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A PAIR OF ARBITRARILY-ORIENTED COPLANAR CRACKS IN AN ANISOTROPIC ELASTIC SLAB

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Abstract

The problem of an anisotropic elastic slab containing two arbitrarily-oriented coplanar cracks in its interior is considered. Using a Fourier integral transform technique, we reduce the problem to a system of simultaneous finite-part singular integral equations which can be solved numerically. Once the integral equations are solved, relevant quantities such as the crack energy can be readily computed. Numerical results for specific examples are obtained.

1. Introduction

Anisotropic materials have numerous applications in modern technology. Fibre-reinforced composites which are widely used in engineering can be reasonably modelled as anisotropic and inextensible along the fibre direction (see, e.g., Spencer [15]). The determination of elastic displacements and stresses in cracked anisotropic materials is, therefore, a subject of considerable practical importance.

The static displacements and stresses in an infinite anisotropic elastic medium containing a single planar crack were obtained by Stroh [16]. From a practical point of view, the usefulness of these displacements and stresses is necessarily restricted to situations where the crack interacts negligibly with the outer boundary of the material. Clements [5] and, more recently, Ang [2, 3] examined the interaction of the crack with the boundary by placing the crack in an infinitely long anisotropic elastic slab. The plane containing the crack is perpendicular and parallel to the boundary of the slab in [2] and [5], respectively, while in [3] it is arbitrarily inclined to the boundary.

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Other related work on cracked anisotropic slabs may also be found in Ang [4], Clements and Tauchert [6], and Hill and Clements [9].

The problem of two or more coplanar cracks in an infinite transversely-isotropic material was examined by Konishi [12] and Krenk [13]. Dhaliwal [7] studied the interaction of two coplanar cracks in an infinitely-long orthotropic slab, with the cracks being parallel to the boundary of the slab.

The present paper considers the problem of two coplanar cracks lying on an arbitrary plane in the interior of a general anisotropic elastic slab. Through the use of a Fourier integral transform technique, the problem is reduced to a system of simultaneous finite-part singular integral equations, which can be solved numerically by using collocation methods described by Loakimidis [10] and Kaya and Erdogan [11]. Once the integral equations are solved, quantities of interest such as the crack energy can be calculated readily. Numerical results for specific cases of the problem are obtained by solving these integral equations.

2. Statement of the problem

Referred to an $Ox_1x_2x_3$ Cartesian coordinate system, consider an anisotropic elastic material which occupies the region between the planes $x_2 = h$ and $x_2 = -h$, where h is a given real positive number. In its interior, the slab contains two coplanar cracks $a < |x_1 \sin\theta - x_2 \cos\theta| < b$, $x_1 \cos\theta + x_2 \sin\theta = 0$, for all x_3, where the angle θ lies between 0° and 360°, and a and b are real positive numbers with $b|\cos\theta| < h$. That is, a normal vector to the plane containing the cracks is given by $[\cos\theta, \sin\theta, 0]$ (see Figure 1). The planes $x_2 = h$ and $x_2 = -h$ are free of tractions, and the cracks are opened up by internal stresses which are independent of the coordinate x_3. The problem is to determine the displacements and the stresses throughout the slab.

3. Equations of anisotropic elasticity

The equilibrium equations for anisotropic elasticity are given by the system of partial differential equations

$$c_{ijkp} \frac{\partial^2 u_k}{\partial x_j \partial x_p} = 0, \tag{3.1}$$

where the Latin subscripts take the values of 1,2 and 3, x_k are the spatial coordinates with respect to a Cartesian coordinate frame, u_k are the Cartesian

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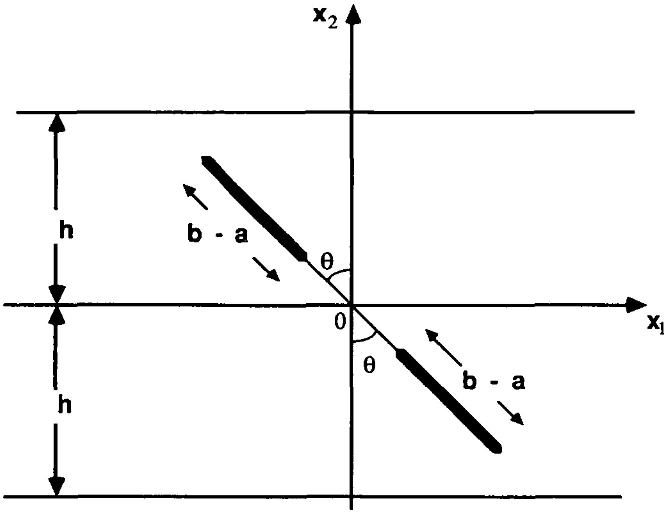

