Canad. Math. Bull. Vol. **58** (1), 2015 pp. 9–18
http://dx.doi.org/10.4153/CMB-2014-051-0

Irreducible Tuples Without the Boundary Property

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Abstract. We examine spectral behavior of irreducible tuples that do not admit the boundary property. In particular, we prove under some mild assumption that the spectral radius of such an m-tuple $(T_1, \dots, T_m)$ must be the operator norm of $T_1^* T_1 + \cdots + T_m^* T_m$. We use this simple observation to ensure the boundary property for an irreducible, essentially normal, joint q-isometry, provided it is not a joint isometry. We further exhibit a family of reproducing Hilbert $\mathbb{C}[z_1, \dots, z_m]$-modules (of which the Drury–Arveson Hilbert module is a prototype) with the property that any two nested unitarily equivalent submodules are indeed equal.

1 Preliminaries

For the set $\mathbb{N}$ of non-negative integers, let $\mathbb{N}^m$ denote the cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times). Let $p \equiv (p_1, \dots, p_m)$ and $n \equiv (n_1, \dots, n_m)$ be in $\mathbb{N}^m$. We write $|p| := \sum_{i=1}^m p_i$ and $p \leq n$ if $p_i \leq n_i$ for $i = 1, \dots, m$. For $n \in \mathbb{N}^m$, we let $n! := \prod_{i=1}^m n_i!$.

For a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. By a *commuting m-tuple* T on $\mathcal{H}$, we mean a tuple $(T_1, \dots, T_m)$ of commuting bounded linear operators $T_1, \dots, T_m$ on $\mathcal{H}$. For a commuting m-tuple T, we interpret T^* to be $(T_1^*, \dots, T_m^*)$, and T^p to be $T_1^{p_1} \cdots T_m^{p_m}$ for $p = (p_1, \dots, p_m) \in \mathbb{N}^m$.

For definitions and basic theory of various spectra including the Taylor spectrum, the reader is referred to [10]. For $T \in B(\mathcal{H})$, we reserve the symbols $\sigma(T)$, $\sigma_{ap}(T)$, and $\sigma_e(T)$ for the Taylor spectrum, approximate point spectrum, essential spectrum of T respectively. It is well known that projection property holds for Taylor and essential spectra.

Let q denote the Calkin map from $B(\mathcal{H})$ into the Calkin algebra $B(\mathcal{H})/K(\mathcal{H})$, where $K(\mathcal{H})$ denotes the ideal of compact operators on $\mathcal{H}$. The symbols $r(T)$ and $r_e(T)$ stand for the spectral radius of T and $q(T)$ respectively. Also, $\|T\|$ (resp. $\|T\|_e$) denotes the operator norm (resp. quotient norm) of T (resp. $q(T)$).

Given a commuting m-tuple $T = (T_1, \dots, T_m)$ on $\mathcal{H}$, we set

$$Q_T(X) := \sum_{i=1}^m T_i^* X T_i \big(X \in B(\mathcal{H}) \big), \tag{1.1}$$

and $Q_T^0(I) = I$. Note that for any integer $n \geq 0$, $Q_T^n(I) = \sum_{|p|=n} \frac{n!}{p!} T^{*p} T^p$.

Received by the editors April 29, 2014.
Published electronically November 3, 2014.
AMS subject classification: **47A13**, 46E22.
Keywords: boundary representations, subnormal, joint p-isometry.

