MULTIVARIATE SEMI-MARKOV MATRICES

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Abstract

Finite matrices with entries $p_{ij}\ F_{ij}(x_1, \cdots, x_k)$, where $\{p_{ij}\}$ is stochastic and $F_{ij}(\cdot)$ is a k-variate probability distribution are discussed. It is shown that the matrix of k-variate Laplace-Stieltjes transforms of the $p_{ij}\ F_{ij}(x_1, \cdots, x_k)$ has a Perron-Frobenius eigenvalue which is a convex function in k variables in a suitably defined region. The values of the partial derivatives near the origin of this maximal eigenvalue are exhibited. They are quantities of interest in a variety of applications in Probability theory.

1. Introduction

A natural combination of the theories of stochastic matrices and of distribution functions, which arises in a large number of problems of analytic Probability theory, is the theory of *semi-Markov matrices*.

In this paper we wish to consider properties of semi-Markov matrices involving multivariate distributions.

DEFINITION. *k-variate semi-Markov matrix*. Let $Q(\boldsymbol{x})$ be an $m \times m$ matrix, whose entries are real valued functions defined on R^k such that each entry $Q_{ij}(\boldsymbol{x})$ may be written as:

$$Q_{ij}(\boldsymbol{x}) = p_{ij}F_{ij}(x_1, \cdots, x_k), \tag{1}$$

where $F_{ij}(x_1, \cdots, x_k)$ is a k-variate probability distribution and where $p_{ij} \geqq 0$, $\sum_{j=1}^{m} p_{ij} = 1$, $i = 1, \cdots, m$, then $Q(\boldsymbol{x})$ is a k-variate semi-Markov matrix.

We note that if $p_{ij} = 0$, the probability distribution $F_{ij}(\cdot)$ may be arbitrarily chosen.

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DEFINITION. *Irreducible semi-Markov matrix.* The semi-Markov matrix $(Q(\boldsymbol{x})$ is called irreducible if and only if the stochastic matrix $P = \{p_{ij}\}$ is irreducible.

DEFINITION. *Nondegenerate k-variate semi-Markov matrix.* The semi-Markov matrix $Q(\boldsymbol{x})$ is nondegenerate k-variate if and only if for every $v = 1, \cdots, k$ there exists a pair of indices (i,j) such that $p_{ij} > 0$ and the corresponding distribution $F_{ij}(x_1, \cdots, x_k)$ has a marginal distribution $F_{ij}(+\infty, \cdots, x_v, \cdots, +\infty)$ which is not degenerate at zero.

The nondegeneracy condition eliminates the case where one or more of the k-variables $x_1, \cdots, x_k$ are actually redundant.

Henceforth we assume that $Q(\boldsymbol{x})$ is an irreducible and nondegenerate k-variate semi-Markov matrix.

We now consider the k-dimensional Lebesgue-Stieltjes integrals:

$$(2)\quad q_{ij}(\xi_1, \cdots, \xi_k) = q_{ij}(\xi) = \int_{R^k} \exp\left[-\sum_{v=1}^{k} \xi_v x_v\right] d_{x_1, \cdots, x_k} Q_{ij}(x_1, \cdots, x_k),$$

which we refer to as the Laplace-Stieltjes transforms of the entries $Q_{ij}(x_1, \cdots, x_k)$ of $Q(\boldsymbol{x})$.

The functions $q_{ij}(\xi_1, \cdots, \xi_k)$ are obviously defined for $Re\, \xi_1 = 0, \cdots, Re\, \xi_k = 0$, but they may not be defined anywhere else. We are mainly interested in the cases where the domain of definition of the $q_{ij}(\xi)$ is larger, as is the case in most applications.

We distinguish the *unilateral* and the *bilateral* cases.