FIGURE 1. A pair of arbitrarily-oriented coplanar cracks in an anisotropic slab.

displacements, and c_{ijkp} are the elastic moduli of the material. The elastic moduli c_{ijkp} satisfy the symmetry conditions

$$c_{ijkp} = c_{jikp} = c_{ijpk} = c_{kpij}, \tag{3.2}$$

and the equality

$$c_{ijkp} h_{ij} h_{kp} > 0, \tag{3.3}$$

where h_{ij} $(i, j = 1, 2, 3)$ are any arbitrary real numbers, not all of which are zero. The usual convention of summing over a repeated index is assumed here only for Latin subscripts.

If the displacements u_k are free of the coordinate x_3, then (3.1) admits solutions of the form (Ang [3])

$$u_k = \mathrm{Re}\left[\sum_{\alpha} A_{k\alpha}(\rho) f_\alpha(z_\alpha(\rho))\right], \tag{3.4}$$

where Re denotes the real part of a complex number, the sum over α is from 1 to 3, $f_\alpha(z_\alpha)$ are differentiable functions of z_α, $z_\alpha = (a_{j1} + \tau_\alpha a_{j2})x_j$, ρ is a real parameter, $a_{ij} = a_{ij}(\rho)$ for our purpose here are taken to be

$$[a_{ij}(\rho)] = \begin{bmatrix} \sin\rho & \cos\rho & 0 \\ -\cos\rho & \sin\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.5}$$

$\tau_\alpha = \tau_\alpha(\rho)$ are the solutions with positive imaginary parts of the characteristic equation,

$$\det[c_{i1k1} + \sigma(\tau)\{c_{i1k2} + c_{i2k1}\} + \sigma^2(\tau) c_{i2k2}] = 0, \tag{3.6}$$

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where $\sigma(\tau) = (a_{21} + \tau a_{22})/(a_{11} + \tau a_{12})$, and $A_{k\alpha} = A_{k\alpha}(\rho)$ are the nonzero solutions of the system

$$[c_{i1k1} + \sigma(\tau_\alpha)\{c_{i1k2} + c_{i2k1}\} + \sigma^2(\tau_\alpha)c_{i2k2}]A_{k\alpha} = 0. \tag{3.7}$$

Note that the inequality in (3.3) is violated if (3.6) admits a real solution. Hence, the solutions of (3.6) are complex and occur in conjugate pairs. It is assumed here that they are distinct.

Using the generalised Hooke's law

$$\sigma_{lk} = c_{lkmp}\frac{\partial u_m}{\partial x_p}, \tag{3.8}$$

we find that the Cartesian stresses σ_{ij} corresponding to the displacements in (3.4) are given by

$$\sigma_{kp} = \operatorname{Re}\left[\sum_\alpha L_{kp\alpha}(\rho)f'_\alpha(z_\alpha(\rho))\right], \tag{3.9}$$

where the prime denotes differentiation with respect to z_α and $L_{ij\alpha} = (a_{p1} + \tau_\alpha a_{p2})c_{ijkp}A_{k\alpha}$.

4. Solution of the problem

Our mathematical objective is to solve (3.1) subject to the conditions

$$\sigma_{k2}(x_1, h) = \sigma_{k2}(x_1, -h) = 0 \quad \text{for } -\infty < x_1 < \infty, \tag{4.1}$$

and

$$a_{p2}(\theta)\sigma_{kp} \to -P_k(a_{p1}(\theta)x_p) \quad \text{as } a_{p2}(\theta)x_p \to 0 \text{ for } a < |a_{p1}(\theta)x_p| < b, \tag{4.2}$$

where $P_k(x)$ are suitably prescribed functions of x. For our discussion here, we assume that the $P_k(x)$ are even functions of x.

