle \frac{\partial \varphi}{\partial t}, \frac{x_k}{2t}\nabla_A^k\varphi\Big\rangle G.$$

Thus

$$\begin{aligned}
I_3 &\geq -\int_{T_R} 2Rt\Big[\Big|\frac{\partial A}{\partial t}\Big|^2 + 2\Big|\frac{\partial \varphi}{\partial t}\Big|^2\Big]\phi^2 G\sqrt{g}\,dz \\
&\qquad - \int_{T_R} 2Rt\Big\langle \frac{\partial A}{\partial t}, \frac{x_k}{2t}\frac{\partial}{\partial x_k}\lrcorner F_A\Big\rangle\phi^2 G\sqrt{g}\,dz + c\int_{T_R} 2Rt\Big|\frac{\partial A}{\partial t}\Big||F_A|\phi\sqrt{g}\,dz \\
&\qquad - \int_{T_R} 4Rt\,\mathrm{Re}\Big\langle \frac{\partial \varphi}{\partial t}, \frac{x_k}{2t}\nabla_A^k\varphi\Big\rangle\phi^2 G\sqrt{g}\,dz + c\int_{T_R} 4Rt\Big|\frac{\partial \varphi}{\partial t}\Big||\nabla_A\varphi|\phi\sqrt{g}\,dz \\
&\geq -\int_{T_R} Rt\Big(\Big|\frac{\partial A}{\partial t} + \frac{x_k}{2t}\frac{\partial}{\partial x_k}\lrcorner F_A\Big|^2 + 2\Big|\frac{\partial \varphi}{\partial t} + \frac{x_k}{2t}\nabla_A^k\varphi\Big|^2\Big)\phi^2 G\sqrt{g}\,dz \\
&\qquad + \int_{T_R} Rt\Big(\Big|\frac{x_k}{2t}\frac{\partial}{\partial x_k}\lrcorner F_A\Big|^2 + 2\Big|\frac{x_k}{2t}\nabla_A^k\varphi\Big|^2\Big)\phi^2 G\sqrt{g}\,dz \\
&\qquad - \int_{T_R} Rt\Big[\Big|\frac{\partial A}{\partial t}\Big|^2 + 2\Big|\frac{\partial \varphi}{\partial t}\Big|^2\Big]\phi^2 G\sqrt{g}\,dz \\
&\qquad - cR\mathcal{SW}(\varphi_0, A_0).
\end{aligned}$$

Here we have used Young's inequality and the energy inequality (2.1). We also recall that $|t| \leq 4R^2$ on T_R, and that $R \leq R_0$. Finally, since, as in [12], $R^{-1}|x|^2 G \leq c(1 + G)$, combining the working above (and recalling that $t < 0$ on T_R), one obtains (2.2). □

Corollary 2.7. *There exists a constant $a > 0$ such that*

$$e^{aR}\Phi(R; \varphi, A) + cR^2\mathcal{SW}(\varphi_0, A_0)$$

where c here represents the same constant as appears in (2.2).

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PROOF. The result follows from (2.2) by multiplying by e^{aR} for some sufficiently large $a > 0$, and integrating from R_a to R_b. □

LEMMA 2.8. *Suppose that* $(\varphi, A) \in C^\infty(P_R(y, s))$ *satisfies* (1.3). *Then there exist constants* δ *and* R_1 *such that if* $R \leq R_1$ *and*

$$\sup_{0<t<s} R^{4-m} \int_{B_R(y)} (|\nabla_A \varphi|^2 + |F_A|^2)\, dV < \delta,$$

then

$$\sup_{P_{R/2}(y,s)} (|\nabla_A \varphi|^2 + |F_A|^2) \leq 256 R^{-4}.$$

PROOF. We begin by choosing $r_0 < R$ so that

$$(R - r_0)^4 \sup_{P_{r_0}(y,s)} (|\nabla_A \varphi|^2 + |F_A|^2) = \max_{0\leq r\leq R} \Big[(R - r)^4 \sup_{P_r(y,s)} (|\nabla_A \varphi|^2 + |F_A|^2)\Big]. \tag{2.3}$$

