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PILOT-WAVE HYDRODYNAMICS: QUANTISATION OF PARTIAL INTEGRABILITY FROM A NONLINEAR INTEGRO-DIFFERENTIAL EQUATION OF THE SECOND ORDER

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Abstract

Vertically vibrating a liquid bath may allow a self-propelled wave-particle entity to move on its free surface. The horizontal dynamics of this walking droplet, under the constraint of an external drag force, can be described adequately by an integro-differential trajectory equation. For a sinusoidal wave field, this equation is equivalent to a closed three-dimensional system of nonlinear ODEs. We explicitly define a stability boundary for the system and a quantised criterion for its partial integrability in the meromorphic category.

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1. Introduction

Pilot-wave hydrodynamics as a field of study was initiated in 2005 with the discovery by Couder *et al.* [3] that a droplet may self-propel along the surface of a vibrating bath of fluid, guided by the waves it generates on each impact. Moláček and Bush [12] developed a hydrodynamically consistent equation of motion for this walking droplet (or 'walker') of mass m given by nondimensional position $\mathbf{x}_p$ governed by its real and analytic wave field $h(\mathbf{x}, t)$. By time averaging over the bouncing period, the vertical dynamics are eliminated from consideration which produces the trajectory equation

$$\kappa\ddot{\mathbf{x}}_p + \dot{\mathbf{x}}_p = \mathcal{F} - \beta\nabla h(\mathbf{x}_p, t),$$

such that $\mathcal{F} = \mathcal{F}(\dot{\mathbf{x}}_p)$ is an external force and

$$\kappa = \frac{m}{DT_F M_e}, \quad \beta = \frac{Fk_F T_F M_e^2}{D}, \quad M_e = \frac{T_d}{T_F(1-\gamma/\gamma_F)},$$

where $F = mgAk_F$, D is the drag coefficient, T_F is the Faraday period, T_d is the decay time of waves without forcing, k_F is the Faraday wavenumber, γ is forcing acceleration,

γ_F is the Faraday instability threshold, A is the amplitude of a single surface wave and t is nondimensional time (see [14]). The guiding potential h is approximated by an integral over the particle's prior trajectory:

$$h(\mathbf{x}, t) = \int_{-\infty}^{t} W(|\mathbf{x} - \mathbf{x}_p(s)|)e^{s-t}\, ds,$$

such that W is real and entire. Writing $\mathbf{x}_p(t)$ as $x(t)\mathbf{j}$, for an arbitrary horizontal unit vector $\mathbf{j}$, we arrive at a dimensionless integro-differential equation:

$$\kappa\ddot{x} + \dot{x} = \mathcal{F} - \beta \int_{-\infty}^{t} W'(x(t) - x(s))e^{s-t}\, ds. \tag{1.1}$$

For vibration amplitudes close to but below the Faraday threshold, waves created by the walker on each bounce extend far in space and decay slowly in time. In this regime, predicting the walker's future dynamics not only requires knowledge of its present state but also of its past, creating memory in the system and nonlocality in time. Durey [4] and Valani *et al.* [16] investigated the high-memory regime using an idealised pilot-wave model that implements a sinusoidal waveform for the waves generated by the droplet and discovered an equivalence in orbital dynamics to the Lorenz system.

For a two-dimensional pilot-wave system in a rotating frame, Oza *et al.* [14] demonstrated quantisation of orbitally unstable and chaotic regions. The differential Galois criterion for partial integrability can be interpreted as a measure of orbital complexity (in the sense that complexity is determined by whether a solution to a given dynamical system $\mathcal{S}$ is expressible in terms of first integrals). So, it is natural that this criterion is connected to classifying whether or not the dynamics of $\mathcal{S}$ are chaotic in its parameter space.

The main methods used in the present paper are due to Ruiz's applications of differential Galois theory [13] and Bogoyavlensky's extension of Liouvillian integrability to dynamical systems [2]. In Section 2, the equivalence of (1.1) to the Lorenz system and its linear stability is provided. In Section 3, we shall formulate a criterion for partial integrability (specifically, 'B-integrability' as defined in [2]), which is the main result.

2. The trajectory equation as a nonlinear differential system in $\mathbb{R}^3$

The following theorem expresses (1.1) as an infinite dimensional system of nonlinear ODEs. Referring to [15, 16], this form of (1.1) is more tractable and, in special cases, such as when considering a waveform in the absence of spatial decay, the infinite system reduces to a finite dimensional system.

