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ON THE SOLUBILITY OF A PRODUCT OF PERMUTABLE SUBGROUPS

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Abstract

Conditions are investigated under which a product $G = HK$ of soluble subgroups H,K, each permutable in G, is soluble. It is shown in particular that this is true if one of the subgroups is nilpotent or if $H \cap K$ is subnormal in both H and K.

A subgroup H of a group G is called permutable (Stonehewer (1972)) if $HK = KH = \langle H, K \rangle$ for all subgroups K of G. Stonehewer (1972, Theorem E) proved that a product $G = HK$ of nilpotent subgroups H and K, each permutable in G, is soluble of derived length bounded in terms of the nilpotency classes of H and K. We shall extend his result to

THEOREM A. *There is an integer valued function $f(c,d)$ such that a product $G = HK$ of a nilpotent permutable subgroup H of class c and a soluble permutable subgroup K of derived length d is soluble of derived length at most $f(c,d)$.*

The fact that there is no analogue of Theorem A for a product of two soluble permutable subgroups was established by Stonehewer (1973, Theorem B). He constructed a non-soluble group which is the product of two metabelian permutable subgroups.

However a study of this example led to the idea that there might be a connection between the solubility of a product G of two soluble permutable subgroups and the way in which the intersection of these subgroups is situated in G. As our second result we shall prove

THEOREM B. *A product G of soluble subgroups H and K, each permutable in G, is soluble if $H \cap K$ is subnormal in both H and K.*

Thus, in particular, a product of two disjoint soluble subgroups, each permutable in G, is soluble. Furthermore it is not difficult to see from the proof that in this case the derived length of G is bounded in terms of the derived lengths of the permutable subgroups.

In the general case the most that can be said is that an inspection of the proof of Theorem B yields a bound on the derived length of G in terms of the derived lengths of H and K and the subnormal indices of $H \cap K$ in H and K.

As a further corollary to Theorem B we have that the product G of soluble subgroups H and K, each permutable in G, is soluble if $H \cap K$ satisfies the minimal condition on normal subgroups. For $H \cap K$ is permutable in K and thus it is subpermutable in G. But a subpermutable subgroup which satisfies the minimal condition on subnormal subgroups is subnormal (Stonehewer (1972, Theorem F)) and therefore $H \cap K$ is subnormal in G. The result now follows on applying Theorem B.

As one might well expect the hypothesis in Theorem B is not a necessary condition for the solubility of a product of soluble permutable subgroups. Examples constructed by Iwasawa (1943) illustrate this fact. Let p be a prime and let A,B be abelian groups of type p^∞ and suppose α is a p-adic integer, $\alpha \equiv 1 \pmod p$ ($\alpha \equiv 1 \pmod 4$ if $p = 2$). Then for $a \in A$ the mapping $a \to a^\alpha$ defines an automorphism of A. Let α act on B as it does on A and form the split extension G of $A \times B$ by $\langle \alpha \rangle$. On setting $H = A\langle \alpha \rangle$ and $K = B\langle \alpha \rangle$ we find that H and K are each permutable in G, G is soluble but $H \cap K = \langle \alpha \rangle$ is not subnormal in either H or K.

I am grateful to Dr. Stonehewer for drawing my attention to this example which also shows, incidentally, that the requirement that $H \cap K$ satisfies the maximal condition on subgroups is insufficient to imply that $H \cap K$ is subnormal in G.

Proof of Theorem A

We first of all deal with the case where H is abelian. Here the assumption that K is permutable in G is unnecessary. In fact we have the

LEMMA. *A product $G = HK$ of a permutable abelian subgroup H and a soluble subgroup K of derived length d is soluble of derived length at most $2d$.*

We prove the lemma by induction on d. If $d = 1$ it is a theorem of Îto (1955). Assume the natural induction hypothesis on d and that the derived length of K is $d + 1$. Let A be the last non-trivial term of the derived series of K. Then G/A^G is soluble with derived length at most $2d$ by the induction hypothesis. Now

$$A^G = A^{KH} = A^H \subseteq AH$$

and AH is metabelian by Îto's theorem. Hence G is soluble of derived length at most $2d + 2 = 2(d + 1)$, as required.