Note further that

$$r_e(T) \leq r(T), \quad \|Q_T(I)\|_e \leq \|Q_T(I)\|.$$

Let T be a commuting m-tuple of bounded linear operators $T_1, \ldots, T_m$ on $\mathcal{H}$. By *the C^*-algebra generated by T* (in symbol, $C^*(T)$), we mean the norm closure of all non-commutative polynomials in the $(2m)$-variables $T_1, \ldots, T_m, T_1^*, \ldots, T_m^*$. By a *unital operator space*, we mean a pair $\mathcal{S} \subseteq \mathcal{B}$ consisting of a linear subspace S of a unital C^*-algebra $\mathcal{B}$, which contains the unit of $\mathcal{B}$ and generates $\mathcal{B}$ as a C^*-algebra, $\mathcal{B} = C^*(\mathcal{S})$. An *irreducible representation* of $\mathcal{B}$ is a unital homomorphism $r\colon \mathcal{B} \to B(\mathcal{H})$ such that $r(\mathcal{B})$ is an irreducible subalgebra of $B(\mathcal{H})$. An irreducible representation $r\colon \mathcal{B} \to B(\mathcal{H})$ is said to be a *boundary representation* for $\mathcal{S}$ if $r|_{\mathcal{S}}$ has a unique completely positive linear extension to $\mathcal{B}$, namely r itself. Recall that ϕ from $\mathcal{B}$ into another C^*-algebra $\mathcal{A}$ is *completely isometric* if $\phi_n\colon M_n(\mathcal{B}) \to M_n(\mathcal{A})$ given by $\phi_n([a_{i,j}]) := [\phi(a_{i,j})]$, $[a_{i,j}] \in M_n(\mathcal{B})$, is isometric for all $n \geq 1$.

We find it convenient here to invoke Arveson's Boundary Theorem for ready reference.

Theorem 1.1 ([1, Theorem 2.1.1]) *Let $\mathcal{S}$ be an irreducible set of operators on a Hilbert space $\mathcal{H}$ such that $C^*(\mathcal{S})$ contains the identity and $C^*(\mathcal{S})$ contains the algebra $K(\mathcal{H})$ of all compact operators on $\mathcal{H}$. Then the identity representation of $C^*(\mathcal{S})$ is a boundary representation for $\mathcal{S}$ if and only if the quotient map $q\colon B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H})$ is not completely isometric on the linear span of $\mathcal{S} \cup \mathcal{S}^*$.*

Definition 1.2 An irreducible commuting m-tuple T has the *boundary property* if the identity representation of the C^*-algebra $C^*(T)$ is a boundary representation for the finite-dimensional operator space spanned by $I, T_1, \ldots, T_m$.

Remark 1.3 Our use of the term boundary property (of tuples) differs from that of [14, Pg 218, Paragraph 1].

A consequence of Arveson's Boundary Theorem gives in particular a sufficient condition ensuring the boundary property for irreducible, essentially normal tuples [1, Theorem 2.2.1]. We state a rather special case of this result, which provides strong motivation for this note.

Theorem 1.4 ([1, Theorem 2.2.1]) *Let T be an irreducible essentially normal m-tuple consisting of bounded linear operators $T_1, \ldots, T_m$. If $r_e(T_i) < \|T_i\|$ for some $i = 1, \ldots, m$, then T has the boundary property.*

Remark 1.5 The above result is applicable to tuples that are not necessarily commuting.

Given a commuting m-tuple $T = (T_1, \ldots, T_m)$, it may happen that T has the boundary property, but the essential spectral radius and norm of T_i are equal for every i.

Example 1.6 Consider the positive definite kernel $\kappa_1(z, w) = \frac{1}{1-\langle z, w\rangle}$ defined on the unit ball $\mathbb{B}_m$ in $\mathbb{C}^m$. The reproducing kernel Hilbert space $\mathscr{H}(\kappa_1)$ is known as the

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Drury–Arveson space, and the commuting m-tuple M_z of multiplication operators $M_{z_1}, \dots, M_{z_m}$ on $\mathscr{H}(\kappa_1)$ is known as the *Drury–Arveson m-shift*. It is well known that M_z admits the boundary property [3, Lemma 7.13]. However, since $\sigma(M_z) = \overline{\mathbb{B}}_m$ and $\sigma_e(M_z) = \partial\mathbb{B}_m$, it follows from the projection property for Taylor and essential spectra that $\sigma(M_{z_i}) = \overline{\mathbb{B}}_1 = \sigma_e(M_{z_i})$, and hence $r(M_{z_i}) = r_e(M_{z_i}) = 1$ for any $i = 1, \dots, m$. Finally, since each M_{z_i} is hyponormal (that is, $M_{z_i}^* M_{z_i} - M_{z_i} M_{z_i}^*$ is positive), by general theory $\|M_{z_i}\| = r(M_{z_i})$, and hence we obtain

