ON THE SPECTRUM OF AN INTEGRAL OPERATOR

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1. Introduction. The integral operator which we will consider in this paper is the operator T defined for suitably restricted functions f on $(0,\infty)$ by

$$(Tf)(x) = \pi^{-1}\int_0^\infty (x-t)^{-1}f(t)\,dt, \tag{1.1}$$

where $x > 0$ and the integral is taken in the Cauchy principal value sense at $t = x$. This operator plays a considerable role in Wiener–Hopf theory; see [**2**; Chapter 5].

Since T is clearly the restriction to $(0,\infty)$ of minus the Hilbert transformation applied to functions which vanish on $(-\infty, 0)$, it follows easily from the theory of the Hilbert transformation, as given in say [**6**; Theorem 101], that T is a bounded operator from $L_p(0,\infty)$ to itself for $1 < p < \infty$.

The spectrum of T on $L_2(0,\infty)$ was found, first by Koppelman and Pincus [**3**] and more recently, using the Mellin transformation, by Del Pace and Venturi [**1**] to be the closed segment of the imaginary axis from $-i$ to i, while its spectrum on $L_p(0,\infty)$ was found by Widom [**7**] to be the circular arc with endpoints $\pm i$ passing through the point $-\cot \pi/p$.

In this paper we shall use the Mellin multiplier technique that we developed in [**5**] to study the spectrum of T on the spaces $\mathscr{L}_{\mu,p}$ and $\mathscr{L}_{\omega,\mu,p}$ defined in that paper. Our notation will be that of [**5**]; other particular notations from [**5**] that we shall use are $\mathfrak{A}_p$, $\mathscr{A}$, $\mathscr{M}$ and $[X]$. We shall show that the spectrum of T on $\mathscr{L}_{\mu,p}$, where $1 < p < \infty$, $0 < \mu < 1$, is the circular arc with endpoints $\pm i$ passing through the point $-\cot \pi\mu$, and that on $\mathscr{L}_{\omega,\mu,p}$, where $\omega \in \mathfrak{A}_p$, the spectrum is a subset of this arc. This is achieved in section three, and is consistent with Widom's result since $L_p(0,\infty) = \mathscr{L}_{1/p,p}$. Naturally we must first study the boundedness of T on $\mathscr{L}_{\mu,p}$ and $\mathscr{L}_{\omega,\mu,p}$ and this is done in section two.

The operator T can be transformed by elementary changes of variable into the finite Hilbert transformation, or Tricomi operator, $T_{a,b}$ where for $-\infty < a < b < \infty$ and suitably restricted f.

$$(T_{a,b}f)(x) = \pi^{-1}\int_a^b (x-t)^{-1}f(t)\,dt, \qquad x \in (a,b), \tag{1.2}$$

the integral again being a Cauchy principal value at $t = x$, and in section four we exploit this fact to determine the spectrum of $T_{a,b}$ on a class of spaces.

In section five we make some concluding remarks, trying to put our technique in its general setting.

2. Boundedness of T. In this section we show that if $1 < p < \infty$, $\omega \in \mathfrak{A}_p$ and $0 < \mu < 1$, then T is a bounded operator on $\mathscr{L}_{\omega,\mu,p}$ to itself. However, first we need a Lemma.

LEMMA. *If* $1<p<\infty$, $0<\mu<1$, $T\in[\mathscr{L}_{\mu,p}]$. *If* $f\in\mathscr{L}_{\mu,p}$ *where* $1<p\leqq 2$ *and* $0<\mu<1$, *then*

$$(\mathcal{M}Tf)(s)=-\cot\pi s(\mathcal{M}f)(s),\ \mathrm{Re}\, s=\mu. \tag{2.1}$$

Proof. From [**6**; Theorem 90], it follows that if $f\in L_2(0,\infty)$, then for $x>0$.

