Glasgow
Mathematical
Journal

RESEARCH ARTICLE

Are free groups of different ranks bi-invariantly quasi-isometric?

Jarek Kędra[1,2] ⓘ and Assaf Libman[1]

[1]University of Aberdeen, Aberdeen, United Kingdom
[2]University of Szczecin, Szczecin, Poland
Corresponding author: Jarek Kędra; Email: kedra@abdn.ac.uk

Abstract
We prove that a homomorphism between free groups of finite rank equipped with the bi-invariant word metrics associated with finite generating sets is a quasi-isometry if and only if it is an isomorphism.

1. Introduction

Let $\mathbf{F}_m$ and $\mathbf{F}_n$ be free groups of rank m and n, respectively, equipped with the bi-invariant word metrics associated with finite generating sets (see Section 2 for definitions).

Theorem 1.1. *Let $n \geq 2$. A homomorphism $\varphi \colon \mathbf{F}_m \to \mathbf{F}_n$ is a quasi-isometry if and only if it is an isomorphism.*

It is well known that if both $\mathbf{F}_m$ and $\mathbf{F}_n$ are equipped with the standard left-invariant word metrics associated with finite generating sets (such metrics are not bi-invariant) then the inclusion of a finite index subgroup is a quasi-isometry [5, Corollary 5.31]. Thus, the above theorem is in contrast with the classical geometric group theory. It can be considered as a form of rigidity where being a homomorphism and a quasi-isometry implies being an isomorphism. It is interesting to what extent this rigidity can be generalised. For example, we do not know whether a general map, not necessarily a homomorphism, between free groups of different ranks can be a quasi-isometry. In particular, the question in the title remains open (see also [7, Question 10.2.4] for a related question). On the other hand, we know that it does not hold for other groups. For example, the abelianisation homomorphism $\mathbf{N} \to \mathbf{Z}^n$, from a torsion-free nilpotent group is a quasi-isometry [1, Theorem 5.8]. Bi-invariant word metrics and their general properties, particularly in connection to free groups, are discussed in [1,4,6].

The proof of Theorem 1.1 splits into three cases:

(a) If φ is an isomorphism then it is a quasi-isometry.
(b) The image of φ is a proper subgroup of finite index in $\mathbf{F}_m$, in which case we prove that φ is distorted by showing that it carries an unbounded set in $\mathbf{F}_m$ to a bounded set in $\mathbf{F}_n$ (Proposition 3.4).
(c) The image of φ has infinite index in $\mathbf{F}_m$, in which case we show that φ is not quasi-surjective (Proposition 3.6).

2. Preliminaries

The material presented in this section is well known and can be found, for example, in Bridson-Haefliger [2] or Calegari [3].

2.1. Word metrics

Let G be a group generated by a set $S \subseteq G$. The corresponding **word norm** is defined by

$$|g|_S = \min \left\{ n \in \mathbf{N} \mid g = s_1 \ldots s_n, \ s_i \in S^{\pm 1} \right\}$$

and the associated metric by $d_S(g, h) = |g^{-1}h|_S$. This metric is left-invariant.

The closure of $S \subseteq G$ to inverses and conjugation is the set

$$\overline{S} = \{ gsg^{-1} : s^{\pm 1} \in S, g \in G \}. \tag{2.1}$$

We say that S **normally generates** G if $\overline{S}$ generates G. In this case, the word norm $|g|_{\overline{S}}$ on G is invariant with respect to conjugation, we denote it by

$$\|g\|_S$$

and call it the **conjugation-invariant word norm** on G associated with S. The metric $d_{\overline{S}}(x, y) = \|x^{-1}y\|_S = \|xy^{-1}\|_S$ on G is bi-invariant. We abusively denote it by d_S (since we are only interested in bi-invariant metrics). If S is finite then the Lipschitz class of d_S is maximal in the sense that for any bi-invariant metric d on G, the identity map $\mathrm{id} \colon (G, d_S) \to (G, d)$ is Lipschitz.

Lemma 2.1. *Let $S \subseteq G$ be a finite set normally generating G. Let d be any bi-invariant metric on G. Then the identity $\mathrm{Id} \colon (G, d_S) \to (G, d)$ is Lipschitz. In particular, bi-invariant word metrics associated with finite normally generating sets are bi-Lipschitz equivalent.*

Proof. Let $C = \max\{d(s, 1) \mid s \in S\} = \max\{d(s, 1) \mid s \in \overline{S}\}$, where the second equality follows from the bi-invariance of d. Let $g, h \in G$ and let $n = d_S(g, h) = \|g^{-1}h\|_S$. This means that $g^{-1}h = s_1 \ldots s_n$ for some $s_i \in \overline{S}$. We have the following estimate, which shows that the identity is Lipschitz.