Let

$$e_0 = \sup_{P_{r_0}(y,s)} (|\nabla_A \varphi|^2 + |F_A|^2) = (|\nabla_A \varphi|^2 + |F_A|^2)(x_0, t_0)$$

for some $(x_0, t_0) \in \bar{P}_{r_0}(y, s)$. We claim that

$$e_0 \leq 16(R - r_0)^{-4}. \tag{2.4}$$

Then

$$\begin{aligned}(R - r)^4 \sup_{P_r(y,s)} (|\nabla_A \varphi|^2 + |F_A|^2) &\leq (R - r_0)^4 \sup_{P_{r_0}(y,s)} (|\nabla_A \varphi|^2 + |F_A|^2)\\ &\leq 16(R - r_0)^4 (R - r_0)^{-4} = 16\end{aligned}$$

for any $r < R$. Choosing $r = \frac{1}{2}R$ in the above, we have the required result. We now prove (2.4). Define $\rho_0 = e_0^{-1/4}$ and suppose by contradiction that $\rho_0 \leq \frac{1}{2}(R - r_0)$. We rescale variables $x = x_0 + \rho_0 \tilde{x}$ and $t = t_0 + \rho_0^2 \tilde{t}$ and set

$$\psi(\tilde{x}, \tilde{t}) = \varphi(x_0 + \rho_0 \tilde{x}, t_0 + \rho_0^2 \tilde{t}),$$
$$B(\tilde{x}, \tilde{t}) = \rho_0 A(x_0 + \rho_0 \tilde{x}, t_0 + \rho_0^2 \tilde{t}),$$

giving

$$|\nabla_B \psi|^2 = \rho_0^2 |\nabla_A \varphi|^2,$$
$$|F_B|^2 = \rho_0^4 |F_A|^2.$$

We define

$$e_{\rho_0}(\tilde{x}, \tilde{t}) = |F_B|^2 + \rho_0^2 |\nabla_B \psi|^2 = \rho_0^4 (|\nabla_A \varphi|^2 + |F_A|^2),$$

so that

$$e_{\rho_0}(\tilde{x}, \tilde{t}) \leq e_{\rho_0}(0, 0) = 1.$$

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We compute

$$\begin{aligned}\sup_{\tilde{P}_1(0,0)} e_{\rho_0}(\tilde{x},\tilde{t}) &= \rho_0^4 \sup_{P_{\rho_0}(x_0,t_0)} (|\nabla_A\varphi|^2 + |F_A|^2)\\ &\le \rho_0^4 \sup_{P_{(R+r_0)/2}(y,s)} (|\nabla_A\varphi|^2 + |F_A|^2)\\ &= \rho_0^4\Big(\frac{R-r_0}{2}\Big)^{-4}\Big(R-\frac{R+r_0}{2}\Big)^4 \sup_{P_{(R+r_0)/2}(y,s)} (|\nabla_A\varphi|^2 + |F_A|^2)\\ &\le \rho_0^4\Big(\frac{R-r_0}{2}\Big)^{-4}(R-r_0)^4 e_0 = 16,\end{aligned}$$

where we have used that $P_{\rho_0}(x_0,t_0) \subset P_{R+r_0/2}(y,s)$, and to get to the last line we have used (2.3). This implies that

$$e_{\rho_0} = \rho_0^4(|\nabla_A\varphi|^2 + |F_A|^2) \le 16$$

on $\bar{P}_1(0,0)$. By Lemma 2.3,

$$\Big(\frac{\partial}{\partial t} + \Delta\Big)(|\nabla_A\varphi|^2 + |F_A|^2 + 1) \le c(|F_A| + 1)(|\nabla_A\varphi|^2 + |F_A|^2 + 1).$$

Then

$$\begin{aligned}\Big(\frac{\partial}{\partial\tilde{t}} + \tilde{\Delta}\Big)(e_{\rho_0} + \rho_0^4) &= \rho_0^6\Big(\frac{\partial}{\partial t} + \Delta\Big)(|\nabla_A\varphi|^2 + |F_A|^2)\\ &\le c\rho_0^6(|F_A| + 1)(|\nabla_A\varphi|^2 + |F_A|^2 + 1)\end{aligned}$$

on $\bar{P}_1(0,0)$. Note that by assumption $\rho_0 < R$, $\rho_0^2|F_A|$ is thus bounded by a constant. Then

$$\Big(\frac{\partial}{\partial\tilde{t}} + \tilde{\Delta}\Big)(e_{\rho_0} + \rho_0^4) \le c(e_{\rho_0} + \rho_0^4)$$

for a constant $c > 0$. We apply Moser's Harnack inequality to give

$$\begin{aligned}1 + \rho_0^4 = e_{\rho_0}(0,0) + \rho_0^4 &\le c\int_{\tilde{P}_1(0,0)} e_{\rho_0}\,d\tilde{x}\,d\tilde{t} + c\rho_0^4\\ &= c\rho_0^{2-m}\int_{P_{\rho_0}(x_0,t_0)} (|\nabla_A\varphi|^2 + |F_A|^2)\,dV\,dt + c\rho_0^4\\ &\le c\sup_{0\le t\le s} R^{4-m}\int_{B_R(y)} (|\nabla_A\varphi|^2 + |F_A|^2)\,dV + cR^4\\ &< c\delta + cR^4,\end{aligned}$$

where we have used the fact that $\rho_0 < R$. Now if we choose R_1 and δ sufficiently small, we have the desired contradiction. □