$$(Tf)(x)=\frac{d}{dx}\pi^{-1}\int_0^\infty f(t)\log|1-x/t|\,dt$$

But then

$$(Tf)(x)=(2\pi)^{-1}\frac{d}{dx}\left\{\int_0^\infty f(t)\log|1-x^2/t^2|\,dt+\int_0^\infty f(t)\log|(t-x)/(t+x)|\,dt\right\},$$

and hence from [**4**; (3.5) and (3.6)]

$$Tf=-\tfrac{1}{2}(H_+f+H_-f),$$

or, on $L_2(0,\infty)$,

$$T=-\tfrac{1}{2}(H_++H_-) \tag{2.2}$$

But from [**4**, Theorem 3.1], if $1<p<\infty$, $H_+\in[\mathscr{L}_{\mu,p}]$ for $-1<\mu<1$ and $H_-\in[\mathscr{L}_{\mu,p}]$ for $0<\mu<2$. Thus from (2.2) if $1<p<\infty$, $0<\mu<1$, $T\in[\mathscr{L}_{\mu,p}]$.

Also, from [**4**; Theorem 3.1 and (3.7) and (3.8)], if $f\in[\mathscr{L}_{\mu,p}]$ where $1<p\leqq 2$, $0<\mu<1$,

$$(\mathcal{M}Tf)(s)=-\tfrac{1}{2}(-\tan(\pi s/2)+\cot(\pi s/2))(\mathcal{M}f)(s))=-\cot\pi s(\mathcal{M}f)(s),$$

$\mathrm{Re}\, s=\mu$, and (2.1) follows, so that the Lemma is proved.

THEOREM 2.1. *Suppose* $1<p<\infty$, $\omega\in\mathfrak{A}_p$ *and* $0<\mu<1$. *Then* $T\in[\mathscr{L}_{\omega,\mu,p}]$.

Proof. If $m(s)=-\cot\pi s$, then m is holomorphic in the strip $0<\mathrm{Re}\, s<1$. It is well known and elementary that if small circles of equal positive radius are drawn about the poles of $\cot\pi s$, then in the closure of the exterior of those circles $|\cot\pi s|$ is bounded and thus if $0<\sigma_1\leqq\sigma_2<1$, then in $\sigma_1\leqq\mathrm{Re}\, s\leqq\sigma_2$, $|m(s)|$ is bounded. Further, if $0<\mu<1$, $|m'(\mu+it)|=|\pi\csc^2\pi(\mu+it)|=O(e^{-2\pi|t|})=O(|t|^{-1})$ as $|t|\to\infty$.

Hence $m\in\mathscr{A}$ with $\alpha(m)=0$, $\beta(m)=1$, and thus by [**5**; Theorem 1], there is a transformation $H_m\in[\mathscr{L}_{\omega,\mu,p}]$ for $1<p<\infty$, $\omega\in\mathfrak{A}_p$ and $0<\mu<1$, such that if $f\in\mathscr{L}_{\mu,p}$, $1<p\leqq 2$, $0<\mu<1$, then

$$(\mathcal{M}H_mf)(s)=m(s)(\mathcal{M}f)(s)=-\cot\pi s(\mathcal{M}f)(s),\quad \mathrm{Re}\, s=\mu$$

But then, from (2.1) on $\mathscr{L}_{\mu,p}$ for $1<p\leqq 2$, $0<\mu<1$, $H_m=T$, and thus extending T to $\mathscr{L}_{\omega,\mu,p}$ by defining it to be H_m, $T\in[\mathscr{L}_{\omega,\mu,p}]$ for $1<p<\infty$, $\omega\in\mathfrak{A}_p$, $0<\mu<1$.

3. The spectrum of T. Let us denote the circular arc with end points $\pm i$ passing through the point $-\cot\pi u$ by $\sigma(\mu)$. Clearly $\lambda\in\sigma(\mu)$ if and only if $\lambda=\pm i$ or $\arg((\lambda-i)/(\lambda+i))=2\pi\mu$, so that $\sigma(\mu)$ is clearly an arc of the Steiner circle of the second kind with poles $\pm i$. The Theorem below shows how the spectrum of T in $\mathscr{L}_{\omega,\mu,p}$ is related to $\sigma(\mu)$.

