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ON FRACTIONAL INTEGRALS EQUIVALENT TO A CONSTANT

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ABSTRACT. The paper is concerned with the Liouville–Riemann and Weyl fractional integrals. Necessary and sufficient conditions are obtained for a function to have a fractional integral which is equivalent to a constant.

1. **Introduction**. Suppose $\lambda > 0$ and f is a measurable function defined on $(0, \infty)$ and Lebesgue integrable on finite intervals $(0, t)$. The λth order *Liouville–Riemann integral* of f, denoted $I_\lambda f$, is given by

$$I_\lambda f(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s)\, ds, \tag{1}$$

for $t > 0$, whenever this expression exists as a Lebesgue integral. It is known that, if $0 < \lambda < 1$, $I_\lambda f(t)$ exists for almost all $t > 0$, and if $\lambda \geq 1$, $I_\lambda f(t)$ exists for all $t > 0$. It is also known that, for $\lambda > 0$ and $\mu > 0$, $I_\lambda(I_\mu f)(t) = I_{\lambda+\mu} f(t)$, whenever the latter integral exists ([4], pp. 177–179).

Suppose h is a measurable function defined on $(1, \infty)$ and $\int_1^\infty v^{\lambda-1} |h(v)|\, dv < \infty$. The λth order *Weyl integral* of h, denoted $W_\lambda h$, is given by

$$W_\lambda h(u) = \frac{1}{\Gamma(\lambda)} \int_u^\infty (v-u)^{\lambda-1} h(v)\, dv, \tag{2}$$

for $u > 1$, whenever this expression exists as a Lebesgue integral. If $0 < \lambda < 1$, $W_\lambda h(u)$ exists for almost all $u > 1$, and if $\lambda \geq 1$, $W_\lambda h(u)$ exists for all $u > 1$ (see Lemma 1 below).

Taking $\lambda = 1$, $I_1 f(t) = \int_0^t f(s)\, ds$, the ordinary Lebesgue integral. It is a well-known result of Lebesgue ([2], Theorem 95) that, for almost all $t > 0$, the derivative $(d/dt) I_1 f(t)$ exists and equals $f(t)$. Consequently, if $I_1 f(t) = c$, a constant, for all $t > 0$, then $f(t) = 0$ for almost all $t > 0$, and $c = 0$. That is, the equation $I_1 f(t) = c$ has an essentially unique solution when $c = 0$, and no solution when $c \neq 0$.

In this note we consider the analogous questions for the Liouville–Riemann and Weyl integrals. What are the solutions of the equations $I_\lambda f(t) = c$, for almost all $t > 0$, and $W_\lambda h(u) = c$, for almost all $u > 1$? We prove the following two theorems.

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THEOREM 1. *Suppose* $\lambda>0$, *c is a constant, f is a measurable function defined on* $(0,\infty)$ *and Lebesgue integrable on finite intervals* $(0,t)$, *and*

$$I_\lambda f(t)=\frac{1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}f(s)\,ds=c, \quad \textit{for almost all} \quad t>0. \tag{3}$$

If $0<\lambda<1$, *then* $f(s)=(c/\Gamma(1-\lambda))s^{-\lambda}$ *for almost all* $s>0$. *If* $\lambda\geq 1$, *then* $f(s)=0$ *for almost all* $s>0$, *and* $c=0$.

THEOREM 2. *Suppose* $\lambda>0$, *c is a constant, h is a measurable function defined on* $(1,\infty)$, $\int_1^\infty v^{\lambda-1}|h(v)|\,dv<\infty$, *and*

$$W_\lambda h(u)=\frac{1}{\Gamma(\lambda)}\int_u^\infty (v-u)^{\lambda-1}h(v)\,dv=c, \quad \textit{for almost all} \quad u>1. \tag{4}$$

Then $h(v)=0$ *for almost all* $v>1$, *and* $c=0$.

2. **Liouville–Riemann integral (Proof of Theorem 1).** Case (a): Suppose $\lambda=1$. Then (3) becomes

$$\int_0^t f(s)=c, \quad \text{for almost all} \quad t>0. \tag{5}$$

The integral in (5) is absolutely continuous, so (5) holds for all $t>0$. Taking the derivatives of both sides gives $f(s)=0$ for almost all $s>0$. Hence $c=0$.

Case (b): Suppose $0<\lambda<1$. Taking the $(1-\lambda)$th integral of both sides of (3), we get

$$I_1 f(t)=I_{1-\lambda}(I_\lambda f)(t)=\frac{c}{\Gamma(1-\lambda)}\int_0^t (t-s)^{-\lambda}\,ds=\frac{c}{(1-\lambda)\Gamma(1-\lambda)}\,t^{1-\lambda}, \tag{6}$$
$$\text{for all} \quad t>0.$$

Taking the derivative of both sides of (6) gives $f(s)=(c/\Gamma(1-\lambda))s^{-\lambda}$, for almost all $s>0$.