$$
\begin{aligned}
d(g, h) &= d(1, g^{-1}h) \\
&= d(1, s_1 \ldots s_n) \\
&\leq d(1, s_1) + d(s_1, s_1 \ldots s_n) && (\text{by triangle inequality}) \\
&= d(1, s_1) + d(1, s_2 \ldots s_n) && (\text{by left-invariance}) \\
&\leq \sum_{i=1}^{n} d(s_i, 1) \leq C\, n = C\, d_S(g, h).
\end{aligned}
$$

$\square$

Let $\mathbf{F}_n = \langle s_1, \ldots, s_n \rangle$ be the free group of rank n. We call the set $\{s_1, \ldots, s_n\}$ the **standard generating set**. In this paper, we are concerned with the bi-invariant word metrics on free groups of finite rank associated with their standard generating sets.

2.2. Quasi-isometries

A map $\varphi \colon (X_1, d_1) \to (X_2, d_2)$ between metric spaces is called a **quasi-isometry** if

1. There exist $C > 0$ and $D \geq 0$ such that for every $x, y \in X$,

$$\frac{1}{C} d_1(x, y) - D \leq d_2(\varphi(x), \varphi(y)) \leq C d_1(x, y) + D \tag{2.2}$$

2.   There exists $B \geq 0$ such that for every $y \in X_2$ there exists $x \in X_1$ such that

$$d_2(\varphi(x), y) \leq B. \tag{2.3}$$

If φ satisfies (2.2) then it is called a **quasi-isometric embedding**; if it satisfies the second inequality of (2.2) then it is called **large-scale Lipschitz**. If φ satisfies (2.3) then it is called **quasi-surjective**. If φ is not a quasi-isometric embedding then it is called **distorted**.

2.3. Quasi-morphisms

A function $\psi : G \to \mathbf{R}$ is called a **quasi-morphism** if there exists $D \geq 0$ such that

$$|\psi(g) - \psi(gh) + \psi(h)| \leq D \tag{2.4}$$

for all $g, h \in G$. The smallest of such D is called the defect of ψ and denoted by D_ψ.

A quasi-morphism is **homogeneous** if $\psi(g^n) = n\psi(g)$ for all $g \in G$ and $n \in \mathbf{Z}$. Every quasi-morphism ψ has an associated homogeneous quasi-morphism defined by

$$\overline{\psi}(g) = \lim_{n \to \infty} \frac{\psi(g^n)}{n}.$$

It is called the **homogenisation** of ψ, see [3, Lemma 2.21]. It is straightforward to check that if ψ is homogeneous then $\overline{\psi} = \psi$. From this, it is easy to deduce that any homogeneous quasi-morphism is a class function, i.e., it is constant on conjugacy classes in G, see [3, Subsection 2.2.3].

Lemma 2.2. *Let G be equipped with the bi-invariant word metric associated with a normally generating subset $S \subseteq G$. If $\psi : G \to \mathbf{R}$ is a homogeneous quasi-morphism bounded on S then it is a Lipschitz function.*

Proof. Suppose that $|\psi(s)| \leq B$ for some $B \geq 0$ and all $s \in S$. Consider some $g \in G$ and set $\|g\|_S = n$. By definition of the bi-invariant word metric, $g = s_1 \ldots s_n$ for some $s_i \in \overline{S}$. Let $D \geq 0$ be the defect of ψ. Since ψ is a class function $|\psi(s)| \leq B$ for all $s \in \overline{S}$ (see (2.1)). Then

$$
\begin{aligned}
|\psi(g)| &= |\psi(s_1 \ldots s_n)| \\
&\leq \sum_{i=1}^{n} |\psi(s_i)| + (n-1)D && \text{(see [3, Lemma 2.17])} \\
&\leq (B+D)n \\
&= (B+D)\|g\|.
\end{aligned}
$$

Let $g \neq h \in G$. Since ψ is homogeneous and since $\|gh^{-1}\|$ is a positive integer

$$
\begin{aligned}
|\psi(g) - \psi(h)| &= |\psi(g) - \psi(gh^{-1}) + \psi(h^{-1}) + \psi(gh^{-1})| \\
&\leq D + |\psi(gh^{-1})| \\
&\leq D + (B+D) \cdot \|gh^{-1}\| && \text{(by the calculation above)} \\
&\leq (B+2D) \cdot \|gh^{-1}\| && \text{(since } \|gh^{-1}\| \text{ is a positive integer)} \\
&= (B+2D) \cdot d(g,h).
\end{aligned}
$$

$\square$

2.4. The little counting quasi-morphism

Consider the free group $\mathbf{F}_n$ with its standard set of generators $S = \{s_1, \ldots, s_n\}$. In what follows the standard word norm $|g|_S$ will be denoted by $\ell(g)$ to make various formulas easier to read.