$$r_e(M_{z_i}) = \|M_{z_i}\| \ (i = 1, \dots, m).$$

It is evident that the spectral radius of a commuting m-tuple T can easily be determined in many situations; for instance, in case the sequence $\{Q_T^k(I)\}$ has polynomial growth. This and the preceding example suggest a possibility of an analog of Theorem 1.4 that takes into consideration the joint spectral behavior of T. Indeed, the main result of this note provides such an analog.

Theorem 1.7 *Let T be an irreducible, essentially normal m-tuple of commuting bounded linear operators $T_1, \dots, T_m$ on $\mathcal{H}$. If $r_e(T) < \sqrt{\|Q_T(I)\|}$, then T has the boundary property.*

We shall obtain this result from a slightly general fact (see Proposition 2.5). The proof of Theorem 1.7 is basically a combination of Arveson's ideas developed in [1,3] with a mild dose of multi-variable spectral theory [16], [10]. As far as the utility of Theorem 1.7 is concerned, we will see that the condition $r_e(T) < \sqrt{\|Q_T(I)\|}$ can be checked quite easily for a subclass of joint q-isometry tuples T that includes, in particular, the Drury–Arveson shift and the Dirichlet shift.

2 Proof of the Main Result

Recall that a commuting m-tuple $T = (T_1, \dots, T_m)$ on a Hilbert space $\mathcal{H}$ is said to be *jointly subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a commuting m-tuple $N = (N_1, \dots, N_m)$ of normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that

$$N_i h = T_i h \quad \text{for every } h \in \mathcal{H} \text{ and } 1 \le i \le m.$$

It is possible to give a spaceless or "C^*-algebra" definition of subnormality (see, for example, [5, Theorem 5.2]).

A commuting m-tuple is a *joint isometry* if $T_1^* T_1 + \dots + T_m^* T_m = I$. It is well known that every joint isometry is jointly subnormal [4].

In the proof of the main result, we need the following spectral radius formula for the Taylor spectrum ([9, 15]). Let T be a commuting m-tuple of bounded linear operators on a Hilbert space. Then

$$(2.1)\qquad r(T) := \sup_{(z_1,\dots,z_n)\in\sigma(T)} \left(|z_1|^2 + \dots + |z_n|^2\right)^{\frac{1}{2}} = \lim_{n\to\infty} \|Q_T^n(I)\|^{\frac{1}{2n}}.$$

Lemma 2.1 *Let T be a commuting m-tuple of bounded linear operators on a Hilbert space. Then $r(T)$ is at most $\sqrt{\|Q_T(I)\|}$.*

Proof Note that Q_T is a positive linear operator on $B(\mathcal{H})$. Now a simple inductive argument on k shows that

$$Q_T^k(I) \le \|Q_T(I)\|^k I \text{ for every integer } k \ge 1. \tag{2.2}$$

Thus $\|Q_T^k(I)\| \le \|Q_T(I)\|^k I$, and hence by (2.1), we get $r(T) \le \sqrt{\|Q_T(I)\|}$. ∎

We next compute spectral radii of subnormal tuples.

Lemma 2.2 *Let T be a jointly subnormal m-tuple on $\mathcal{H}$ with a minimal normal extension N on $\mathcal{K}$. Then*

$$r(T) = r(N) = \sqrt{\|Q_T(I)\|} = \sqrt{\|Q_N(I)\|}.$$

Proof The proof involves repeated applications of the spectral radius formula (2.1). We divide the proof into a number of small observations:

(a) $r(N) = \sqrt{\|Q_N(I)\|}$: Since $Q_N^k(I) = Q_N(I)^k$ for any positive integer k, by (2.1), $r(N) = \sqrt{\|Q_N(I)\|}$.

(b) $r(N) \le r(T)$: By the spectral inclusion for jointly subnormal tuples [16], $\sigma(N) \subseteq \sigma(T)$. It follows that $r(N) \le r(T)$.