Let the displacements and the stresses be respectively given by

$$\begin{aligned} u_k = \operatorname{Re}\int_0^\infty \sum_\alpha \{&2A_{k\alpha}(\phi)E_\alpha(\xi)\sinh(i\xi z_\alpha(\phi)) \\ &+ H(a_{p2}(\theta)x_p)A_{k\alpha}(\theta)G^+_\alpha(\xi)\exp(i\xi z_\alpha(\theta)) \\ &+ H(-a_{p2}(\theta)x_p)A_{k\alpha}(\theta)G^-_\alpha(\xi)\exp(-i\xi z_\alpha(\theta))\}\,d\xi, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \sigma_{km} = \operatorname{Re}\int_0^\infty \sum_\alpha \{&2L_{km\alpha}(\phi)E_\alpha(\xi)\cosh(i\xi z_\alpha(\phi)) \\ &+ H(a_{p2}(\theta)x_p)L_{km\alpha}(\theta)G^+_\alpha(\xi)\exp(i\xi z_\alpha(\theta)) \\ &- H(-a_{p2}(\theta)x_p)L_{km\alpha}(\theta)G^-_\alpha(\xi)\exp(-i\xi z_\alpha(\theta))\}i\xi\,d\xi, \end{aligned} \tag{4.4}$$

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where $\phi = \pi/2$, $i = (-1)^{1/2}$, $H(x)$ is the Heaviside unit-step function and $E_\alpha(\xi)$, $G^+_\alpha(\xi)$ and $G^-_\alpha(\xi)$ are arbitrary functions yet to be determined.

Our choice of u_k and σ_{kj} in (4.3) and (4.4) requires us to impose the continuity conditions

$$[a_{p2}(\theta)\sigma_{kp}]^+ - [a_{p2}(\theta)\sigma_{kp}]^- = 0 \quad \text{for } |a_{p1}(\theta)x_p| \geq 0, \tag{4.5}$$

and

$$[u_k]^+ - [u_k]^- = 0 \quad \text{for } 0 < |a_{p1}(\theta)x_p| < a,\ b < |a_{p1}(\theta)x_p| < \infty, \tag{4.6}$$

where $[f(x_1, x_2)]^+$ and $[f(x_1, x_2)]^-$ denote the limiting values of $f(x_1, x_2)$ as $a_{j2}(\theta)x_j$ approaches 0 from above and below respectively.

From (4.4), it is easy to verify that conditions (4.5) hold if we choose

$$G^+_\alpha(\xi) = M_{\alpha p}(\theta)\psi_p(\xi) \quad \text{and} \quad G^-_\alpha(\xi) = M_{\alpha p}(\theta)\overline{\psi}_p(\xi), \tag{4.7}$$

where the bar denotes the complex conjugate of a complex number, ψ_p are arbitrary functions yet to be determined and $[M_{\alpha p}(\theta)]$ is the inverse of $[a_{j2}(\theta)L_{kj\alpha}(\theta)]$.

Using (4.3) and (4.7), we find that

$$\begin{aligned}[u_k]^+ - [u_k]^- = \operatorname{Re}\int_0^\infty \sum_\alpha \{A_{k\alpha}(\theta)M_{\alpha p}(\theta) - \overline{A}_{k\alpha}(\theta)\overline{M}_{\alpha p}(\theta)\} \\ \times \psi_p(\xi)\exp(i\xi a_{q1}(\theta)x_q)\,d\xi.\end{aligned} \tag{4.8}$$

Since $P_k(x)$ are assumed to be even functions of x, in order to satisfy conditions (4.6), it is sufficient to let

$$\psi_p(\xi) = i\int_a^b r_p(t)\cos(\xi t)\,dt, \tag{4.9}$$

where $r_p(t)$ are real arbitrary functions to be found. Then, from (4.8), we obtain

$$\begin{aligned}[u_k]^+ - [u_k]^- = i\frac{\pi}{2}\sum_\alpha \{A_{k\alpha}(\theta)M_{\alpha p}(\theta) - \overline{A}_{k\alpha}(\theta)\overline{M}_{\alpha p}(\theta)\}r_p(a_{q1}(\theta)x_q) \\ \text{for } a < a_{p1}(\theta)x_p < b.\end{aligned} \tag{4.10}$$

By using a Fourier inversion theorem which is given in Sneddon [14], we find that conditions (4.1) are satisfied if and only if

$$\int_{-\infty}^\infty \sigma_{k2}(\xi, h)\exp(-i\lambda\xi)\,d\xi = \int_{-\infty}^\infty \sigma_{k2}(\xi, -h)\exp(-i\lambda\xi)\,d\xi = 0, \tag{4.11}$$

where λ is some real constant.