Definition 2.3. *Let $1 \neq w \in \mathbf{F}_n$ be presented as a reduced word. The **little counting function** $c_w \colon \mathbf{F}_n \to \mathbf{N}$ is defined by*

$$c_w(g) = \max\{k\text{:the reduced form of } g \text{ contains } k \text{ disjoint copies of } w\}$$

*The **little counting quasi-morphism** $\psi_w \colon \mathbf{F}_n \to \mathbf{Z}$ is the function*

$$\psi_w(g) = c_w(g) - c_{w^{-1}}(g).$$

It is shown in [3, Section 2.3.2] that ψ_w is indeed a quasi-morphism with defect at most 2.

Let $g \in \mathbf{F}_n$. If $w \neq 1$, it is clear that $c_{w^{\pm 1}}(g) \leq \lfloor \ell(g)/\ell(w) \rfloor$ and therefore

$$|\psi_w(g)| \leq \lfloor \ell(g)/\ell(w) \rfloor.$$

Moreover, $\ell(g^k) \leq k \cdot \ell(g)$ and therefore $|\psi_w(g^k)| \leq \left\lfloor \frac{k \cdot \ell(g)}{\ell(w)} \right\rfloor$. Since $\frac{1}{k} \cdot \left\lfloor \frac{k \cdot \ell(g)}{\ell(w)} \right\rfloor \xrightarrow{k \to \infty} \frac{\ell(g)}{\ell(w)}$, the homogenisation of ψ_w satisfies

$$|\overline{\psi_w}(g)| \leq \frac{\ell(g)}{\ell(w)}. \tag{2.5}$$

By [3, Lemma 2.27] copies of w in g are disjoint from those of w^{-1}. Assume that $1 \neq w \in \mathbf{F}_n$ is cyclically reduced. Then for any $k > 0$, the word representing w^k is the concatenation of k copies of w which contains k disjoint copies of w leaving no room for copies of w^{-1}. It follows that $\psi_k(w^k) = c_w(w^k) - c_{w^{-1}}(w^k) = k$. We deduce that if w is cyclically reduced then

$$\overline{\psi_w}(w) = 1. \tag{2.6}$$

3. Proof of the main result

Throughout, free groups are equipped with the conjugation-invariant word norm associated with their standard generating sets. All groups are assumed to be equipped with conjugation-invariant norms.

The next two lemmas are the core of the proof of Theorem 1.1. Given a homomorphism $i \colon \mathbf{F}_m \to \mathbf{F}_n$, if its image is a proper subgroup of finite index we will prove the existence of a quasi-morphisms which satisfies the conditions of the first lemma, and if the index is infinite we will prove the existence of a quasi-morphism satisfying the conditions of the second lemma.

A function $f \colon G \to \mathbf{R}$ is called homogeneous if $f(g^n) = n \cdot f(g)$ for all $n > 0$.

Lemma 3.1. *Let $\rho \colon H \to G$ be a homomorphism. Suppose that there exists a Lipschitz homogeneous function $\phi \colon H \to \mathbf{R}$ and elements $h_1, h_2 \in H$ such that*

(i) $\phi(h_1) \neq \phi(h_2)$.
(ii) $\rho(h_1), \rho(h_2)$ are conjugate in G.

Then ρ is not a quasi-isometric embedding.

Proof. We construct an unbounded $Y \subseteq H$ such that $\rho(H)$ is bounded in G. Set

$$Y = \{h_1^k h_2^{-k} : k > 0\}.$$

Denote $g_1 = \rho(h_1)$ and $g_2 = \rho(h_2)$. By assumption, there exists $g \in G$ such that $g_2 = g g_1 g^{-1}$. Then $\rho(Y)$ is bounded in G since for any $k > 0$

$$\|\rho(h_1^k h_2^{-k})\|_G = \|g_1{}^k \cdot g_2{}^{-k}\|_G = \|g_1{}^k g g_1{}^{-k} g^{-1}\|_G \leq \|g_1{}^k g g_1{}^{-k}\|_G + \|g^{-1}\|_G \leq 2\|g\|.$$

Next, let $C > 0$ be the Lipschitz constant of ϕ. Then for any $k > 0$

$$C \cdot \|h_1^k h_2^{-k}\|_H = C \cdot d_H(h_1^k, h_2^k) \geq |\phi(h_1^k) - \phi(h_2^k)| = k \cdot |\phi(h_1) - \phi(h_2)|.$$

Since $\phi(h_1) \neq \phi(h_2)$ it follows that $\|h_1^k h_2^{-k}\|_H \xrightarrow{k \to \infty} \infty$, hence Y is unbounded. $\square$

Lemma 3.2. *Let $\rho\colon H \to G$ be a homomorphism. Assume that there exists a Lipschitz homogeneous function $\phi\colon G \to \mathbf{R}$ such that*

(i) $\phi \neq 0$.
(ii) $\phi \circ \rho = 0$.

Then ρ is not quasi-surjective.