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3. Singularity analysis

Let (φ, A) be a smooth solution on $[0, T)$. Suppose that there exists some $R \le R_1$ such that

$$R^{4-m} \int_{B_R(y)} (|\nabla_A \varphi|^2 + |F_A|^2)\, dV < \delta,$$

for all $x_0 \in M$ and $t_0 = T$. Then by Lemma 2.8, $|\nabla_A \varphi|^2$ and $|F_A|^2$ are uniformly bounded on $M \times [0, T)$. As in [5], using Lemma 2.4 we can show that φ and A are smooth at $t = T$. In conjunction with the local existence result, we can extend (φ, A) to a global smooth solution.

We define the singular set

$$\Sigma = \bigcap_{0<R\le R_1} \Big\{ x_0 \in M : \limsup_{t\to T} R^{4-m} \int_{B_R(x_0)} (|\nabla_A \varphi|^2 + |F_A|^2)\, dV \ge \delta \Big\}.$$

By the above discussion, $(\varphi(T), A(T))$ is smooth on $M\backslash\Sigma$. Let Σ' be defined as for Σ, but with δ replaced by a smaller constant. Clearly $\Sigma \subseteq \Sigma'$. Furthermore, if $x \in M\backslash\Sigma$ then by smoothness at x, $x \in M\backslash\Sigma'$. Thus replacing δ with with any smaller constant defines the same set. If $x \in M\backslash\Sigma$, then by Lemma 2.8, $B_R(x) \in M\backslash\Sigma$ for some R. Thus Σ is closed. Unlike in the four-dimensional case [5], we cannot conclude at this point that the singular set is finite. We can instead show that Σ has finite $(m-4)$-dimensional Hausdorff measure $\mathcal{H}^{m-4}$. Explicitly, for $x_0 \in \Sigma$,

$$\delta \le \limsup_{t\to T} R^{4-m} \int_{B_R(x_0)} e(\varphi, A)\, dV \tag{3.1}$$

for any R. The family $\mathcal{F} = \{B_R(x_0) : x_0 \in \Sigma\}$ covers Σ, and by Vitali's covering lemma, there exists a finite subfamily $\mathcal{F}' = \{B_R(x_j)\}$ such that any two balls in $\mathcal{F}'$ are disjoint and $\{B_{5R}(x_j)\}$ covers Σ. Then using (3.1),

$$\begin{aligned}\sum_j (5R)^{m-4} &\le \frac{5^m}{\delta} \sum_j \limsup_{t\to T} \int_{B_R(x_j)} (|\nabla_A \varphi|^2 + |F_A|^2)\, dV \\ &\le C\mathcal{SW}(\varphi_0, A_0),\end{aligned}$$

where $\{B_{5R}(x_j)\}$ covers Σ. It follows that $\mathcal{H}^{m-4}(\Sigma)$ is finite, as claimed.

To establish Theorem 1.1, we show that $\Sigma = \emptyset$. Suppose by contradiction that Σ is nonempty. Since the flow is smooth on $[0, T)$, we can find sequences $x_n \in M$, $t_n \to T$, $R_n \to 0$ such that

$$\begin{aligned}\delta &> R_n^{4-m} \mathcal{SW}_{B_{R_n}(x_n)}(\varphi(t_n), A(t_n)) \\ &= \sup_{0\le t\le t_n, x\in M} R_n^{4-m} \mathcal{SW}_{B_{R_n}(x)}(\varphi(t), A(t)) > \frac{\delta}{2}\end{aligned} \tag{3.2}$$

for each n, where $\mathcal{SW}_{B_R(x)}$ is defined by

$$\mathcal{SW}_{B_R(x)}(\varphi, A) = \int_{B_R(x)} |\nabla_A \varphi|^2 + \frac{1}{2}|F_A|^2 + \frac{S}{4}|\varphi|^2 + \frac{1}{8}|\varphi|^4\, dV.$$