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Using the result (Gradshtein and Ryzhik [8])

$$\int_0^\infty \frac{x\sin(ax)\cos(bx)\,dx}{x^2+\beta^2} = \frac{\pi}{2}\exp(-a\beta)\cosh(b\beta)$$
$$\text{for } 0 < b < a \text{ and } \operatorname{Re}\beta > 0, \quad (4.12)$$

we find that, from (4.4), (4.7) and (4.9), conditions (4.11) can be written as (for $\lambda > 0$)

$$\sum_\alpha \{L_{2k\alpha}(\phi)E_\alpha(\lambda)\exp[i\lambda\tau_\alpha(\phi)h] - \overline{L}_{k2\alpha}(\phi)\overline{E}_\alpha(\lambda)\exp[i\lambda\overline{\tau}_\alpha(\phi)h]\}$$
$$= -i\sum_\alpha D_\alpha^2(\theta)L_{2k\alpha}(\theta)M_{\alpha p}(\theta)\exp[i\lambda hD_\alpha(\theta)(-a_{12}(\theta)+\tau_\alpha(\theta)a_{11}(\theta))]$$
$$\times\int_a^b r_p(t)\cosh[i\lambda tD_\alpha(\theta)]\,dt, \quad (4.13)$$

where $D_\alpha(\theta) = [a_{11}(\theta) + \tau_\alpha(\theta)a_{12}(\theta)]^{-1}$.

Equations (4.13) (together with their complex conjugates) can now be solved for $E_\alpha(\lambda)$. We obtain (for $\lambda > 0$)

$$E_\alpha(\lambda) = \exp[i\lambda h\tau_\alpha(\phi)]\int_a^b Y_{\alpha k}(\lambda)K_{kp}(\lambda, t)r_p(t)\,dt, \quad (4.14)$$

where $[Y_{\alpha k}(\lambda)]$ is the inverse of $[Z_{\alpha k}(\lambda)]$ and

$$Z_{k\beta}(\lambda)$$
$$= \sum_\alpha\{\overline{L}_{k2\alpha}(\phi)\overline{M}_{\alpha q}(\phi)L_{q2\beta}(\phi)\exp[2i\lambda h(\tau_\beta(\phi)-\overline{\tau}_\alpha(\phi))] - \delta_{\alpha\beta}L_{k2\beta}(\phi)\},$$

$$K_{kp}(\lambda, t) = i\sum_\alpha\sum_\beta\{\delta_{\alpha\beta}\overline{D}_\beta^2(\theta)\overline{L}_{k2\beta}(\theta)\overline{M}_{\beta p}(\theta)\cosh[i\lambda t\overline{D}_\beta(\theta)]$$
$$\times\exp[-i\lambda h\overline{D}_\beta(\theta)(-a_{12}(\theta)+\overline{\tau}_\beta(\theta)a_{11}(\theta))]$$
$$-\overline{M}_{\alpha q}(\phi)D_\beta^2(\theta)\overline{L}_{k2\alpha}(\phi)L_{q2\beta}(\theta)M_{\beta p}(\theta)\cosh[i\lambda tD_\beta(\theta)]$$
$$\times\exp[i\lambda h\{D_\beta(\theta)(-a_{12}(\theta)+\tau_\beta(\theta)a_{11}(\theta)) - 2\overline{\tau}_\alpha(\phi)\}]\}, \quad (4.15)$$

where $\delta_{\alpha\beta}$ denotes the Kronecker-delta.