Proof. By assumption, there exists $g \in G \setminus \rho(H)$ such that $\phi(g) \neq 0$. Let $C > 0$ be the Lipschitz constant of ϕ. For any $h \in H$ and any $n > 0$

$$C \cdot d_G(g^n, \rho(h)) \geq |\phi(g^n) - \phi(\rho(h))| = |\phi(g^n)| = n \cdot |\phi(g)|.$$

It follows that $d(g^n, \rho(H)) \geq C^{-1}|\phi(g)| \cdot n \xrightarrow{n\to\infty} \infty$. This completes the proof. $\square$

We now specialise to the case when $H = \mathbf{F}_m$ and $G = \mathbf{F}_n$ and prove Theorem 1.1. The maps ϕ in Lemmas 3.1 and 3.2 will be obtained as homogeneous quasi-morphisms. The Lipschitz condition in these Lemmas is a consequence of Lemma 2.2. To achieve condition (i) in Lemma 3.1, we need the next lemma.

Lemma 3.3 (Separation Lemma). *Let $g, h \in \mathbf{F}_n$ be such that the subgroups $\langle g \rangle$ and $\langle h \rangle$ are not conjugate. Then there exists a homogeneous quasi-morphism $\psi\colon \mathbf{F}_n \to \mathbf{R}$ such that $\psi(g) \neq \psi(h)$.*

Proof. Since homogeneous quasi-morphisms are class functions, we can assume without losing generality that g and h are cyclically reduced and that $\ell(h) \leq \ell(g)$. Then $g \neq 1$ since otherwise also $h = 1$ which contradicts the hypothesis.

Let $\psi_g\colon \mathbf{F}_n \to \mathbf{Z}$ be the little counting quasi-morphism, see Definition 2.3, and let $\overline{\psi}_g$ denote its homogenisation. Since g is cyclically reduced, it follows from (2.6) that

$$\overline{\psi}_g(g) = 1. \tag{3.1}$$

We complete the proof by showing that $|\overline{\psi}_g(h)| < 1$. If $\ell(h) < \ell(g)$ then by (2.5)

$$\left|\overline{\psi}_g(h)\right| \leq \ell(h)/\ell(g) < 1$$

and we are done. If $\ell(h) = \ell(g)$ we claim that $\psi_g(h^k) = 0$ for all $k \geq 0$. Otherwise, if $\psi_g(h^k) \neq 0$ for some k then either g or g^{-1} must be a subword of the cyclically reduced word h^k. Since $\ell(h) = \ell(g)$ this implies that the reduced word representing $g^{\pm 1}$ is a cyclic permutation of h, i.e., h is conjugate to $g^{\pm 1}$. In particular $\langle g \rangle$ is conjugate to $\langle h \rangle$ which contradicts the hypothesis. It follows that $\overline{\psi}_g(h) = 0$ and this completes the proof. $\square$

We recall that centralisers in the free group are cyclic, generated by elements which are not proper powers. Also, no element of the free group other than the identity is conjugate to its inverse [8, Proposition 2.19].

Proposition 3.4. *Let $n \geq 2$. Any homomorphism $\rho\colon \mathbf{F}_m \to \mathbf{F}_n$ whose image is a proper subgroup of finite index in $\mathbf{F}_n$ is not a quasi-isometric embedding.*

Proof. Set $H = \mathbf{F}_m$ and $G = \mathbf{F}_n$. We prove the existence of ϕ and h_1, h_2 in Lemma 3.1.

Since $[G:\rho(H)] < \infty$, there exists $N \leq \rho(H)$ such that N is normal and of finite index in G [10, Corollary 4.16]. Let x, y be distinct generators of $\mathbf{F}_n$. Since G/N is finite, there exist $k, \ell > 0$ such that $x^k, y^\ell \in N$. Set $g_1 = x^k y^\ell$. Since g_1 is not a proper power in $\mathbf{F}_n$, the centraliser $C_{\mathbf{F}_n}(g_1)$ of g_1 in $\mathbf{F}_n$ is the cyclic subgroup generated by g_1:

$$C_{\mathbf{F}_n}(g_1) = \langle g_1 \rangle \leq N \leq \rho(H).$$

Choose some $u \in G \setminus \rho(H)$ and set $g_2 = ug_1u^{-1}$. By construction, g_1 and g_2 are conjugate in G and $g_2 \in N$ since N is normal. We claim that the subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are not conjugate in $\rho(H)$. If they are then there exists $v \in \rho(H)$ such that

$$\langle g_2 \rangle = \langle vg_1v^{-1} \rangle.$$

Since $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are infinite cyclic, either $g_2 = vg_1v^{-1}$ or $g_2 = vg_1^{-1}v^{-1}$. If $g_2 = vg_1v^{-1}$ then $vg_1v^{-1} = ug_1u^{-1}$ which implies that $v^{-1}u \in C_{\mathbf{F}_n}(g_1) \leq \rho(H)$, so $u \in v \cdot \rho(H) = \rho(H)$, contradicting the choice of u. If $g_2 = vg_1^{-1}v^{-1}$ then $vg_1^{-1}v^{-1} = ug_1u^{-1}$ which implies that g_1^{-1} is conjugate in $\mathbf{F}_n$ to g_1, which is again a contradiction since $g_1 \neq 1$. It follows that $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are not conjugate in $\rho(H)$ as claimed.