THEOREM 2.1. *Let*

$$M_n = -\int_{-\infty}^{t} W^{(n+1)}(x(t) - x(s))e^{s-t}\, ds$$

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for all $n \in \mathbb{N}$. Equation (1.1) holds if and only if

$$\begin{aligned}\dot{x} &= v,\\ \dot{v} &= \frac{1}{\kappa}(\beta M_0 - v + \mathcal{F}),\\ \dot{M}_n &= -M_n - W^{(n+1)}(0) + vM_{n+1}.\end{aligned} \tag{2.1}$$

PROOF. The first two linear ODEs are easily verified, since (1.1) may be expressed as

$$\ddot{x} = \frac{1}{\kappa}(\beta M_0 - \dot{x} + \mathcal{F}).$$

Given that the integrand of M_n is analytic, we may apply the Leibniz integral rule. Observe that for all $n \geq 0$,

$$\begin{aligned}\dot{M}_n &= -\frac{\partial}{\partial t}\int_{-\infty}^{t} W^{(n+1)}(x(t)-x(s))e^{s-t}\,ds\\ &= -W^{(n+1)}(0) - \int_{-\infty}^{t} e^{s-t}(\dot{x}W^{(n+2)}(x(t)-x(s)) - W^{(n+1)}(x(t)-x(s)))\,ds\\ &= -W^{(n+1)}(0) + \int_{-\infty}^{t} W^{(n+1)}(x(t)-x(s))e^{s-t}\,ds - \dot{x}\int_{-\infty}^{t} W^{(n+2)}(x(t)-x(s))e^{s-t}\,ds\\ &= -M_n - W^{(n+1)}(0) + \dot{x}M_{n+1},\end{aligned}$$

as claimed. □

2.1. The linear stability of a sinusoidal waveform system. According to Durey [4], we may define $W(s) = \cos(s)$ for the high memory regime, such that letting $\sigma = 1/\kappa$, $r = \beta$ and $b = 1$, (2.1) is reduced to the vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$\dot{\mathbf{P}} = V(\mathbf{P}) = (\sigma(Y - X + \mathcal{F}), rX - Y - XZ, XY - bZ), \tag{2.2}$$

where $X = v$, $Y = \beta M_0$, $Z = \beta(1 - M_1)$ and $\mathbf{P} = (X, Y, Z)$. For all $\mathcal{F}$ such that $\mathcal{F}(0) = 0$, an equilibrium point occurs at $\mathbf{P}^* = (0, 0, 0)$ and the Jacobian matrix of V at $\mathbf{P}^*$, letting $\mathcal{F}'(X) = \partial\mathcal{F}/\partial X$, is

$$JV_{\mathbf{P}^*} = \begin{pmatrix} \sigma(\mathcal{F}'(0) - 1) & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{pmatrix}.$$

So, we now consider the linearised system to first order in $\epsilon \in \mathbb{R}$, where $\mathbf{P} = \mathbf{P}^* + \epsilon\mathbf{u}$:

$$\dot{\mathbf{u}} = JV_{\mathbf{P}^*}\mathbf{u}. \tag{2.3}$$

Setting $\det(\lambda I - JV_{\mathbf{P}^*}) = 0$ for $\lambda \in \mathbb{R}$ yields the characteristic equation

$$0 = (\lambda + b)(\lambda^2 + \lambda(1 + \sigma - \sigma\mathcal{F}'(0)) + \sigma(1 - \mathcal{F}'(0) - r)). \tag{2.4}$$

Hence, we have $\lambda_0 = -b$ and may derive the remaining eigenvalues explicitly:

$$\lambda_{1,2} = \tfrac{1}{2}\Big(\sigma\mathcal{F}'(0) - \sigma - 1 \pm \sqrt{4\sigma(\mathcal{F}'(0) + r - 1) + (\sigma\mathcal{F}'(0) - \sigma - 1)^2}\Big).$$

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This explicit formula for eigenvalues is useful when calculating the linearised approximation $\mathbf{P}_t \approx \sum_{i=0}^{2} C_i e^{\lambda_i t}\mathbf{v}_i$ for the stable region given in Theorem 2.3, where $\mathbf{v}_i$ is the corresponding eigenvector to λ_i and $C_i \in \mathbb{R}$. Note that the accuracy of the approximation is determined by how close the initial conditions are to $\mathbf{P}^*$, which follows from Lemma 2.2.