From (4.4), (4.7), (4.9) and (4.14) together with the limit (which can be worked out using a results in Gradshtein and Ryzhik [8])

$$\lim_{y\to 0^+}\int_0^\infty 2\xi\exp(-\xi y)\cos(\xi t)\cos(\xi u)\,d\xi = -(t-u)^{-2} - (t+u)^{-2}, \quad (4.16)$$

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conditions (4.2) give rise to the system of simultaneous finite-part singular integral equations

$$\oint_a^b \frac{r_k(t)\,dt}{(t-u)^2} + \int_a^b \left(\frac{r_k(t)}{(t+u)^2} + \Omega_{ks}(t\,,\,u) r_s(t) \right) dt = -2P_k(u) \quad \text{for } a < u < b\,, \tag{4.17}$$

where $\oint$ denotes that the integral is to be interpreted in Hadamard finite-part sense and

$$\Omega_{ks}(t\,,\,u) = 4\int_0^\infty \mathrm{Re} \sum_\alpha L_{kp\alpha}(\phi) a_{p2}(\theta) Y_{\alpha q}(\xi) K_{qs}(\xi\,,\,t) \times \exp[i\xi h \tau_\alpha(\phi)] \cosh(i\xi[a_{11}(\theta) + a_{21}(\theta)\tau_\alpha(\phi)]u) i\xi\, d\xi\,. \tag{4.18}$$

Equations (4.17) can be solved numerically by employing the collocation techniques described in [10] and [11]. We make the approximation

$$r_k(t) \approx \sqrt{1 - \left(\frac{t-a-d}{d}\right)^2} \sum_{n=1}^N \alpha_k^n U_{n-1}\left(\frac{t-a-d}{d}\right) \quad \text{for } a < t < b\,, \tag{4.19}$$

where $2d = b - a$, $U_n(x)$ is the nth order Chebyshev polynomial of the second kind and α_k^n $(k = 1\,,\,2\,,\,3\,;\; n = 1\,,\,2\,,\,\ldots\,,\,N)$ are constant coefficients to be determined.

Substituting (4.19) into (4.17), we obtain for $-1 < s < 1$

$$\sum_{n=1}^N \frac{\alpha_q^n}{d} \{[-\pi n U_{n-1}(s) + F^n(s)]\delta_{kq} + J_{kq}^n(s)\} = -2P_k(ds + a + d)\,, \tag{4.20}$$

where

$$F^n(s) = \int_{-1}^1 \frac{\sqrt{1-r^2}\,U_{n-1}(r)\,dr}{(r+s+2+2ad^{-1})^2}\,,$$

$$J_{kq}^n(s) = \int_{-1}^1 d^2 \sqrt{1-r^2}\,U_{n-1}(r) \Omega_{kq}(rd+a+d\,,\, sd+a+d)\,dr\,. \tag{4.21}$$

The integrals in (4.21) can be accurately calculated using the numerical quadrature (25.4.40) in Abramowitz and Stegun [1].

Choosing s in (4.20) to be given by

$$s = s_p = \cos(\{2p-1\}\pi/(2N)) \quad \text{for } p = 1\,,\,2\,,\,\ldots\,,\,N\,, \tag{4.22}$$

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we find that equations (4.20) give rise to a system of $3N$ linear algebraic equations in ($3N$ unknowns) α_k^n. The constants α_k^n can be determined by solving these equations.

5. Calculation of crack energy

The crack energy U for the crack $a < x_1 \sin\theta - x_2 \cos\theta < b$, $x_1 \cos\theta + x_2 \sin\theta = 0$, is defined by the integral

$$U = \frac{1}{2}\int_a^b P_k(\phi)\Delta u_k(\phi)\, d\phi, \tag{5.1}$$

where $P_k(\phi)$ give the traction distribution over the crack surface and $\Delta u_k(\phi)$ are the differences between the displacements u_k on the top and the bottom faces of the crack as given in (4.10). On a practical note, the crack energy is an important quantity for examining the stability of the crack.

The use of (4.10), (4.19) and (5.1) yields

$$U \approx \frac{\pi}{4}\sum_\beta \{A_{k\beta}(\theta)M_{\beta p}(\theta) - \overline{A}_{k\beta}(\theta)\overline{M}_{\beta p}(\theta)\}id \times \sum_{n=1}^N \alpha_p^n \int_{-1}^1 \sqrt{1-r^2}P_k(rd+a+d)U_{n-1}(r)\, dr. \tag{5.2}$$

For the special case where $P_k(x) = T_k$, where T_k are constants, using the orthogonality relation for Chebyshev polynomials, (5.2) becomes

$$U \approx \frac{\pi^2}{8}\sum_\beta \{A_{k\beta}(\theta)M_{\beta p}(\theta) - \overline{A}_{k\beta}(\theta)\overline{M}_{\beta p}(\theta)\}id\alpha_p^1 T_k. \tag{5.3}$$

6. Specific examples

In this section, the analysis in Section 4 is applied to solve a problem involving a particular transversely-isotropic slab which contains two coplanar cracks.