Let $h_1, h_2 \in H$ be in the preimages of g_1, g_2, respectively. Then $\langle h_1 \rangle$ and $\langle h_2 \rangle$ cannot be conjugate in H. By the Separation Lemma 3.3 there exists a homogeneous quasi-morphism $\psi \colon H \to \mathbf{R}$ such that $\psi(h_1) \neq \psi(h_2)$. By Lemma 2.2 ψ is Lipschitz. Then ρ is not a quasi-isometric embedding by Lemma 3.1. □

We now turn to deal with homomorphisms $\mathbf{F}_m \to \mathbf{F}_n$ whose images have infinite index. We need a machinery to construct maps ϕ as in Lemma 3.2.

A **killer word** for a subgroup $G \leq \mathbf{F}_n$ is a reduced word $w \in \mathbf{F}_n$ which is not a subword in any $g \in G$. The reason for this terminology is that the little counting quasi-morphism ψ_w vanishes on G. In fact, $\psi_u|_G = 0$ for any reduced word u which contains w.

Lemma 3.5. *Let $n \geq 2$. Any $G \leq \mathbf{F}_n$ of infinite index and finite rank admits a killer word.*

Proof. A direct consequence of [9, Lemmas 4.1 and 4.6]. □

Pagliantini and Rolli's argument in [9] is indirect. In the Appendix, we give a constructive proof of Lemma 3.5 which gives an effective way to generate killer words.

Proposition 3.6. *Let $n \geq 2$. Any homomorphism $\rho \colon \mathbf{F}_m \to \mathbf{F}_n$ with image of infinite index is not quasi-surjective.*

Proof. Set $H = \mathbf{F}_m$ and $G = \mathbf{F}_n$. By Lemma 3.5 there exists a reduced word $w' \in \mathbf{F}_n$ which is not a subword in any $u \in \rho(H)$. Since $n \geq 2$, we may multiply w' on the left or on the right by an appropriate generator of $\mathbf{F}_n$ to obtain a cyclically reduced w which is not a subword in any $u \in \rho(H)$. Clearly, the same is then true for w^{-1}. As a result, the little counting quasi-morphism $\psi_w \colon G \to \mathbf{R}$ vanishes on $\rho(H)$. Consequently, the same holds for its homogenisation $\overline{\psi_w}$, which is non-trivial by (3.1). By Lemma 2.2 $\overline{\psi}_w$ is Lipschitz and by construction $\overline{\psi}_w \circ \rho = 0$. The result follows from Lemma 3.2. □

Remark 3.7. *Consider $\mathbf{F}_2 = \mathbf{Z} * \mathbf{Z}$ with the canonical set $S = \{a, b\}$ of free generators. Let $\pi \colon \mathbf{Z} * \mathbf{Z} \to \mathbf{Z}_2 * \mathbf{Z}_2$ be the canonical quotient and $T = \{\bar{a}, \bar{b}\}$ the image of S. Let G be the kernel of π. Then G is a free group, since it is a subgroup of a free group, of infinite rank. One easily checks that $\operatorname{diam}(\mathbf{Z}_2 * \mathbf{Z}_2, \| \|_T) = 2$ and that any $x \in \mathbf{Z}_2 * \mathbf{Z}_2$ lifts to $\tilde{x} \in \mathbf{F}_2$ such that $\|x\|_T = \|\tilde{x}\|_S$. From this it follows that $d(y, G) \leq 2$ for any $y \in \mathbf{F}_2$, in particular the inclusion $i \colon G \to \mathbf{F}_2$ is quasi-surjective. This shows that the assumption in Proposition 3.6 that the groups are of finite ranks is essential.*

On the other hand, $G \lhd \mathbf{F}_2$ and the elements a^2b^2 and $ab^2a = a^{-1}a^2b^2a$ are conjugate in $\mathbf{F}_2$ but not in G. They are also not proper powers. By Lemmas 3.3, 2.2 and 3.1 the inclusion $i \colon G \to \mathbf{F}_2$ is not a quasi-isometric embedding where G is equipped with the conjugation-invariant word norm with respect to some (infinite) generating set. ◇

We remark that any $1 \neq N \lhd \mathbf{F}_n$ must be unbounded. Indeed, N must contain a cyclically reduced $w \neq 1$, then by (2.6) $\overline{\psi_w}(w^k) = k$, and since $\overline{\psi_w}$ is Lipschitz by Lemma 2.2, $\|w^k\| \xrightarrow{k \to \infty} \infty$.