(c) $r(T) \le \sqrt{\|Q_N(I)\|}$: Let $P_{\mathcal{H}}$ denote the orthogonal projection of $\mathcal{K}$ on $\mathcal{H}$. Then

$$Q_T^k(I)h = P_{\mathcal{H}} Q_N^k(I)h \ (h \in \mathcal{H})$$

(see, for instance, [6, Proposition 3.4]). It follows that

$$\|Q_T^k(I)\| \le \|Q_N^k(I)\| = \|Q_N(I)\|^k \text{ for every positive integer } k.$$

Another application of (2.1) yields $r(T) \le \sqrt{\|Q_N(I)\|}$.

(d) $\sqrt{\|Q_T(I)\|} \le r(T)$: It is observed in the proof of [7, Proposition 4.9] that $r(T) \ge \sqrt{\|Q_T(I)\|}$, provided T satisfies

$$\langle Q_T^k(I)h,\, h\rangle \le \langle Q_T^{k-1}(I)h,\, h\rangle^{\frac{1}{2}} \langle Q_T^{k+1}(I)h,\, h\rangle^{\frac{1}{2}} \tag{2.3}$$

for all $h \in \mathcal{H}$ and for all integers $k \ge 1$. However, every jointly subnormal m-tuple T satisfies (2.3).

By (a)–(c), we obtain $r(T) = r(N) = \sqrt{\|Q_N(I)\|}$. On the other hand, (d) and Lemma 2.1 yield $r(T) = \sqrt{\|Q_T(I)\|}$. ∎

Let $T = (T_1, \dots, T_m)$ be a commuting m-tuple on $\mathcal{H}$ and let

$$q \colon B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H})$$

be the Calkin map. We say that the m-tuple $T = (T_1, \dots, T_m)$ is *essentially normal* (resp. *essentially joint isometry*, resp. *essentially subnormal*) if

$$q(T) := \big(q(T_1), \dots, q(T_m)\big)$$

is normal (resp. joint isometry, resp. jointly subnormal).

Remark 2.3 Clearly an essentially normal m-tuple is essentially subnormal. It follows from [4, Proposition 2] that an essentially joint isometry is also essentially subnormal.

Lemma 2.4 *Let T be a commuting m-tuple on $\mathcal{H}$. If T is essentially subnormal, then $r_e(T) = \sqrt{\|Q_T(I)\|_e}$.*

Proof Apply Lemma 2.2 to the m-tuple $q(T)$, where q is the Calkin map. ■

As recorded earlier, the main result of this note may be considered as a joint spectral analog of [1, Theorem 2.2.1].

Proposition 2.5 *Let T be an irreducible commuting m-tuple of bounded linear operators $T_1, \dots, T_m$ on $\mathcal{H}$. Suppose that T is either essentially normal or an essentially joint isometry. If T does not admit the boundary property, then*

$$r(T) = r_e(T) = \sqrt{\|Q_T(I)\|} = \sqrt{\|Q_T(I)\|_e}.$$

Proof The irreducible C^*-algebra $C^*(T)$ contains either the compact operator

$$T_i^* T_i - T_i T_i^* \quad \text{or} \quad I - \sum_{i=1}^{n} T_i^* T_i.$$

By [2, Corollary 2], $C^*(T)$ contains all the compact operators on $\mathcal{H}$. Let $\mathcal{S} := \operatorname{span}\{I, T_1, \dots, T_m\}$ and let $\mathcal{L}$ denote the linear span of $\mathcal{S} \cup \mathcal{S}^*$. In view of Arveson's Boundary Theorem, it suffices to check that if the quotient map $q\colon B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H})$ is completely isometric on $\mathcal{L}$, then $r(T) = r_e(T) = \sqrt{\|Q_T(I)\|} = \sqrt{\|Q_T(I)\|_e}$.