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By the compactness of M, passing to a subsequence we have $x_n \to x_0$ where $x_0 \in \Sigma$ by Lemma 2.8. We define the region

$$D_n = \{(y, s) : R_n y + x_n \in B_{i(M)/2}(x_n),\ s \in [-R_n^{-2} t_n, 0]\} =: U_n \times [-R_n^{-2} t_n, 0].$$

Note that as $n \to \infty$, $D_n \to \mathbb{R}^m \times (-\infty, 0]$. Furthermore, truncating the sequence if necessary, we can arrange that $B_{i(M)/2}(x_n) \subset B_{i(M)}(x_0)$. We rescale to

$$\varphi_n(y, s) = \varphi(R_n y + x_n, R_n^2 s + t_n),$$

$$A_n(y, s) = R_n A(R_n y + x_n, R_n^2 s + t_n),$$

which are defined on D_n. We have

$$|\nabla_{A_n}\varphi_n|^2 = R_n^2 |\nabla_A \varphi|^2,$$
$$|F_{A_n}|^2 = R_n^4 |F_A|^2.$$

If we choose our local coordinates on $B_{i(M)}(x_0)$ to be orthonormal coordinates, then the metric on the rescaled space is simply $g_{ij} = \delta_{ij}$. From (3.2),

$$\int_{B_1(0)} R_n^2 |\nabla_{A_n}\varphi_n|^2 + |F_{A_n}|^2 + R_n^4\Big(\frac{S}{4}|\varphi_n|^2 + \frac{1}{8}|\varphi_n|^4\Big)\, dy > \frac{\delta}{2} \tag{3.3}$$

for each n and $s = 0$. Next, from Lemma 2.8 and (3.2),

$$\sup_{D_n}(|\nabla_{A_n} R_n \varphi_n|^2 + |F_{A_n}|^2) \le K_1, \tag{3.4}$$

where K_1 is independent of n. We consider the rescaled equations

$$\frac{\partial R_n\varphi_n}{\partial s} = R_n^3 \frac{\partial \varphi}{\partial t} = -\nabla_{A_n}^* \nabla_{A_n} R_n\varphi_n - \frac{1}{4}[R_n^2 S + |R_n\varphi_n|^2] R_n \varphi_n,$$
$$\frac{\partial A_n}{\partial s} = R_n^3 \frac{\partial A}{\partial t} = -d^* F_{A_n} - i\,\mathrm{Im}\langle \nabla_{A_n} R_n\varphi_n, R_n\varphi_n\rangle. \tag{3.5}$$

Noting the similarity of these equations to (1.3), we use (3.4) and results identical to Lemma 2.4 and Corollary 2.5 to find that

$$\sup_{D_n}(|\nabla_{A_n}^{(k+1)} R_n\varphi_n|^2 + |\nabla_M^{(k)} F_{A_n}|^2) \le K_{k+1}$$

for each $k \ge 0$. Thus by a result of Uhlenbeck ([15, Theorem 1.3]; see also [6]), passing to a subsequence and using an appropriate gauge, we have C^∞ convergence $R_n\varphi_n \to \tilde{\varphi} = 0$ (since φ_n is bounded), $A_n \to \tilde{A}$ where $\tilde{\varphi}$ and $\tilde{A}$ are defined on $\mathbb{R}^m \times (-\infty, 0]$. Then as $n \to \infty$ in (3.3),

$$\int_{B_2(0)} |F_{\tilde{A}}|^2\, dy \ge \frac{\delta}{2} \tag{3.6}$$

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for $s = 0$. Since $R_n\varphi_n \to 0$, from (3.5), $\tilde{A}$ satisfies the equation

$$\frac{\partial \tilde{A}}{\partial s} = -d^* F_{\tilde{A}}$$

on $\mathbb{R}^m \times (-\infty, 0]$. Using the Bianchi identity $dF_{\tilde{A}} = 0$, this implies that

$$\frac{\partial F_{\tilde{A}}}{\partial s} = -\Delta F_{\tilde{A}}$$

on $\mathbb{R}^m \times (-\infty, 0]$, where $\Delta = d^*d + dd^*$. Since the solution to the heat equation converges to constant data in infinite time, the only possible solution to this equation satisfying (3.4) is $F_{\tilde{A}} =$ constant. See, for example, [3, Theorem 9 of Ch. 2]. In the notation of [3], choose $k = 1$ and $t = 0$, note that for us $\|u\|_{L^1(C(x,t;r))} \leq cr^{n+2}$, and let $r \to \infty$.