Proof of Theorem 1.1. Let $\varphi\colon \mathbf{F}_m \to \mathbf{F}_n$ be a homomorphism. If it is an isomorphism then φ is a quasi-isometry since it is (quasi) surjective and since both φ and φ^{-1} are Lipschitz by Lemma 2.1

Suppose that φ is not an isomorphism. If φ is not injective then $\ker\varphi$ is an unbounded subset whose image is bounded, so φ is not a quasi-isometric embedding. If φ is not surjective then it cannot be a quasi-isometry by Propositions 3.4 and 3.6. □

Acknowledgements

References

[1] Michael Brandenbursky, Światosław R. Gal, Jarek Kędra and Michał Marcinkowski, The cancellation norm and the geometry of bi-invariant word metrics, *Glasg. Math. J* **58**(1) (2016), 153–176.

[2] Martin R. Bridson and André Haefliger, **Metric spaces of non-positive curvature**, *Grundlehren math. Wiss*, vol. **319** (Springer, Berlin, 1999).

[3] Danny Calegari scl, *MSJ Mem*, **20** (Mathematical Society of Japan, Tokyo, 2009).

[4] Danny Calegari and Dongping Zhuang, Stable *W*-Length. In Topology and Geometry in Dimension Three. Triangulations, Invariants, and Geometric Structures. (*Conference in honor of William Jaco's 70th birthday*, American Mathematical Society (AMS), Stillwater, OK, USA, Providence, RI, June 4-6, 2010), 145–169, 2011

[5] Cornelia Druţu and Michael Kapovich, **Geometric group theory**, With an appendix by Bogdan Nica, vol. **63** Colloquium Publications, (American Mathematical Society (AMS), American Mathematical Society Providence, RI, 2018).

[6] Henry Jaspars, The stable conjugation-invariant word norm is rational in free groups, *Proc. Am. Math. Soc* **152**(6) (2024), 2403–2412.

[7] Arielle Leitner and Federico Vigolo, An invitation to coarse groups, *Lect. Notes math*, vol. **2339** (Springer, Cham, 2023).

[8] Roger C. Lyndon and Paul E. Schupp, Combinatorial group theory, *Class. Math*, reprint of the 1977ed. (Springer, Berlin, 2001).

[9] Cristina Pagliantini and Pascal Rolli, Relative second bounded cohomology of free groups, *Geom. Dedicata* **175**(1) (2015), 267–280.

[10] John S. Rose, *A course on group theory*, (Dover Publications, New York, NY, repr. of the 1978 orig. edition1994).

[11] John R. Stallings, Topology of finite graphs, *Invent. Math* **71**(3) (1983), 551–565.

Appendix A. Killer words

The valence of a vertex v in a graph Γ is k if a neighbourhood of v is homeomorphic to a point with k whiskers. The **initial** and **terminal** vertices of a simplicial path γ in Γ are denoted $\mathrm{ini}(\gamma)$ and $\mathrm{term}(\gamma)$. A simplicial path in Γ is called **reduced** if it has no backtracks. A subpath $\gamma' \subseteq \gamma$ is called a *p*refix of γ if $\gamma = \epsilon_1 \ldots \epsilon_n$ is the concatenation of simplicial edges and $\gamma' = \epsilon_1 \ldots \epsilon_k$ for some $0 \le k \le n$. A simplicial path γ is called **self-intersecting** if the sequence of vertices $v_0, v_1, \ldots, v_n$ it visits contains repetitions. In particular, $\mathrm{length}(\gamma) \ge 1$.

Lemma A.1. *Let Γ be a connected graph with finitely many vertices, possibly with multiple edges and loops. Suppose that the valence of every $v \in V(\Gamma)$ is at least 2. Let γ be a reduced simplicial path in Γ. Then every vertex of Γ is the terminus of some reduced simplicial path $\tilde{\gamma}$ with prefix γ.*

Proof. Let u be the terminus of γ. Let Π be the set of all simplicial paths π in Γ starting at u such that $\gamma\pi$ has no backtracks, thus, $\gamma\pi$ is reduced with prefix γ. Set

$$T = \{\mathrm{term}(\pi) : \pi \in \Pi\}.$$

Since $\mathrm{term}(\gamma\pi) = \mathrm{term}(\pi)$, we will complete the proof by proving that $T = V(\Gamma)$.