We now proceed with the proof of Theorem A by induction on c, the nilpotency class of H. The case $c = 1$ is immediate from the lemma so we suppose that $c > 1$ and that $f(c - 1, d)$ exists for all d. Again from the lemma we may take $f(c,1) = 2c$ and we assume that $f(c,r)$ exists for $r < d$ and that K is soluble of derived length d (at least 2).

Let A be the centre of H. Then by hypothesis G/A^G is soluble of derived length at most $f(c - 1, d)$. Also $A^G = A^K \subseteq AK$. We shall show that AK is soluble of derived length at most $f(c,d - 1) + d + 2$ in consequence of which we can put

$$f(c,d) = f(c - 1, d) + f(c,d - 1) + d + 2$$

thus completing the induction step.

Let B be the $d - 1$-st term of the derived series of K and set $I = H \cap K$. Then

$$(AK) \cap (HB) = ((AK) \cap H)B = IAB = M, \quad \text{say.}$$

Also

$$I^M = I^{AB} = I^B \subseteq K$$

and M/I^M is metabelian by Îto's theorem. It follows at once that M is soluble of derived length at most $d + 2$. Let $N = B^{KA} = B^A \subseteq M$, so that N is soluble of derived length at most $d + 2$. Set $X = H \cap (AK) = AI$. Clearly N is normal in AK and, denoting factors modulo N by bars, we have $\overline{AK} = \overline{XK}$, where $\bar{X}$ and $\bar{K}$ are permutable in $\overline{AK}$ and $\bar{K}$ is soluble of derived length at most $d - 1$. Hence by the induction hypothesis $\overline{AK}$ is soluble of derived length at most $f(c, d - 1)$. It now follows that AK is soluble of derived length at most $f(c, d - 1) + d + 2$, as required.

Proof of Theorem B

If H and K are both abelian then G is metabelian by Îto's theorem. We may therefore assume that H is not abelian and proceed by induction on the sum of the derived lengths of H and K.

Suppose first that $H \cap K$ is normal in H. Then, setting $H \cap K = I$, we have $I^G = I^{HK} = I^K \subseteq K$, from which it follows that I^G is soluble.

Set $J = H'K$ where H' is the derived subgroup of H. Then $J \cap H = H'I$ and this is permutable in J since H is permutable in G. Denote factors modulo I^K by bars. Then $\bar{J}$ is the product of $\bar{H}'$ and $\bar{K}$, each of which is permutable in $\bar{J}$.

It is easy to see that $\bar{H}' \cap \bar{K} = 1$, which is certainly subnormal in $\bar{H}'$ and $\bar{K}$. $\bar{H}'$ has derived length less than that of H and so by the induction hypothesis $\bar{J}$ is soluble. Therefore J is soluble.

Now $H'^G = H'^K \subseteq J$, whence H'^G is soluble. Using bars now to denote factors modulo H'^G we have $\bar{G} = \overline{HK}$ with both $\bar{H}$ and $\bar{K}$ permutable in $\bar{G}$ and it is routine to verify that $\bar{H} \cap \bar{K} = \overline{H \cap K}$. Therefore $\bar{H} \cap \bar{K}$ is subnormal in $\bar{H}$ and in $\bar{K}$. By our induction hypothesis $\bar{G}$ is soluble and hence G is soluble as required.

Assume now that $I = H \cap K \lhd^m H$ for some $m > 1$ and that the natural induction hypothesis on m holds. Set $V = FK$ where $F = I^H$. Then $V \cap H$ is permutable in V and therefore $V \cap H = F.K \cap H = F$ is permutable in V. Also K is permutable in V. Moreover $I = F \cap K \lhd^{m-1} F$ and I subnormal in K and so by the induction hypothesis on m we have that V is soluble.

Now $F^G = F^{HK} = F^K \subseteq V$, so that F^G is soluble. Use bars to denote factors modulo F^K. It is easy to check that $\bar{H} \cap \bar{K} = 1$. By the case $m = 1$ we deduce that $\bar{G}$ is soluble. Since F^K is soluble we have G soluble, as required.

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