THEOREM 3.1. *Suppose* $1<p<\infty$, $\omega \in \mathfrak{A}_p$ *and* $0<\mu<1$. *Then on* $\mathcal{L}_{\omega,\mu,p}$ *the spectrum of* T *is a subset of* $\sigma(\mu)$, *while on* $\lambda_{\mu,p}$ *the spectrum of* T *is equal to* $\sigma(\mu)$.

Proof. Suppose $\lambda \notin \sigma(\mu)$. Then there is a γ, $0 \leqq \gamma < 1$, $\gamma \neq \mu$, so that $\arg((\lambda-i)/(\lambda+i)) = 2\pi\gamma$. We show first that if $m_\lambda(s) = \lambda + \cot \pi s$, then $1/m_\lambda \in \mathcal{A}$, with $\alpha(1/m_\lambda) = \gamma$, $\beta(1/m_\lambda) = \gamma+1$ if $\gamma < \mu$ and with $\alpha(1/m_\lambda) = \gamma - 1$, $\beta(1/m_\lambda) = \gamma$ if $\gamma > \mu$.

Suppose that $0 \leqq \gamma < \mu$. Then $m_\lambda(s)$ has no zeros in the strip $\gamma < \operatorname{Re} s < \gamma+1$. For $m_\lambda(s)$ has a zero on the line $\operatorname{Re} s = \gamma$, namely at the point

$$s = (2\pi i)^{-1} \log((\lambda-i)/(\lambda+i)) = \gamma + (2\pi i)^{-1} \log |(\lambda-i)/(\lambda+i)|,$$

and it is easy to see that $\cot \pi s$ takes on a value only once in a strip of the form $\eta < \operatorname{Re} s \leqq \eta+1$. Thus (*i*) $1/m_\lambda(s)$ is holomorphic in the strip $\gamma < \operatorname{Re} s < \gamma+1$. Suppose $\gamma < \sigma_1 \leqq \sigma_2 < \gamma+1$. Then $-\cot \pi(\sigma_2+it) = (i \cot \pi\sigma_2 \coth \pi t + 1)/(\cot \pi\sigma_2 - i \coth \pi t) = (\tanh \pi t + \cot \pi\sigma_2)/(\cot \pi\sigma_2 \tanh \pi t - i)$, and thus as t increases from $-\infty$ to ∞, $w = -\cot \pi(\sigma_2+it)$ describes the arc $\arg\,(w-i)/(w+i)) = 2\pi\sigma_2$ from $-i$ to i. Similarly, as t runs from ∞ to $-\infty$, $w = -\cot \pi(\sigma_1+it)$ describes the arc $\arg((w-i)/(w+i)) = 2\pi\sigma_1$ from i to $-i$. Thus since $-\cot \pi(\sigma+it) \to \pm i$ as $t \to \pm\infty$ uniformly in σ for $\sigma_1 \leqq \sigma \leqq \sigma_2$, the values taken on by $-\cot \pi s$ in the strip $\sigma_1 \leqq \operatorname{Re} s \leqq \sigma_2$ lie in the set

$$W = \{w \mid 2\pi\sigma_1 \leqq \arg((w-i)/(w+i)) \leqq 2\pi\sigma_2\}$$

and thus since $\arg((\lambda-i)/(\lambda+i)) = 2\pi\gamma$ and $\gamma < \sigma_1 < \sigma_2 < \gamma+1$, λ is at a positive distance from W so that $|\lambda + \cot \pi s|$ is bounded away from zero in $\sigma_1 \leqq \operatorname{Re} s \leqq \sigma_2$. Hence (ii) $|(1/m_\lambda(s))|$ is bounded in $\sigma_1 \leqq \operatorname{Re} s \leqq \sigma_2$. Finally (iii) if $\gamma < \sigma < \gamma+1$ and $\operatorname{Re} s = \sigma$, $\dfrac{d}{ds}(m_\lambda(s))^{-1} = \pi(m_\lambda(s))^{-2} \csc^2 \pi s$ and $|m_\lambda(\sigma+it)|^{-2}$ is bounded and $|\csc^2 \pi(\sigma+it)| = O(|t|^{-1})$ as $|t| \to \infty$. Thus, if $0 \leqq \gamma < \mu$, $1/m_\lambda(s)\mathcal{A}$ with $\alpha(1/m_\lambda) = \gamma$, $\beta(1/m_\lambda) = \gamma+1$.