The elastic behaviour of a transversely-isotropic material is characterised by five independent constants A, N, F, C and L. If the transverse planes of the material are parallel to the Ox_2x_3 plane, the only nonzero c_{ijkp} are related to these five constants by

$$\begin{gathered} A = c_{2222}, \qquad C = c_{1111}, \qquad F = c_{1122} = c_{2211} = c_{1133} = c_{3311}, \\ (A-N)/2 = c_{2323} = c_{3232} = c_{3223} = c_{2332}, \\ L = c_{1212} = c_{2121} = c_{2112} = c_{1313} = c_{3131} = c_{3113}. \end{gathered} \tag{6.1}$$

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Consequently, the system (3.1) reduces to

$$
\begin{gathered}
C\frac{\partial^2 u_1}{\partial x_1^2} + L\frac{\partial^2 u_1}{\partial x_2^2} + (F+L)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} = 0, \\
A\frac{\partial^2 u_2}{\partial x_2^2} + L\frac{\partial^2 u_2}{\partial x_1^2} + (F+L)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} = 0, \\
L\frac{\partial^2 u_3}{\partial x_1^2} + \frac{1}{2}(A-N)\frac{\partial^2 u_3}{\partial x_2^2} = 0,
\end{gathered}
\tag{6.2}
$$

and the characteristic equation (3.6) becomes

$$\left[\frac{1}{2}(A-N)\sigma^2(\tau) + L\right][AL\sigma^4(\tau) - (F^2 + 2FL - AC)\sigma^2(\tau) + CL] = 0. \tag{6.3}$$

If we take

$$\sigma^2(\tau_3) = 2L/(N-A), \tag{6.4}$$

then τ_1 and τ_2 can be obtained from the roots of the quartic factor in $\sigma(\tau)$ in (6.3). From (3.7), we have

$$[A_{k\alpha}(\rho)] = \begin{bmatrix} V_1(\rho) & V_2(\rho) & 0 \\ i & i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.5}$$

where $V_k(\rho) = -i\sigma(\tau_k(\rho))(F+L)/(C + L\sigma^2(\tau_k(\rho)))$. Other constants such as $L_{ij\alpha}$ and $M_{\alpha p}$ which are of relevance to our computation here can be calculated directly using (6.5).

Firstly, consider the case where the cracks are subject to an antiplane deformation, i.e. we take

$$P_1 = 0,\ P_2 = 0 \quad \text{and} \quad P_3 = s_0(\sin\theta + \cos\theta), \tag{6.6}$$

where s_0 is a real positive constant.

For the antiplane case, the coefficients α_1^n and α_2^n are zero. We solve (4.20) for α_3^n, and use (5.3) to compute the non-dimensionalised crack energy $LU/(s_0 d)^2$. In our computation, we employ no more than 10 terms in the series approximation (4.19). For $R = 2L/(A-N) = 1.334$ (titanium), $a/d = 0.250$ and $h/d = 2.500$, 3.500 and 4.500 as well as for h/d tending to infinity, we plot $LU/(s_0 d)^2$ against θ in Figure 2. From Figure 2, it is obvious that, for a fixed value of θ, the crack energy increases as h/d decreases. Also, note that, for a given value of h/d, the crack energy is a maximum when $\theta = \theta_0$, where $0 < \theta_0 < \pi/2$. For the case where h/d tends to infinity, $\theta_0 = \pi/4$. For the cases considered here, it is apparent