Assume that q is completely isometric on $\mathcal{L}$. Consider the $m \times m$ matrix A in $M_m(\mathcal{S})$ given by

$$A := \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ T_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_m & 0 & \cdots & 0 \end{pmatrix}.$$

Note that $A^*A = \sum_{i=1}^{m} T_i^* T_i = Q_T(I)$. Since q is completely isometric, we have $\|A\| = \|A\|_e$. This gives $\|Q_T(I)\| = \|A^*A\| = \|A^*A\|_e = \|Q_T(I)\|_e$. By equation (2.2), for every positive integer k,

$$\|Q_T^k(I)\| \leq \|Q_T(I)\|^k I = \|Q_T(I)\|_e^k I.$$

An application of the spectral radius formula gives $r(T) \leq \sqrt{\|Q_T(I)\|_e}$. Since T is essentially subnormal, by Lemma 2.4, $r_e(T) = \sqrt{\|Q_T(I)\|_e}$. Thus we have $r(T) \leq r_e(T)$. Since the essential spectrum is a subset of the Taylor spectrum, we have $r(T) = r_e(T)$. Finally, since $\|Q_T(I)\| = \|Q_T(I)\|_e$, we obtain the desired conclusion. ■

Remark 2.6 If T is an essentially joint isometry, then $\|Q_T(I)\|_e = 1$. It follows that $r(T) = r_e(T) = 1$, and $\sum_{i=1}^{m} T_i^* T_i \leq I$.

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Let $\mathcal{H}$ be a Hilbert space and let T be a commuting m-tuple of bounded linear operators $T_1, \dots, T_d$. Then $\mathcal{H}$ can be considered as a *Hilbert module* over the polynomial ring $\mathbb{C}[z_1, \dots, z_d]$, where the module action is given by

$$(p, h) \in \mathbb{C}[z_1, \dots, z_d] \times \mathcal{H} \longrightarrow p(T)h \in \mathcal{H}.$$

In the main result, we used the spectral theory to study boundary representations. We now reverse this procedure and use boundary representations to get spectral information (*cf.* [11, Theorem 4.9(a)]).

Corollary 2.7 *Let Ω be a bounded domain in $\mathbb{C}^m$. Consider the Hilbert module $\mathscr{H}(\kappa)$ associated with the reproducing kernel $\kappa(z, w)$ $(z, w \in \Omega)$ and the multiplication tuple M_z on $\mathscr{H}(\kappa)$. Suppose that M_z is an essentially normal jointly subnormal m-tuple such that $\sigma(M_z) = \overline{\Omega}$. Then*

$$r(M_z) = r_e(M_z) = \sqrt{\|Q_{M_z}(I)\|} = \sqrt{\|Q_{M_z}(I)\|_e}.$$

Proof By [14, Theorem 3.2], M_z does not have the boundary property. The desired conclusion follows from the preceding result. ∎

An m-variable weighted shift $T = (T_1, \dots, T_m)$ with respect to an orthonormal basis $\{e_n\}_{n\in\mathbb{N}^m}$ of a Hilbert space $\mathcal{H}$ is defined by

$$T_i e_n := w_n^{(i)} e_{n+\epsilon_i} \ (1 \leq i \leq m),$$

where ϵ_i is the m-tuple with 1 in the i-th place and zeros elsewhere.

Remark 2.8 Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a bounded sequence of positive numbers. Consider the m-variable weighted shift $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ with the weight multi-sequence

$$w_n^{(i)} = \delta_{|n|}\sqrt{\frac{n_i + 1}{|n| + m}} \ (n \in \mathbb{N}^m, 1 \leq i \leq m).$$

If $\lim_{k\to\infty} \delta_k^2 - \delta_{k-1}^2 = 0$ and $\limsup_{k\to\infty} \delta_k < \sup_k \delta_k$, then T admits the boundary property. This is precisely [14, Proposition 4.9]. Alternatively, it may be obtained from [8, Theorem 3.4(5)] and the main result.

3 Boundary Property for Joint q-isometries

Definition 3.1 Let Q_T be as given in (1.1). For an integer $q \geq 1$, let

$$B_q(Q_T) := \sum_{s=0}^{q} (-1)^s \binom{q}{s} Q_T^s(I).$$

If $B_q(Q_T) = 0$, then T is a *joint q-isometry*.