LEMMA 2.2 (Hartman–Grobman theorem, [6]). *If all the eigenvalues λ_j of $JV_{\mathbf{P}^*}$ have nonzero real part, then the nonlinear flow is topologically conjugate to the flow of the linearised system in a neighbourhood of $\mathbf{P}^*$. If $\Re(\lambda_j) > 0$ for at least one j, the nonlinear flow is asymptotically unstable. If $\Re(\lambda_j) < 0$ for all j, the nonlinear flow is asymptotically stable.*

THEOREM 2.3. *The asymptotically stable region S of (2.2) is*

$$r < 1 - \mathcal{F}'(0),$$

such that $\sigma(\mathcal{F}'(0) - 1) < 1$ and $r, \sigma \geq 0$. Letting ζ denote the stability boundary, the remaining asymptotically unstable region $U := \mathbb{R}^3 \setminus (S \cup \zeta)$ satisfies $r > 1 - \mathcal{F}'(0)$.

PROOF. If $\lambda = i\omega$ is a root with $\omega \in \mathbb{R}$, then according to (2.4),

$$0 = (\sigma - \sigma\mathcal{F}'(0) - r\sigma - \omega^2) + i\omega(1 + \sigma - \sigma\mathcal{F}'(0)). \tag{2.5}$$

Separately considering real and imaginary parts, the stability boundary satisfies

$$\begin{aligned} 0 &= \sigma - \sigma\mathcal{F}'(0) - r\sigma - \omega^2 \\ &= \omega(1 + \sigma - \sigma\mathcal{F}'(0)). \end{aligned}$$

Suppose that $\sigma(\mathcal{F}'(0) - 1) < 1$. For $\omega \lessgtr \pm\sqrt{\sigma(1 - \mathcal{F}'(0) - r)}$, we must have each real part of $\lambda_{1,2}$ less than zero, which is the criterion for asymptotic stability according to Lemma 2.2 (the converse applies for asymptotic instability). Hence, simplifying the imaginary part of (2.5), we have

$$\begin{aligned} &0 \lessgtr \pm(1 + \sigma - \sigma\mathcal{F}'(0))\sqrt{\sigma(1 - \mathcal{F}'(0) - r)}, \\ &r < 1 - \mathcal{F}'(0). \end{aligned}$$

□

3. Nonlinear differential Galois theory and an integrability criterion

We will present an argument for algebraic B-integrability introduced by Morales-Ruiz [13] and subsequently developed by Huang *et al.* [7] for its application to the Lorenz system. From (2.2), we consider the system

$$\dot{\mathbf{P}} = V(\mathbf{P}), \tag{3.1}$$

such that $\mathbf{P} \in \mathcal{M}$ is an $(n = 3)$-dimensional complex analytic manifold with $t \in \mathbb{C}$.

DEFINITION 3.1. System (3.1) is completely integrable if it possesses $n - 1$ functionally independent first integrals Φ_1, Φ_2.

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DEFINITION 3.2. System (3.1) is B-integrable if it possesses k functionally independent first integrals $\Phi_1, \ldots, \Phi_k$ and $(n-k)$ vector fields $w_1 = V, \ldots, w_{n-k}$ such that

$$\begin{aligned} 0 &= [w_i, w_j] \\ &= w_j[\Phi_i], \end{aligned}$$

with $1 \leq i \leq k \leq n$ and $1 \leq j \leq n-k$. Note that $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the Lie bracket for a vector space $\mathfrak{g}$.

B-integrability was introduced by Bogoyavlenskij [2] and it was shown that if a system is B-integrable, then it is integrable by quadrature. Intuitively, B-integrability is a generalisation of Liouvillian integrability from Hamiltonian systems to dynamical systems (see [2]). Referring to Llibre *et al.* [11], a completely integrable system is orbitally equivalent to a linear differential system.