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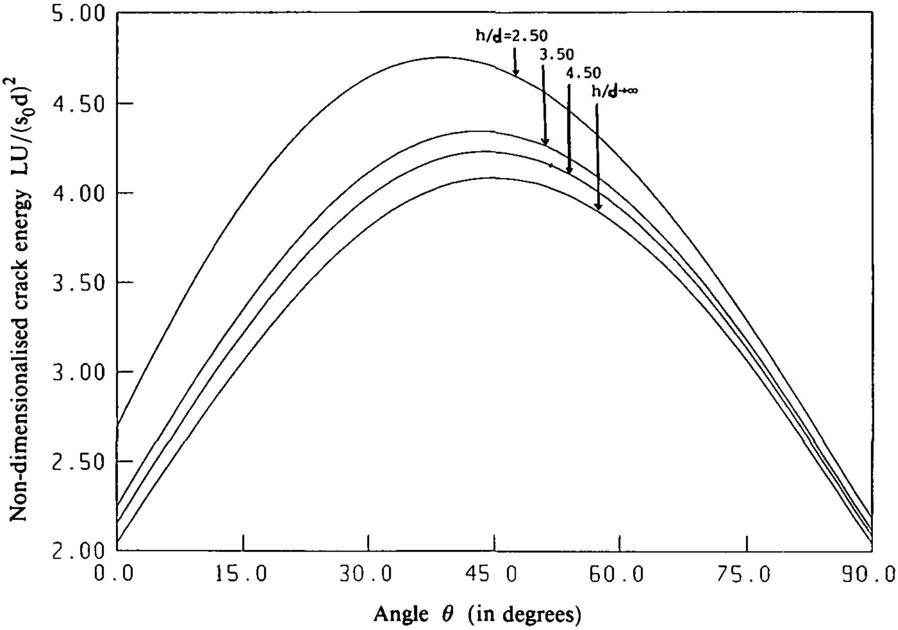

FIGURE 2. Variations of $LU/(s_0 d)^2$ with θ for $a/d = 0.25$, $R = 1.334$ (titanium) $h/d = 2.50$, 3.50 and 4.50 as well as h/d tending to infinity.

that decreasing the value of h/d has the effect of decreasing θ_0. For $a/d = 0.500$, $h/d = 5.000$ and $R = 0.500$, 1.000 (isotropic case) and 2.000, the non-dimensionalised crack energy is plotted against θ in Figure 3. It is clear from Figure 3 that the crack energy for $R = 1.000$ is less or greater than that for $R = 2.000$ or $R = 0.500$ respectively.

We now consider the case where the cracks are opened up by constant tensile tractions acting on their faces. Specifically, we choose $P_k(x)$ as

$$P_1 = T_0 \cos\theta, \qquad P_2 = T_0 \sin\theta \quad \text{and} \quad P_3 = 0, \tag{6.7}$$

where T_0 is a given positive constant.

To obtain some numerical results, we use the elastic constants for titanium. These constants are given by $A/L = 3.469$, $N/L = 1.970$, $C/L = 3.876$ and $F/L = 1.478$. Using these constants, we solve (4.20) and employ (5.3) to calculate the non-dimensionalised crack energy $LU/(T_0 d)^2$. The results for $a/d = 0.250$ and selected values of θ and h/d are given in Table 1. From the table, it is obvious that for a given value of θ, decreasing h/d has the effect of increasing the crack energy. Note that for $h/d = 4.500$ as well as for h/d tending to infinity, the crack energy is an increasing function of θ while for $h/d = 2.500$ it is a decreasing function of θ. In Table

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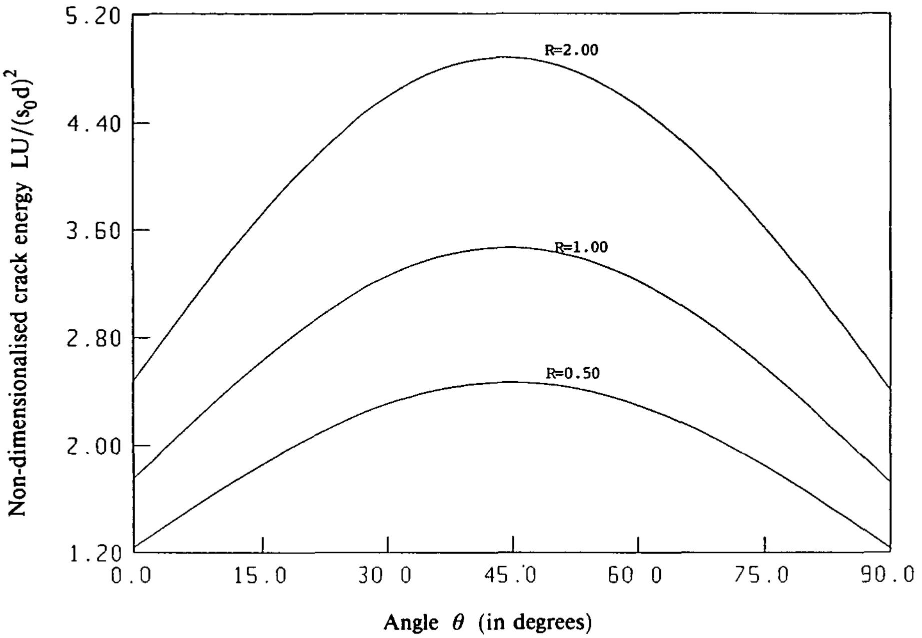