A joint 1-isometry is nothing but a joint isometry. The Drury–Arveson m-shift is a joint m-isometry [12], but it is not a joint isometry unless $m = 1$.

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Proposition 3.2 *Let T be an irreducible essentially normal commuting m-tuple of bounded linear operators $T_1, \dots, T_m$ on $\mathcal{H}$. If T is a joint q-isometry that is not a joint isometry, then T has the boundary property.*

Proof Suppose T is a joint q-isometry that is not a joint isometry. By [7, Lemma 4.3], a joint q-isometry T is a joint isometry if and only if $\sum_{i=1}^{m} T_i^* T_i \leq I$. It follows that $\|Q_T(I)\| > 1$. On the other hand, the spectral radius of a joint p-isometry is always 1, as observed in [12, Proposition 3.1]. Hence, by Proposition 2.5, T admits the boundary property. ■

We now illustrate the usefulness of Proposition 3.2 by exhibiting a concrete family of multiplication tuples M_z acting on reproducing kernel Hilbert spaces. We first recall the definition of complete NP kernels.

A reproducing kernel κ on the unit ball $\mathbb{B}_m$ is called a *complete Nevanlinna–Pick (NP) kernel* if $\kappa(\cdot, 0) = 1$ and if there exists a sequence $\{a_n\}$ of analytic functions a_n on $\mathbb{B}_m$ such that

$$1 - \frac{1}{\kappa(z, w)} = \sum_{n \geq 0} a_n(z)\overline{a_n(w)} \text{ for all } z, w \in \mathbb{B}_m.$$

The Drury–Arveson kernel $\frac{1}{1-\langle z, w\rangle}$ and the Dirichlet kernel $-\frac{\log(1-\langle z, w\rangle)}{\langle z, w\rangle}$ are two important examples of complete NP kernels.

In the application of Proposition 3.2, we need a suitable modification of [14, Theorem 5.1] (see also [3, Lemma 7.13]).

Lemma 3.3 *Let $\mathcal{H}(\kappa)$ denote a reproducing kernel Hilbert space with complete NP kernel $\kappa(z, w)$ on the open unit ball $\mathbb{B}_m$ in $\mathbb{C}^m$. Assume that there is a set $\mathcal{P} \subseteq \mathcal{H}(\kappa) \cap C(\overline{\mathbb{B}})$ that is dense in $\mathcal{H}(\kappa)$ and satisfies*

$$\text{(3.1)} \qquad \lim_{\lambda \to z} \frac{\|p\kappa(\cdot, \lambda)\|}{\|\kappa(\cdot, \lambda)\|} = |p(z)| \textit{ for all } p \in \mathcal{P} \textit{ and for } [\sigma] \textit{a.e. } z \in \partial\mathbb{B}_m,$$

where σ denotes the normalized surface area measure supported on the unit sphere $\partial\mathbb{B}_m$. Let M_z denote the multiplication m-tuple on $\mathcal{H}(\kappa)$ and let $\mathcal{M}$ be an invariant subspace of M_z. Then the m-tuple $S := M_z|_{\mathcal{M}}$ is irreducible.

Proof We imitate the argument of [14, Theorem 5.1]. Suppose that there exists an orthogonal projection $P_{\mathcal{N}}$ from $\mathcal{M}$ onto a proper subspace $\mathcal{N}$ of $\mathcal{M}$ such that $P_{\mathcal{N}} S_i = S_i P_{\mathcal{N}}$. Note that $\mathcal{N}$ and its orthogonal complement $\mathcal{N}'$ in $\mathcal{M}$ are z-invariant subspaces of $\mathcal{H}(\kappa)$. It follows that $\|P_{\mathcal{M}}\kappa(\cdot, \lambda)\|^2 = \|P_{\mathcal{N}}\kappa(\cdot, \lambda)\|^2 + \|P_{\mathcal{N}'}\kappa(\cdot, \lambda)\|^2$ for every $\lambda \in \mathbb{B}_m$. On the other hand, by [13, Theorem 1.2], for $[\sigma]$ a.e. $z \in \partial\mathbb{B}_m$,

