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THE SPECTRAL RADIUS OF A NON-NEGATIVE MATRIX

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ABSTRACT. A max min formula for the spectral radius of a non-negative matrix is derived from a characterization of non-singular M-matrices in terms of diagonal stability.

The matrices in this note are real and square. A matrix A is said to be in a class Z if it is of the form $A = \lambda I\text{-}B$, where B is non-negative ($B_{ij} \geq 0$). A matrix A in Z is called a non-singular M-matrix if $\lambda > \rho(B)$, the spectral radius of B. A matrix A is said to be diagonally stable if there exists a positive diagonal matrix D ($D_{ii} > 0$) such that $AD + DA^t$ is positive definite. Diagonally stable matrices arise in the study of predator-prey systems, e.g. Krikorian [3], and of composite dynamical systems, e.g. Araki [1]. An excellent survey of the theory and many applications of M-matrices is given in Plemmons [4].

In this note we obtain an expression for the spectral radius, $\rho(B)$, of a non-negative matrix B. The results follow from two characterizations of diagonally stable matrices. The first is general. The second is restricted to the class Z.

LEMMA 1. (*Barker, Berman and Plemmons* [2]) *A is diagonally stable if and only if for every non-zero positive semidefinite matrix S, there exists an index i such that $(SA)_{ii}$ is positive.*

LEMMA 2. (*Tartar* [5], *Araki* [1] *If $A \varepsilon Z$, then diagonal stability is a necessary and sufficient condition for A to be a non-singular M-matrix.*

Combining the two results yields the desired expression for $\rho(B)$.

THEOREM. *Let B be a non-negative matrix.*
Then

$$\rho(B) = \max \min \frac{(SB)ii}{S_{ii}}$$

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where the minimum is computed over all indices i such that $S_{ii} > 0$ *and the maximum is taken over all non-zero positive semidefinite matrices S.*

Proof. For every $\varepsilon > 0$, $(\rho(B)+\varepsilon)\ I\text{-}B$ is a non-singular M-matrix. Thus $\rho(B) \geq \max\min(SB_{ii}/S_{ii})$. On the other hand $\rho(B) \leq \max\min(SB_{ii}/S_{ii})$ since $\rho(B)I\text{-}B$ is not a non-singular M-matrix.

Notice that choosing $S = (S_{ij}) = (1)$, yields the well known lower bound

$$\rho(B) \geq \min_j \sum_i b_{ij}.$$

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