In the *unilateral* case, we assume that all $F_{ij}(x_1, \cdots, x_k)$ corresponding to indices i, j such that $p_{ij} > 0$, concentrate all their mass on the positive orthant $x_1 \geqq 0, \cdots, x_k \geqq 0$. In this case all integrals in (2) exist for all ξ with $Re\, \xi_1 \geqq 0, \cdots, Re\, \xi_k \geqq 0$. Moreover all the functions $q_{ij}(\xi_1, \cdots, \xi_k)$ are jointly analytic in $Re\, \xi_1 > 0, \cdots, Re\, \xi_k > 0$ and any function obtained by setting some but not all of its variables equal to zero is analytic inside the corresponding part of the boundary of the set $Re\, \xi_1 > 0, \cdots, Re\, \xi_k > 0$. The latter statement is obvious if we realize that setting one or more, but not all of the ξ-variables equal to zero, corresponds to taking the Laplace-Stieltjes transforms of suitable 'marginal' distributions of $Q_{ij}(x_1, \cdots, x_k)$.

The *bilateral* case encompasses all distributions not in the unilateral case.

In our discussion of the bilateral case we shall assume that there exist $2k$ real numbers ξ_i' and ξ_i'', $i = 1, \cdots, k$ such that:

$$(3)\qquad -\infty \leqq \xi_i'' < 0 < \xi_i' \leqq +\infty, \qquad i = 1, \cdots, k$$

and such that in the 'box':

$$(4)\qquad \xi_i'' \leqq \xi_i \leqq \xi_i', \qquad i = 1, \cdots, k,$$

all functions $q_{ij}(\xi_1, \cdots, \xi_k)$ are analytic in $\xi_1, \cdots, \xi_k$.

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In order to discuss both cases simultaneously, we shall refer to the *domain D* in the unilateral case as the open positive orthant $\xi_1 > 0, \cdots, \xi_k > 0$ and in the bilateral case as the box $\xi_1'' \leqq \xi_1 \leqq \xi_1', \cdots, \xi_k'' \leqq \xi_k \leqq \xi_k'$.

2. The Perron-Frobenius eigenvalue of $q(\xi)$

The matrix $q(\xi)$ with entries $q_{ij}(\xi_1, \cdots, \xi_k)$ is an irreducible, nonnegative matrix for every real point ξ in the domain D or on its boundary. It follows from the classical theory of nonnegative matrices, [1, 4], that $q(\xi)$ has an eigenvalue of maximum modulus, which is real, positive and of geometric and algebraic multiplicity one. Denoting this, the Perron-Frobenius eigenvalue, by $\rho(\xi) = \rho(\xi_1, \cdots, \xi_k)$, we set out to discuss the properties of $\rho(\xi)$ as a function of ξ over the domain D. In the simpler case where $k = 1$, this was done by H. D, Miller [3].

We shall assume that the reader is familiar with the basic properties of nonnegative matrices as discussed in the references listed above.

LEMMA 1. *All functions $q_{ij}(\xi)$, $i, j = 1, \cdots, m$ are convex functions over the domain D and its boundary, i.e. for ξ and η in the closure $\bar{D}$, we have:*

$$q_{ij}[\alpha\xi + (1-\alpha)\eta] \leqq \alpha q_{ij}(\xi) + (1-\alpha) q_{ij}(\eta) \tag{5}$$

for all $0 \leqq \alpha \leqq 1$, and all $i, j = 1, \cdots, m$.

Moreover if $\xi \neq \eta$ and $0 < \alpha < 1$, strict inequality must hold in (5) for at least one pair (i, j).

PROOF. Since for all real k-tuples $(x_1, \cdots, x_k)$, the function $\exp\left[-\sum_{v=1}^k \xi_v x_v\right]$ is strictly convex over the domain $\bar{D}$, the inequality (5) follows immediately from the definition of $q_{ij}(\xi)$.