4. Proof of Theorem 1.1

As in [9, 14], the term $\mathscr{F}(R; \varphi, A)$ in Lemma 2.6 can be used to further analyze the singularity (see, for example, [14, Lemma 3.3.2]). However, we are already in a position to show that the existence of a singularity implies a contradiction. Noting that $G_{(x_n,t_n)} \geq cR_n^{-m}$ on $B_{rR_n}(x_n) \times [t_n - 4(rR_n)^2, t_n - (rR_n)^2]$, we consider for any $r \in (0, \infty)$,

$$\begin{aligned} r^{2-m} & \int_{B_r(0)\times[-4r^2,-r^2]} |F_{\tilde{A}}|^2 \, dy \, ds \\ &= \lim_{n\to\infty} (rR_n)^{2-m} \int_{B_{rR_n}(x_n)\times[t_n-4(rR_n)^2, t_n-(rR_n)^2]} |F_A|^2 \, dV \, dt \\ &\leq c \lim_{n\to\infty} (rR_n)^2 \int_{B_{rR_n}(x_n)\times[t_n-4(rR_n)^2, t_n-(rR_n)^2]} |F_A|^2 G_{(x_n,t_n)} \, dV \, dt \\ &\leq c \lim_{n\to\infty} (rR_n)^2 \int_{T_{rR_n}(x_0,T)} e(\varphi, A) G_{(x_0,T)} \phi^2 \, dV \, dt. \end{aligned}$$

However, the latter expression is bounded by Lemma 2.6. Thus

$$\int_{B_r(0)\times[-4r^2,-r^2]} |F_{\tilde{A}}|^2 \, dy \, ds \leq cr^{m-2}.$$

But since $|F_{\tilde{A}}|$ is constant and nonzero by (3.6), this implies that $r^4 \leq c$. This is impossible for r sufficiently large. This proves Theorem 1.1. □

References

[1] A. H. Bilge, T. Dereli and S. Kocak, 'Seiberg–Witten type monopole equations on 8-manifolds with spin(7) holonomy, as minimizers of a quadratic action', *J. High Energy Phys.* **4**(003).

[2] N. Değirmenci and N. Özdemir, 'Seiberg–Witten-like equations on 7-manifolds with g_2-structure', *J. Nonlinear Math. Phys.* **12**(4) (2005), 457–461.

https://doi.org/10.1017/S1446788712000304 Published online by Cambridge University Press

[3] L. C. Evans, *Partial Differential Equations* (American Mathematical Society, Providence, RI, 1998).
[4] Y.-H. Gao and G. Tian, 'Instantons and the monopole-like equations in eight dimensions', *J. High Energy Phys.* **5**(36).
[5] M.-C. Hong and L. Schabrun, 'Global existence for the Seiberg–Witten flow', *Comm. Anal. Geom.* **18**(3) (2010), 433–474.
[6] M.-C. Hong and G. Tian, 'Asymptotical behaviour of the Yang–Mills flow and singular Yang–Mills connections', *Math. Ann.* **330** (2004), 441–472.
[7] J. Jost, *Riemannian Geometry and Geometric Analysis* (Springer, Berlin, 1995).
[8] F. H. Lin, 'Gradient estimates and blow-up analysis for stationary harmonic maps', *Ann. of Math.* **149** (1999), 785–829.
[9] F. H. Lin and C. Y. Wang, 'Harmonic and quasi-harmonic spheres', *Comm. Anal. Geom.* **10**(2) (1999), 397–429.
[10] X. Peng, J. Jost and G. Wang, 'Variational aspects of the Seiberg–Witten functional', *Calc. Var. Partial Differential Equations* **3** (1996), 205–218.
[11] A. Scorpan, *The Wild World of 4-Manifolds* (American Mathematical Society, Providence, RI, 2005).
[12] C.-L. Shen and Y. Chen, 'Monotonicity formula and small action regularity for Yang–Mills flows in higher dimensions', *Calc. Var. Partial Differential Equations* **2**(4) (1994), 389–403.
[13] M. Struwe, 'On the evolution of harmonic maps in higher dimensions', *J. Diff. Geom.* **28** (1999), 485–502.
[14] G. Tian, 'Gauge theory and calibrated geometry, I', *Ann. of Math.* **151** (2000), 193–268.
[15] K. Uhlenbeck, 'Connections with l^p-bounds on curvature', *Commun. Math. Phys.* **83** (1982), 31–42.

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