Let $\hat{\mathbf{P}} = \hat{\mathbf{P}}(t)$ denote a nonequilibrium solution of (3.1). A form of the variational equation was briefly introduced as (2.3). For the known particular solution, the variational equation along the phase curve Γ of $\hat{\mathbf{P}}$ is

$$\dot{\nu} = JV_{\hat{\mathbf{P}}} \cdot \nu,$$

such that $\nu \in T_\Gamma\mathcal{M}$ and $T_\Gamma\mathcal{M}$ is a vector bundle of $T\mathcal{M}$ restricted on Γ. With the normal bundle $N = T_\Gamma\mathcal{M}/\Gamma$, a natural projection $\pi : T_\Gamma\mathcal{M} \to N$ and $\eta \in N$, the variational equations may be reduced to

$$\dot{\eta} = \pi_*(T(V)(\pi^{-1}\eta)). \tag{3.2}$$

The differential Galois group of (3.2) can be defined as a matrix group G, with $G \subset \mathrm{GL}(n-1, \mathbb{C})$ acting on the fundamental solutions of (3.2) such that it does not change polynomial and differential relations between them.

The following theorem is a powerful result which extends Galoisian obstructions to mermorphic integrability from Hamiltonian systems, provided in [13], to the more general non-Hamiltonian case.

THEOREM 3.3 [1]. *System (3.1) is B-integrable, in the meromorphic category, in a neighbourhood of Γ if and only if the identity element G^0 of the differential Galois group of the normal variational equations along Γ is abelian.*

The following theorem provides necessary conditions for (3.1) to possess a certain number of first integrals. In the proof of Theorem 3.7, we will show that (3.1) violates condition (1) and provide a criterion for when condition (3) is violated.

THEOREM 3.4 [10]. *If (3.1) has m functionally independent meromorphic first integrals in a neighbourhood of Γ, then the Lie algebra $\mathcal{G}$ of the differential Galois group G of (3.2) has m meromorphic invariants and G^0 has at most $(n-m-1)(n-1)$ generators. Hence:*

(1) *if $m = 2$, then (3.1) is completely integrable, $\mathcal{G} = 0$ and $G^0 = \{\mathbf{1}\}$, which is the identity element;*

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(2) *if $m = 1$, then $\mathcal{G}$ and G^0 have at most two generators;*
(3) *if $m = 1$, then $\mathcal{G}$ and G^0 are solvable.*

The point $\hat{\mathbf{P}} = (0, 0, e^{-bt})$ is a nonequilibrium solution to (3.1). Hence, the variational equations along Γ are

$$\begin{pmatrix} \dot{\nu} \\ \dot{\eta} \\ \dot{\vartheta} \end{pmatrix} = \begin{pmatrix} \sigma(\mathcal{F}'(0) - 1) & \sigma & 0 \\ r - e^{-bt} & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} \nu \\ \eta \\ \vartheta \end{pmatrix}, \tag{3.3}$$

such that $\mathbf{P} = (\nu, \eta, \vartheta + e^{-bt})$ is in (3.1). Trivially, this may be reduced to the closed subsystem

$$\begin{pmatrix} \dot{\nu} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \sigma(\mathcal{F}'(0) - 1) & \sigma \\ r - e^{-bt} & -1 \end{pmatrix} \begin{pmatrix} \nu \\ \eta \end{pmatrix},$$
$$0 = \ddot{\nu} + (\sigma - \sigma\mathcal{F}'(0) + 1)\dot{\nu} - \sigma(r + \mathcal{F}'(0) - 1 - e^{-bt})\nu.$$

Applying the change of variable $\tau = e^{-bt}$, the above equation is expressed with rational coefficients:

$$\frac{d^2\nu}{d\tau^2} + \frac{b + \sigma(\mathcal{F}'(0) - 1) - 1}{b\tau}\frac{d\nu}{d\tau} - \frac{\sigma(r + \mathcal{F}'(0) - \tau - 1)}{b^2\tau^2}\nu = 0.$$

Hence, letting $\nu(\tau) = \chi(\tau)\tau^{(1-\sigma(\mathcal{F}'(0)-1)-b)/2b}$,

$$\frac{d^2\chi}{d\tau^2} = \frac{(\sigma - \sigma\mathcal{F}'(0) + 1)^2 + 4\sigma(r + \mathcal{F}'(0) - 1) - b^2 - 4\sigma\tau}{4b^2\tau^2}\chi. \tag{3.4}$$

Using the convention of Kovačič [9], since the coefficient of χ is a rational function, when we refer to the poles of the coefficient, we mean the poles in $\mathbb{C}$. If $r = z_1/z_2$ with $z_1, z_2 \in \mathbb{C}[\tau]$ relatively prime, then the poles of r are the zeros of z_2 and the order of the pole is the multiplicity of the zero of z_2. By the order of the coefficient at ∞, we mean the order of ∞ as a zero of the coefficient so that the order of the coefficient of χ at ∞ is $\deg z_2 - \deg z_1$.