To prove the next statement we must clearly consider only those pairs (i, j) for which $p_{ij} > 0$. The corresponding Laplace-Stieltjes transform $q_{ij}(\xi_1, \cdots, \xi_k)$ is strictly convex with respect to all the variables which explicitly occur in it. The variables ξ_r which do not explicitly occur in $q_{ij}(\xi_1, \cdots, \xi_k)$ correspond to variables x_r in $F_{ij}(x_1, \cdots, x_k)$ with respect to which the marginal distributions are degenerate at zero.

The nondegeneracy assumption may be restated as saying that every variable ξ_v, $v = 1, \cdots, k$ must occur explicitly in at least one of the functions $q_{ij}(\xi_1, \cdots, \xi_k)$.

Let now $\xi \neq \eta$. In particular $\xi_v \neq \eta_v$. Let (i, j) be a pair such that $q_{ij}(\xi_1, \cdots, \xi_k)$ contains ξ_v explicitly, then for $0 < \alpha < 1$

$$q_{ij}[(1-\alpha)\eta + \alpha\xi] < \alpha q_{ij}(\xi) + (1-\alpha) q_{ij}(\eta),$$

since $q_{ij}(\cdot)$ is jointly strictly convex in all variables upon which it explicitly depends.

DEFINITION. *Superconvex Matrices*. Let f be a positive function defined on the convex set $\Gamma \in K$. Then f is *superconvex* if $\log f$ is a convex function on Γ. Clearly, f is superconvex if and only if for each $\xi, \eta \in \Gamma$,

$$f(\alpha\xi+\beta\eta) \leqq [f(\xi)]^{\alpha}[f(\eta)]^{\beta}; \quad \alpha+\beta = 1, \ \alpha \geqq 0, \ \beta \geqq 0.$$

A matrix $A(\xi) = [A_{ij}(\xi)]$ is *superconvex* if for each (i,j), $A_{ij}(\xi)$ is superconvex on Γ.

The proofs of the following lemmas can be found in reference (2) *or* (3).

Lemma 2. *If f is superconvex on Γ, then it is convex there.*

Lemma 3. *Let $\gamma(\xi)$ be any non constant positive linear function on Γ. Then $\gamma(\xi)$ is not superconvex.*

Following Kingman (2) we let C denote the class of all superconvex functions along with the function which is identically zero on Γ.

Lemma 4. *C is closed under addition, multiplication and raising to any positive power. If for each n, $f_n \in C$, so does* $\limsup_{n\to\infty} f_n$.

Lemma 5. *Let $A(\xi)$ be a superconvex matrix on Γ and let $\rho(\xi)$ denote its largest eigenvalue. Then $\rho(\xi) \in C$.*

Lemma 6. *Let $A(\xi)$ be a superconvex matrix on Γ and suppose $\rho(\xi)$ is not a constant function. Then $\rho(\xi)$ is strictly convex on Γ.*

Proof. By lemma's 2 and 5, $\rho(\xi)$ is convex on Γ, Suppose now that $\rho(\xi)$ is in fact linear. Then by lemma 3, since ρ is not constant, $\rho(\xi)$ is not superconvex. This contradiction implies that $\rho(\xi)$ is strictly convex on Γ.

Theorem 1. *Let $\xi = \sigma + i\,\tau$ where $\xi \in D$.*

(a) *The Perron Frobenius eigenvalue, $\rho(\xi)$ is analytic at $\xi = \sigma$ in the domain D.*

(b) *$\rho(\sigma)$ is a strictly convex function of σ in $\bar{D}$, suitably continuous on the boundary.*

Proof. (a) As in the univariate case, Miller [5], for each real σ, $\rho(\sigma)$ is a simple root of the determinantal equation $|zI-q(\sigma)| = 0$. Since $|zI-q(\sigma)|$ is an analytic function of the $k+1$ complex variables, $z, \sigma_1, \cdots, \sigma_k$, the result follows from the implicit functions theorem for analytic functions.