Similarly if $\mu < \gamma < 1$, $1/m_\lambda(s) \in \mathcal{A}$ with $\alpha(1/m_\lambda) = \gamma - 1$ and $\beta(1/m_\lambda) = \gamma$.

But obviously $m_\lambda(s)$ is the multiplier of $\lambda I - T$, and hence by [**5**; Theorem 1] since $\alpha(1/m_\lambda) < \mu < \beta(1/m_\lambda)$, $(\lambda I - T)^{-1}$ exists and is in $[\mathcal{L}_{\omega,\mu,p}]$. Thus if $\lambda \notin \sigma(\mu)$, λ is in the resolvent set of T and hence the spectrum of T is a subset of $\sigma(\mu)$.

To show that on $\mathcal{L}_{\mu,p}$, $\sigma(\mu)$ equals the spectrum of T, suppose first that $1 < p \leqq 2$, $0 < \mu < 1$ and that $\lambda \in \sigma(\mu)$, $\lambda \neq \pm i$. Then if λ is in the resolvent set of T, for any $g \in \mathcal{L}_{\mu,p}$ the equation $(\lambda I - T)f = g$ has a solution $f \in \mathcal{L}_{\mu,p}$. Taking Mellin transforms it follows that $(\mathcal{M}f)(s) = (\mathcal{M}g)(s)/(\lambda - \cot \pi s)$, $\operatorname{Re} s = \mu$. Since $\mathcal{M}$ maps $\mathcal{L}_{\mu,p}$ into $L_{p'}(-\infty, \infty)$, where $p' = p/(p-1)$, it follows that for any $g \in \mathcal{L}_{\mu,p}$, $(\mathcal{M}g)(\mu+it)/(\lambda + \cot \pi(\mu+it)) \in L_{p'}(-\infty, \infty)$.

However since $\arg((\lambda-i)/(\lambda+i)) = 2\pi\mu$, $\lambda + \cot \pi s$ has a simple zero at $s = \mu + (2\pi i)^{-1} \log |(\lambda-i)/(\lambda+i)| = \mu + it_0$. Choose real numbers a and b so that $a < t_0 < b$ and let

$$g(x) = \pi^{-1} x^{-\mu} e^{-\frac{1}{2}i(a+b)\log x} \sin(\tfrac{1}{2}(b-a)\log x)/\log x$$

Then $g \in \mathcal{L}_{\mu,p}$ since

$$\int_0^\infty |x^\mu g(x)|^p\,dx = \pi^{-p} \int_0^\infty |\sin(\tfrac{1}{2}(b-a)\log x)/\log x|^p\,dx/x$$
$$= \pi^{-p} \int_{-\infty}^\infty |\sin(\tfrac{1}{2}(b-a)t)/t|^p\,dt < \infty$$

Also

$$\lim_{R\to\infty}\int_{1/R}^{R} x^{\mu+it-1}g(x)\,dx=\pi^{-1}\lim_{R\to\infty}\int_{-\log R}^{\log R}\cos(t-\tfrac{1}{2}(a+b)u\sin\tfrac{1}{2}(b-a)u\,du/u$$

$$=(2\pi)^{-1}\lim_{R\to\infty}\int_{-\log R}^{\log R}(\sin(t-a)u-\sin(t-b)u)\,du/u=\tfrac{1}{2}(\operatorname{sgn}(t-a)-\operatorname{sgn}(t-b))$$

$$=\begin{cases}0, & t<a\\ 1, & a<t<b\\ 0, & t>b\end{cases}$$

Thus $(\mathcal{M}g)(\mu+it)$ equals the characteristic function of (a,b) a.e., and hence since, as noted, $(\mathcal{M}g)(\mu+it)/(\lambda+\cot\pi(\mu+it))$ is in $L_{p'}(-\infty,\infty)$, we must have

$$\int_a^b|\lambda+\cot\pi(\mu+it)|^{-p'}\,dt<\infty,$$

a contradiction.