Since the valence of every $v \in V(\Gamma)$ is at least 2, it follows that Π contains arbitrarily long paths. Since $V(\Gamma)$ is finite, Π must contain self-intersecting paths and we choose such a path π of minimum length. Say π is the concatenation of simplicial edges $\epsilon_1 \dots \epsilon_n$ visiting the vertices $v_0, \dots, v_n$. The minimality of $\mathrm{length}(\pi)$ implies that $v_n = v_k$ for some $0 \leq k < n$ (otherwise we can discard the last edge ϵ_n in π). We now continue the path π by backtracking along π from v_k down to $v_0 = u$ and obtain in this way a loop $\lambda = \pi\overline{\epsilon_k} \dots \overline{\epsilon_1}$ from u to u. The minimality of $\mathrm{length}(\pi)$ implies that λ has no backtracks because a backtrack in λ, if it exists, can only occur at the juncture of π with $\overline{\epsilon_k}$ which implies that $k \geq 1$ and that $v_{n-1} = v_{k-1}$ which contradicts the minimality of $\mathrm{length}(\pi)$. Thus, $\lambda \in \Pi$, and $\mathrm{length}(\lambda) \geq 1$ (since it contains the self-intersecting π as a prefix). We are now ready to prove that $T = V(\Gamma)$.

First, $T \neq \emptyset$ since Π contains the trivial path from u to u. Assume that $T \subsetneq V(\Gamma)$. Choose $v \in V(\Gamma) \setminus T$ at distance 1 from some $v' \in T$. Let ϵ be a simplicial edge from v' to v. Since $v' \in T$, there exists $\pi \in \Pi$ such that $v' = \mathrm{term}(\pi)$. If $v' = u$ we choose π to be the loop λ we constructed above. In either case, whether $v' = u$ or not, $\mathrm{length}(\pi) \geq 1$. Clearly, $\pi\epsilon$ has no backtracks because π has this property and all the vertices of π belong to T while $\mathrm{term}(\epsilon) \notin T$. Since $\mathrm{length}(\pi) \geq 1$ it is clear that $\gamma\pi\epsilon$ is reduced because $\gamma\pi$ is. We deduce that $\pi\epsilon \in \Pi$. But then $v = \mathrm{term}(\pi\epsilon) \in T$ which is a contradiction. $\square$

Recall that a **killer word** for $G \leq \mathbf{F}_n$ is a reduced word which is not a subword in any $g \in G$.

Theorem A.2. *Let $G \leq \mathbf{F}_n$ be a finitely generated subgroup of infinite index. Then G admits a killer word* $w \in \mathbf{F}_n$.

Proof.   Let $S = \{s_1, \dots, s_n\}$ be the standard generating set of $\mathbf{F}_n$. Let $X = \bigvee \mathbf{S}^1$ be a graph with one vertex $x \in X$ and n edges, all of which are loops, so that $\pi_1(X, x) \cong \mathbf{F}_n$. The graph X is directed and each edge is labelled by a generator $s \in S$. Moreover, the valence of the vertex x is equal to $2n$. Let $p \colon (\widetilde{X}, \tilde{x}) \to (X, x)$ be a covering projection corresponding to the subgroup $G \leq \pi_1(X, x)$. We equip $\widetilde{X}$ with the directed labelled graph structure such that the projection p is a morphism of directed labelled graphs.

Since G is finitely generated $\widetilde{X}$ contains a finite connected subgraph Γ which contains $\tilde{x}$, has no vertices of valence 1 and is a deformation retract of $\widetilde{X}$. This is the *core* graph in the terminology of Stallings [11]. The restriction $p \colon \Gamma \to X$ of the covering map is not a covering any more, since otherwise G would be of finite index. It implies that some vertices of Γ have valence smaller than $2n$. We call them **bad**.

Recall that the edges of the graph $\widetilde{X}$ are oriented and labelled by the letters of the alphabet S. Hence, the labels of the edges in a simplicial path π in $\widetilde{X}$ determine a word w in the alphabet $S^{\pm 1}$ which is reduced if and only if the path π is reduced. Conversely, any vertex $v \in V(\widetilde{X})$ and any word w in the alphabet $S^{\pm 1}$ determine a unique simplicial path in $\widetilde{X}$ which we denote by $\mathrm{path}(v, w)$. To see how this path is constructed, observe that $\widetilde{X}$ is a covering of X so for any $v \in V(\widetilde{X})$ and any $s \in S$ there is a unique directed edge $e_s \in \Gamma$ with label s emanating from v and a unique directed edge $e_{s^{-1}} \in \Gamma$ with label s terminating at v. In the first case we obtain the simplicial edge $\epsilon_s = e_s$, and in the second the simplicial edge $\epsilon_{s^{-1}} = \overline{e_{s^{-1}}}$, both have initial vertex v. If $w = w_1 \dots w_n$ is a word in the alphabet $S^{\pm 1}$, then $\mathrm{path}(v, w)$ is the concatenation of the simplicial edges $\epsilon_{w_1}, \dots, \epsilon_{w_n}$ described above, where ϵ_{w_1} starts at v and for every $2 \leq i \leq n$ the simplicial edge ϵ_{w_i} starts at $\mathrm{term}(\epsilon_{w_{i-1}})$.