(b) We need only show that $q_{ij}(\sigma)$ is a superconvex function for each (i,j). This follows at once since

$$\int_D e^{(\alpha\sigma+\beta\sigma')\cdot X} dQ(X) \leqq \left[\int_D e^{\sigma\cdot X} dQ(X)\right]^{\alpha} \left[\int_D e^{-\sigma'\cdot X} dQ(X)\right]^{\beta}$$

for $\xi = \sigma + i\tau$, $\xi' = \sigma' + i\tau'$, $\xi, \xi' \in D$, and $\sigma \cdot X = \sigma_1 X_1 + \cdots + \sigma_k X_k$. This is just Hölder's inequality for a Banach space with a finite measure. Consequently $q(\sigma)$

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is a superconvex matrix and so $\rho(\sigma)$ is convex. By lemma 1 $\rho(\sigma)$ is not constant and so by lemma 6 $\rho(\sigma)$ is strictly convex on D.

By suitably continuous on the boundary $\bar{D}$ we mean that if $\xi^* = \sigma^* + i\tau^* \in \bar{D}$ and if $\xi_n \to \xi^*$ where $\xi_n \in D$ then $\rho(\sigma_n) \to \rho(\sigma^*)$. Hence we have $\rho(\sigma)$ is strictly convex on $\bar{D}$.

The entries of $q(\xi)$ are all suitably continuous on the boundary and hence $\rho(\xi)$ is suitably continuous on the boundary, since convergence of a sequence of positive matrices entails convergence of their Perron-Frobenius eigenvalues to that of the limit matrix.

The theorem 1 implies in particular that $\rho(\xi)$ is a continuously differentiable function of ξ in D. In the unilateral case one may easily verify that $\rho(\xi)$ is also suitably differentiable at all boundary points of the positive orthant D, with the possible exception of the origin.

In many applications, see Neuts [6], the quantities

(11) $$M_j = \left[\frac{\partial}{\partial \xi_j} \rho(\xi_1, \cdots, \xi_k)\right]_{\xi = \mathbf{0}}$$

play a fundamental role. In the unilateral case, the derivatives at $\mathbf{0}$ are to be understood in the same 'suitable' sense as in theorem 1.

We denote by $\alpha_i^{(\nu)}$, the mean with respect to the variable x_ν of the probability distribution $H_i(x_1, \cdots, x_k)$ defined by:

(12) $$H_i(x_1, \cdots, x_k) = \sum_{j=1}^{m} p_{ij} F_{ij}(x_1, \cdots, x_k), \qquad i = 1, \cdots, m$$

i.e. $\alpha_i^{(\nu)}$ is given by:

(13) $$\alpha_i^{(\nu)} = \int_{R^k} x_\nu \, d_{x_1, \ldots, x_k} H_i(x_1, \cdots, x_k),$$

provided the integral (13) converges absolutely. In this case $\alpha_i^{(\nu)}$ is also given by:

(14) $$\alpha_i^{(\nu)} = - \left[\frac{\partial}{\partial \xi_\nu} \sum_{j=1}^{m} q_{ij}(\xi_1, \cdots. \xi_k)\right]_{\xi = 0}$$

where the derivative is in the suitable sense in the unilateral case.

Furthermore, let $\pi_1, \cdots, \pi_m$ be the stationary probabilities associated with the matrix P, i.e. the row-vector $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_m)$ is the unique solution to the equations:

(15) $$\boldsymbol{\pi} = \boldsymbol{\pi} P, \qquad \boldsymbol{\pi} \cdot \boldsymbol{e} = 1,$$

where $\boldsymbol{e}$ is the columnvector with all its components equal to one.

THEOREM 2. *The quantities M_j are given by*:

(16) $$M_j = - \sum_{i=1}^{m} \pi_i \alpha_i^{(j)}.$$

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In the unilateral case, this is provided the means $\alpha_i^{(j)}$, $i = 1, \cdots, m$ exist. In the bilateral case, our earlier assumptions encompass the existence of these means.