$$\lim_{\lambda \to z} \frac{\|P_{\mathcal{M}}\kappa(\cdot, \lambda)\|^2}{\|\kappa(\cdot, \lambda)\|^2} = \lim_{\lambda \to z} \frac{\|P_{\mathcal{N}}\kappa(\cdot, \lambda)\|^2}{\|\kappa(\cdot, \lambda)\|^2} = \lim_{\lambda \to z} \frac{\|P_{\mathcal{N}'}\kappa(\cdot, \lambda)\|^2}{\|\kappa(\cdot, \lambda)\|^2} = 1,$$

(see the discussion prior to [13, Theorem 1.2]). This certainly yields a contradiction, and hence S is irreducible. ■

Lemma 3.4 *If T is an essentially normal joint q-isometry, then T is an essentially joint isometry.*

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Proof Let q denote the Calkin map. Note that $q(T)$ is a normal joint q-isometry, and hence $q(T)$ is a joint isometry. ∎

A special case of the following result, in which $\mathscr{H}(\kappa)$ is the Drury–Arveson space, was first obtained in [14, Theorem 5.1].

Proposition 3.5 *Let m be a positive integer bigger than 1 and let $\{a_k\}_{k\in\mathbb{N}}$ be a non-increasing sequence of positive numbers such that $\binom{m+k-1}{k}/a_k$ is a non-constant polynomial in k of degree at most m. Let κ be a complete NP kernel given by*

$$\kappa(z,w) := \sum_{k=0}^{\infty} a_k \langle z,\, w\rangle^k \ (z, w \in \mathbb{B})$$

and let $\mathscr{H}(\kappa)$ denote the reproducing kernel Hilbert space associated with the kernel κ. Then for every invariant subspace $\mathcal{M}$ of the multiplication m-tuple M_z on $\mathcal{H}(\kappa)$, the m-tuple $S := M_z|_{\mathcal{M}}$ has the boundary property.

Remark 3.6 We note that in case of the Drury–Arveson kernel $a_k = 1$ for all $k \geq 1$ and that of Dirichlet kernel $a_k = \frac{1}{k+1}$ for all $k \geq 1$. Thus the conclusion of Proposition 3.5 holds true for the Drury–Arveson m-shift and the Dirichlet m-shift. On the other hand, in the case of a Szegö kernel, $a_k = \binom{m+k-1}{k}$; as expected, Proposition 3.5 is not applicable.

Proof Let $\kappa(z, w)$ be a reproducing kernel of the form

$$\kappa(z,w) := \sum_{k=0}^{\infty} a_k \langle z,\, w\rangle^k,$$

where a_k are positive numbers such that $\binom{m+k-1}{k}/a_k$ is a non-constant polynomial in k of degree at most m. As noted in [13, Section 4], κ is a complete NP kernel satisfying (3.1) of Corollary 3.3 provided $\sum_{k=0}^{\infty} a_k = \infty$ and $\frac{a_{k+1}}{a_k} \to 1$. By hypothesis, we have $a_k = \frac{(k+1)(k+2)\cdots(k+m-1)}{p(k)}$ for some polynomial p of degree d, where $1 \leq d \leq m$. It follows that $a_k \approx k^{m-d-1}$, and hence $\sum_{k=0}^{\infty} a_k = \infty$.

Let M_z denote the multiplication m-tuple acting on the reproducing kernel Hilbert space $\mathscr{H}(\kappa)$ associated with the kernel κ. It is easy to see that M_z is an m-variable weighted shift with weight multi-sequence

$$\left\{ \sqrt{\frac{a_{|\alpha|}}{a_{|\alpha|+1}}} \sqrt{\frac{\alpha_i + 1}{|\alpha| + 1}} : 1 \leq i \leq m, n \in \mathbb{N}^m \right\}.$$