LEMMA 3.5. *The differential Galois group G of (3.4) is infinite.*

PROOF. The coefficient of χ has two poles at 0 and ∞ with orders 2 and 1. According to the theorem provided in Section 2.1 of [9], the conditions for cases 1 and 3 do not hold and hence, G is not finite. □

LEMMA 3.6. *The identity element G^0 of G is solvable if and only if*

$$\frac{2\sqrt{(\sigma - \sigma\mathcal{F}'(0) - 1)^2 + 4\sigma r}}{|b|} \in \mathbb{Z}_{odd}.$$

PROOF. Letting $s = -2\sqrt{\sigma\tau}/|b|$ and $\hat{\chi} = \chi/s$, (3.4) becomes the Bessel equation

$$\frac{d^2\hat{\chi}}{ds^2} + \frac{1}{s}\frac{d\hat{\chi}}{ds} + \left(1 - \frac{\rho^2}{s^2}\right)\hat{\chi} = 0, \tag{3.5}$$

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such that

$$\rho = \sqrt{\frac{(\sigma - \sigma\mathcal{F}'(0) - 1)^2 + 4\sigma r}{b^2}}.$$

Since $\hat{\chi}' + s^{-1}\hat{\chi} = 0$ has the nontrivial solution $s^{-1} \in \mathbb{C}$, we have $G \subset \mathrm{SL}(2, \mathbb{C})$. Referring to [8], when $2\rho \in \mathbb{Z}_{\mathrm{odd}}$, we may (replacing ρ by $-\rho$ if necessary) suppose that $\mu = \rho - \frac{1}{2} \geq 0$ so that

$$\psi_{\pm} = e^{\pm is} \sum_{0 \leq k \leq \mu} \frac{(\mu + k)!}{k!\,(\mu - k)!} (\pm i)^k 2^{-k} s^{-k-1/2}$$

is a fundamental system of solutions to (3.5), such that each solution is exponential over $\mathbb{C}$ and their product is in $\mathbb{C}$.

Hence, the differential Galois group G is a diagonal group

$$\mathcal{D} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle|\; a \in \mathbb{C}, a \neq 0 \right\},$$

which has a solvable (abelian) identity element G^0 if $2\rho \in \mathbb{Z}_{\mathrm{odd}}$. Otherwise, when $2\rho \notin \mathbb{Z}_{\mathrm{odd}}$, [8] demonstrates that $G = \mathrm{SL}(2, \mathbb{C})$. Kovačič's algorithm may also be applied to derive the Galois group of (3.5) (see [5, 9]). □

THEOREM 3.7. *If (3.1) is B-integrable in the meromorphic category, then*

$$r = r_n := \frac{(b(2n+1))^2 - 4(\sigma(\mathcal{F}'(0) - 1) + 1)^2}{16\sigma}, \tag{3.6}$$

such that $\sigma, b \neq 0$ and $n \in \mathbb{Z}$.

PROOF. Since the differential Galois group of (3.3) is a normal subgroup of the differential Galois group of (3.4), the differential Galois group of (3.3) is also infinite by Lemma 3.5. The identity element G^0 is a normal subgroup of G with finite index. Hence, G^0 is a trivial subgroup if and only if G is finite [13], which implies that $G^0 \neq \{\mathbf{1}\}$ and (3.1) is not completely integrable with meromorphic first integrals by Theorem 3.4.

According to Lemma 3.6, if

$$\frac{2\sqrt{(\sigma - \sigma\mathcal{F}'(0) - 1)^2 + 4\sigma r}}{|b|} \neq 2n + 1, \tag{3.7}$$

the identity elements of the differential Galois groups of (3.5) and (3.3) are not solvable. Therefore, the identity element G^0 of (3.3) is not abelian. After rearranging (3.7) and considering the contrapositive, the criterion in (3.6) follows trivially in view of Theorems 3.3 and 3.4. □

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