Hence λ cannot be in the resolvent set of T and must then be in the spectrum of T, and since the spectrum is closed $\sigma(\mu)$ must be in the spectrum of T, and consequently that spectrum is $\sigma(\mu)$.

If $p>2$, then the same result follows since T and its adjoint have the same spectrum, the adjoint of T is $-T$, the adjoint space of $L_{\mu,p}$ is $L_{1-\mu,p'}$, $p'<2$ and $\sigma(1-\mu)=-\sigma(\mu)$.

One might remark that it is easy to see that on $\mathcal{L}_{\mu,p}$, $1<p<\infty$, $0<\mu<1$, the spectrum of T consists entirely of continuous spectrum. Also, it is easy to show that if $f(x)=x^{-\frac{1}{2}}$, then $Tf=0$. Hence since $0\notin\sigma(T)$ on $L_{\omega,\mu,p}$, $1<p<\infty$, $0<\mu<1$, unless $\mu=\frac{1}{2}$, it follows that $f\notin L_{\omega,\mu,p}$, $1<p<\infty$, $0<\mu<1$ unless $\mu=\frac{1}{2}$, and thus if $\nu>-\frac{3}{2}$, $\nu\neq-1$ and $\omega\in\mathfrak{A}_p$ where $1<p<\infty$, then

$$\int_0^\infty\omega(x)x^\nu\,dx=\infty$$

4. The spectrum and boundedness of $T_{a,b}$. If f is suitably restricted and $g=T_{a,b}f$ and if we let $F(x)=(x+1)^{-1}f((bx+a)/(x+1))$, and $G(x)=(x+1)^{-1}g((bx+a)/(x+1))$, then $G=TF$. The following theorem follows immediately.

THEOREM 4.1. *Suppose* $1<p<\infty$, $\omega\in\mathfrak{A}_p$ *and* $0<\mu<1$. *Then on the space of functions* f, *measurable on* (a,b), *and normed by the norm*

$$\|f\|_{\omega,\mu,p}=\left\{\int_a^b\omega((x-a)/(b-x))\,|(x-a)^\mu(b-x)^{1-\mu}f(x)|^p\,dx/((b-x)(x-a))\right\}^{1/p},$$

to itself, $T_{a,b}$ *is a bounded operator and its spectrum is a subset of* $\sigma(\mu)$; *if* $\omega(x)\equiv1$, *the spectrum of* $T_{a,b}$ *is* $\sigma(\mu)$.

5. Concluding remarks. The technique that we have used here to analyze the spectrum of T seems to be of considerably more general applicability. Indeed if $m\in\mathcal{A}$

and T_m is the transformation associated with m by [**5**; Theorem 1], and if $m_\lambda = \lambda - m$, where $\lambda \in \mathbf{C}$, then clearly $\lambda I - T_m$ is associated with m_λ and $m_\lambda \in \mathscr{A}$, so that if $1/m_\lambda \in \mathscr{A}$ and $\max(\alpha(m), \alpha(1/m_\lambda)) < \mu < \min(\alpha(m), \alpha(1/m_\lambda))$, then if $1 < p < \infty$ and $\omega \in \mathfrak{A}_p$, $(\lambda I - T_m)^{-1} \in [\mathscr{L}_{\omega,\nu,p}]$, so that λ is in the resolvent set of T_m.

The only barrier to this method seems to be showing that $\alpha(1/m_\lambda) < \mu < \beta(1/m_\lambda)$. which requires that the range of $m(\mu + it)$, $-\infty < t < \infty$, be known. In the case of the T of sections one to three, it was possible to find this because of the simplicity of the corresponding m, but for a more complicated m this could be very difficult.

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