PROOF. Let $x(\xi)$ and $y(\xi)$ be right and left eigenvectors of $q(\xi)$ corresponding to $\rho(\xi)$, normalized such that $\boldsymbol{y}(\xi) \cdot \boldsymbol{x}(\xi) = 1$, and $\boldsymbol{y}(\xi) \cdot e = 1$. It is known that such a normalization is possible and uniquely determines $\boldsymbol{x}$ and $\boldsymbol{y}$ for every $\boldsymbol{\xi}$. Moreover as ξ tends (suitably) to $\mathbf{0}$, we have that $\boldsymbol{y}(\xi) \to \boldsymbol{\pi}$ and $\boldsymbol{x}(\xi) \to \boldsymbol{e}$, componentwise. The components of $\boldsymbol{x}(\xi)$ and $\boldsymbol{y}(\xi)$ are (suitably) continuously differentiable functions of $\boldsymbol{\xi}$ in $\bar{D}$.

We have that:

$$(17) \qquad \sum_{j=1}^{m} q_{vj}(\xi_1, \cdots. \xi_k)x_j(\xi_1, \cdots, \xi_k) = \rho(\xi_1, \cdots, \xi_k)x_v(\xi_1, \cdots, \xi_k),$$

for $v = 1, \cdots, m$ and all $\boldsymbol{\xi}$ in $\bar{D}$.

Differentiation with respect to ξ_i yields.

$$(18) \qquad \begin{aligned} &\rho(\xi_1, \cdots, \xi_k)\frac{\partial}{\partial \xi_i} x_v(\xi_1, \cdots, \xi_k) + x_v(\xi_1, \cdots, \xi_k)\frac{\partial}{\partial \xi_i}\rho(\xi_1, \cdots, \xi_k) \\ &= \sum_{j=1}^{m} x_j(\xi_1, \cdots. \xi_k)\frac{\partial}{\partial \xi_i} q_{vj}(\xi_1, \cdots, \xi_k) + \sum_{j=1}^{m} q_{vj}(\xi_1, \cdots, \xi_k)\frac{\partial}{\partial \xi_i} x_j(\xi_1, \cdots. \xi_k). \end{aligned}$$

Upon letting $\xi \to \mathbf{0}$ (suitably) and noting that $\rho(\mathbf{0}) = 1$, we obtain.

$$(19) \qquad \left[\frac{\partial}{\partial \xi_i} x_v(\xi)\right]_{\boldsymbol{\xi}=\mathbf{0}} + M_i = -\alpha_v^{(i)} + \sum_{j=1}^{m} p_{vj}\left[\frac{\partial}{\partial \xi_i} x_j(\xi)\right]_{\boldsymbol{\xi}=\mathbf{0}}$$

for $v = 1, \cdots, m$.

Multiplying by π_v in (19), summing on v and applying (15), it follows that:

$$(20) \qquad M_i = -\sum_{v=1}^{m} \pi_v \alpha_v^{(i)}.$$

REMARK. Formally, the quantities M_i appear in the same manner as the first moment does from the Laplace-Stieltjes transform of a probability distribution. A natural question to ask is whether $\rho(\xi_1, \cdots, \xi_k)$ is itself the transform of a probability distribution. The answer is negative in general. Consider the following example of a 2×2 univariate semi-Makov matrix

$$p_{11} = p_{22} = 0, \quad p_{12} = p_{21} = 1.$$

It is easy to see that:

$$\rho(\xi) = [f_1(\xi) \cdot f_2(\xi)]^{\frac{1}{2}},$$

where $f_1(\xi)$ and $f_2(\xi)$ are the Laplace-Stieltjes transforms of the probability distributions $F_{12}(\cdot)$ and $F_{21}(\cdot)$. It is well-known that $f_1(\xi)$ and $f_2(\xi)$ can be chosen so that their product is not the square of a Laplace-Stieltjes transform of a probability distribution, e.g.:

$$f_1(\xi) = e^{-\xi}, \qquad f_2(\xi) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\xi}.$$

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