FIGURE 3. Variations of $LU/(s_0 d)^2$ with θ for $a/d = 0.25$, $R = 1.334$ (titanium) and $h/d = 2.50$, 3.50 and 4.50 as well as h/d tending to infinity.

2, for $\theta = \pi/3$ and $h/d = 3/2$, we examine the effect of changing a/d $(0 < a/d < 1)$ on the crack energy $LU/(T_0 d)^2$. There is a critical value of a/d for which the crack energy is minimum, and as a/d tends to 0 (from above) or 1 (from below) the crack energy tends to infinity.

TABLE 1. Non-dimensionalised crack energy for $a/d = 0.250$ and selected values of θ and h/d (plane problem involving a titanium slab).

$LU/(T_0 d)^2$				
h/d θ	2.500	3.500	4.500	$h/d \to \infty$
0°	1.656	1.316	1.243	1.157
30°	1.584	1.358	1.284	1.173
60°	1.527	1.398	1.333	1.206
90°	1.474	1.392	1.341	1.223

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TABLE 2. Non-dimensionalised crack energy for $\theta = \pi/3$, $h/d = 3/2$ and selected values of a/d (plane problem involving a titanium slab).

a/d	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900
$LU/(T_0 d)^2$	2.300	2.117	2.090	2.150	2.286	2.508	2.823	3.224	3.692

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References

[1] M. Abramowitz and I. A. Stegun (eds), *Handbook of Mathematical Functions* (Dover, New York, 1970).

[2] W. T. Ang, "A cracked anisotropic elastic slab", *Int. J. Engng. Sc.* **26** (1988) 277–283.

[3] W. T. Ang, "An arbitrarily-oriented plane crack in an anisotropic elastic slab", *Engng. Fract. Mech.* **32** (1989) 965–972.

[4] W. T. Ang, "Transient response of a crack in an anisotropic strip", *Acta Mechanica* **70** (1987) 97–109.

[5] D. L. Clements, "A crack in an anisotropic elastic slab", *Quart. Appl. Maths* **34** (1977) 437–443.

[6] D. L. Clements and T. R. Tauchert, "Deformation of an anisotropic slab containing a crack", *Acta Mechanica* **32** (1979) 55–61.

[7] R. S. Dhaliwal, "Two coplanar cracks in an infinitely long orthotropic slab", *Utilitas Mathematica* **4** (1973) 115–128.

[8] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, San Diego, 1980).

[9] D. L. Hill and D. L. Clements, "On deformations of cracked anisotropic slabs", *J. Elast.* **14** (1984) 403–413.

[10] N. I. Loakimidis, "A new singular integral equation for the classical crack problem in plane and antiplane elasticity", *Int. J. Fract.* **21** (1983) 115–122.

[11] A. C. Kaya and F. Erdogan, "On the solution of integral equations with strongly singular kernels", *Quart. Appl. Math.* **45** (1987) 105–122.

[12] Y. Konishi, "Two coplanar cracks in an infinite transversely isotropic medium", *Int. J. Engng. Sc.* **10** (1972) 917–923.

[13] S. Krenk, "The stress distribution in an infinite plate with co-linear cracks", *Int. J. Solids Structures* **11** (1975) 449–460.

[14] I. N. Sneddon, *Fourier Transforms* (McGraw-Hill, New York, 1951).

[15] A. J. M. Spencer, *Deformations of fibre-reinforced materials* (Oxford University Press, 1972).

[16] A. N. Stroh, "Dislocations and cracks in anisotropic elasticity", *Phil. Mag.* **3** (1958) 625–646.

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