An application of [7, Lemma 3.1] yields that M_z is a joint q-isometry if and only if the one-variable weighted shift with weight sequence $\{\sqrt{a_k/a_{k+1}}\sqrt{k+m/k+1}\}$ is a q-isometry. It is well known that a one-variable weighted shift with weight-sequence $\{\delta_k : k \in \mathbb{N}\}$ is a q-isometry if and only if $\delta_0^2\delta_1^2\cdots\delta_{k-1}^2$ is a polynomial in k of degree less than or equal to $q-1$. It follows that M_z is a joint q-isometry if and only if $\frac{a_0}{a_k}\binom{m+k-1}{k}$ is a polynomial in k of degree less than or equal to $q-1$. By assumption, M_z is a $(d+1)$-isometry. By [8, Corollary 5.6], M_z is essentially normal. Hence by the preceding lemma, M_z is an essentially joint isometry. In particular, $\frac{a_{k+1}}{a_k} \to 1$. Thus all hypotheses of Lemma 3.3 are satisfied, and hence we conclude that S is irreducible.

If M_z is a joint q-isometry then so is S. Also, if M_z is an essentially joint isometry then so is S. By Proposition 3.2, S admits the boundary property provided it is not a joint isometry. To complete the proof, it suffices to check that M_z is not a joint isometry on any non-zero invariant subspace. Suppose that there exists $f(z) = \sum_{\alpha \geq 0} b_\alpha \frac{z^\alpha}{\|z^\alpha\|}$ in $\mathscr{H}$ such that $\sum_{i=1}^m \|M_{z_i} f\|^2 = \|f\|^2$. Since $\{z^n\}$ is orthogonal,

$$\sum_{i=1}^{m} \sum_{\alpha \geq 0} |b_\alpha|^2 \Big(\frac{a_{|\alpha|}}{a_{|\alpha|+1}} \frac{\alpha_i + 1}{|\alpha| + 1} \Big) = \sum_{\alpha \geq 0} |b_\alpha|^2,$$

hence

$$\sum_{\alpha \geq 0} b_\alpha^2 \Big(\frac{a_{|\alpha|}}{a_{|\alpha|+1}} \frac{|\alpha| + m}{|\alpha| + 1} - 1 \Big) = 0.$$

Since $\frac{a_k}{a_{k+1}} \geq 1$ and $m \geq 2$, we have $b_\alpha = 0$ for all α, and consequently $f = 0$. ■

Remark 3.7 Note that $\kappa(z, w)$ is a complete NP kernel provided that $\frac{a_{k+1}}{a_k} \uparrow 1$ and $\binom{m+k-1}{k}/a_k$ is a non-constant polynomial in k of degree at most m (the reader is referred to [13]). The conclusion of the proposition holds even for $m = 1$ in this case.

Let κ_1 and κ_2 denote the Drury–Arveson kernel and Dirichlet kernel respectively in dimension $m \geq 2$. Note that Theorem 3.5 is applicable to the kernel $\kappa_1 + \rho\kappa_2$ for every $\rho \in \mathbb{N}$ such that $\rho \leq m - 2$. In particular, the Hilbert reproducing $\mathbb{C}[z_1, \dots, z_m]$-module $\mathscr{H}(\kappa_1 + \kappa_2)$ associated with the kernel $\kappa_1 + \kappa_2$ has nested rigidity in dimension 3 (see Corollary 3.8).

We conclude the note with an application to function theory, which may be obtained by combining Proposition 3.5 with [14, Corollary 2.5].

Corollary 3.8 *Under the hypotheses of Proposition 3.5, the reproducing Hilbert $\mathbb{C}[z_1, \dots, z_m]$-module $\mathscr{H}(\kappa)$ has nested rigidity: if for submodules $\mathcal{M}, \mathcal{N}$ of $\mathscr{H}(\kappa)$ such that $\mathcal{M} \subseteq \mathcal{N}$, $M_z|_{\mathcal{M}}$ is unitarily equivalent to $M_z|_{\mathcal{N}}$, then $\mathcal{M} = \mathcal{N}$.*

Acknowledgments The author thanks Sudipta Dutta and Akash Anand for some helpful discussions.

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