We are now ready to construct a killer word for G. Before we start, recall that by construction every $g \in G$ is represented by a reduced simplicial loop in Γ based at $\tilde{x}$.

Let $v_1, \dots, v_m$ be an enumeration of the vertices of Γ. We construct by induction reduced words $w_0, \dots, w_m$, such that each is a prefix of its successor, with the property that for any $0 \leq k \leq m$,

$$1 \leq i \leq k \implies \mathrm{path}(v_i, w_k) \nsubseteq \Gamma.$$

To start the induction, set w_0 to be the empty word. The condition holds vacuously. Assume that w_{k-1} has been constructed for some $1 \leq k \leq m$. Consider $\pi = \mathrm{path}(v_k, w_{k-1})$. It is reduced since w_{k-1} is. If $\pi \nsubseteq \Gamma$

then set $w_k = w_{k-1}$. Then $\text{path}(v_k, w_k) \not\subseteq \Gamma$ by assumption and $\text{path}(v_i, w_k) \not\subseteq \Gamma$ for all $1 \leq i \leq k-1$ by construction of w_{k-1}. If $\pi \subseteq \Gamma$, use Lemma A.1 to continue it to a reduced path $\pi' \subseteq \Gamma$ whose terminus is a bad vertex u'. Continue π' along a simplicial edge ϵ in $\widetilde{X}$ not in Γ. Then $\pi'\epsilon$ is a reduced path and let w_k be the word associated with it. Now, w_{k-1} is a prefix of w_k since π is a prefix of π'. Therefore $\text{path}(v_i, w_k) \not\subseteq \Gamma$ for all $1 \leq i \leq k-1$ by construction of w_{k-1}, and $\text{path}(v_k, w_k) = \pi'\epsilon \not\subseteq \Gamma$ by construction. This completes the induction step of the construction. Set $w = w_m$. By construction, $\text{path}(v, w) \not\subseteq \Gamma$, for any $v \in V(\Gamma)$. We will finish the proof by showing that w is a killer word for G. Consider an arbitrary $g \in G$ presented as a reduced word in $\mathbf{F}_n$. We claim that w cannot be a subword of g. Let $\gamma \subseteq \Gamma$ be a reduced simplicial loop based at $\widetilde{x}$ which represents g. Let u be the word in the alphabet $S^{\pm 1}$ that γ determines. Then $\gamma = \text{path}(\widetilde{x}, u)$ and since $\mathbf{F}_n$ is free, $g = u$ as reduced words. Suppose that w is a subword of g. Then $u = u'wu''$ for some subwords u', u'' of u. Let v' be the terminus of $\text{path}(\widetilde{x}, u')$ and v'' the terminus of $\text{path}(v', w)$. Then $v' \in V(\Gamma)$ since $\text{path}(\widetilde{x}, u') \subseteq \gamma \subseteq \Gamma$, and

$$\gamma = \text{path}(\widetilde{x}, u) = \text{path}(\widetilde{x}, u') \cdot \text{path}(v', w) \cdot \text{path}(v'', u'').$$

In particular, $\text{path}(v', w) \subseteq \Gamma$ which is a contradiction, since $v' \in V(\Gamma)$. $\qquad\square$

The proof of Theorem A.2 gives an algorithm to find killer words which we demonstrate in the next example.

Example A.3. *Let $G \leq \mathbf{F}_2 = \langle a, b \rangle$ be generated by $aba^{-1}b^{-1}$, b^4 and a^3. The corresponding graph Γ is pictured below. All vertices except v_0 are bad and are marked red.*
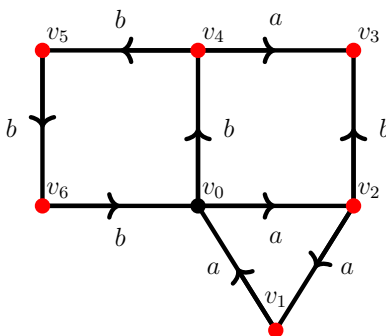
The path from v_0 to v_1 defines an element a^{-1}. Augmenting it with b yields an element $a^{-1}b$ that cannot be obtained as a path starting at v_0. The path $v_1v_2v_3$ defines an element $a^{-1}b$ and augmenting it with a yields a word $a^{-1}ba$ that cannot be obtained as a path from either v_0 or v_1. It can, however, be obtained as a path $v_2v_0v_4v_3$, so the new word is $a^{-1}ba^2$. It cannot be obtained from a path starting at v_3, v_4, v_5 or v_6. So $a^{-1}ba$ is a killer word for G.

This word is not cyclically reduced, but multiplying it on the right by the generator b gives a cyclically reduced killer word $a^{-1}ba^2b$. $\diamond$