Case (c): Suppose $\lambda>1$. Equation (3) can be written as $I_1(I_{\lambda-1}f)(t)=c$, for almost all $t>0$. From case (a), we conclude that $I_{\lambda-1}f(t)=0$ for almost all $t>0$, and $c=0$. From case (b) we then conclude that $f(s)=0$ for almost all $s>0$. This completes the proof of Theorem 1.

If $c=0$, Theorem 1 is a special case of the following theorem of Titchmarsh ([3], Theorem 152).

THEOREM 3. *Suppose f and g are integrable on finite intervals* $(0,t)$, *and* $\int_0^t g(t-s)f(s)\,ds=0$ *for almost all* $t>0$. *Then either* $f(s)=0$ *for almost all s, or* $g(s)=0$ *for almost all s.*

Theorem 1 for arbitrary c, and $0<\lambda<1$, can be deduced from Theorem 3 by considering the function $f(s)-(c/\Gamma(1-\lambda))s^{-\lambda}$.

3. **Weyl integral.** The following lemma establishes the existence of the Weyl integral $W_\lambda h(u)$, for almost all $u>1$.

LEMMA 1. *Suppose* $\lambda>0$, h *is a measurable function defined on* $(1,\infty)$, *and* $\int_1^\infty v^{\lambda-1}|h(v)|<\infty$. *If* $0<\lambda<1$, *then* $W_\lambda h(u)$, *given by* (2), *exists for almost all* $u>1$. *If* $\lambda\geq 1$, *then* $W_\lambda h(u)$ *exists for all* $u>1$.

Proof. If $\lambda\geq 1$, the absolute convergence of the integral in (2) follows from $\int_u^\infty (v-u)^{\lambda-1}|h(v)|\,dv\leq\int_u^\infty v^{\lambda-1}|h(v)|\,dv<\infty$. If $0<\lambda<1$, we consider

$$\begin{aligned}\int_1^\infty u^{-2}W_\lambda\,|h|\,(u)\,du&=\frac{1}{\Gamma(\lambda)}\int_1^\infty u^{-2}\,du\int_u^\infty (v-u)^{\lambda-1}|h(v)|\,dv\\&=\frac{1}{\Gamma(\lambda)}\int_1^\infty |h(v)|\,dv\int_1^v (v-u)^{\lambda-1}u^{-2}\,du,\end{aligned}\tag{7}$$

by Fubini's theorem. Letting $a=\max(1,v/2)$, the inner integral in (7) can be written as $\int_1^a (v-u)^{\lambda-1}u^{-2}\,du+\int_a^v (v-u)^{\lambda-1}u^{-2}\,du$, whereupon it is seen to be $\leq Hv^{\lambda-1}$, for a suitable constant H independent of v. The lemma follows.

A Weyl integral may be transformed into a Liouville–Riemann integral by the following substitution (cf. [1], p. 175).

LEMMA 2. *Suppose* $\lambda>0$, h *is a measurable function defined on* $(1,\infty)$, *and* $\int_1^\infty v^{\lambda-1}|h(v)|\,dv<\infty$. *Define* $f(u)=u^{-\lambda-1}h(1/u)$, *for* $0<u<1$. *Then* $f\in L(0,1)$ *and*

$$I_\lambda f(t)=t^{\lambda-1}W_\lambda h\left(\frac{1}{t}\right),\quad \textit{for}\quad 0<t<1,\tag{8}$$

whenever either integral exists.

Proof. Making the substitution $u=1/v$, we have

$$\int_0^1 |f(u)|\,du=\int_0^1 u^{-\lambda-1}\left|h\left(\frac{1}{u}\right)\right|du=\int_1^\infty v^{\lambda-1}|h(v)|\,dv<\infty,\quad\text{and}$$

$$\begin{aligned}I_\lambda f(t)&=\frac{1}{\Gamma(\lambda)}\int_0^t (t-u)^{\lambda-1}f(u)\,du=\frac{1}{\Gamma(\lambda)}\,t^{\lambda-1}\int_{1/t}^\infty \left(v-\frac{1}{t}\right)^{\lambda-1}h(v)\,dv\\&=t^{\lambda-1}W_\lambda h\left(\frac{1}{t}\right).\end{aligned}$$

We can now prove Theorem 2.

Proof of Theorem 2. Case (a): Suppose $0<\lambda<1$. Define $f(u)=u^{-\lambda-1}h(1/u)$, for $0<u<1$, so that (8) holds. Equation (4) then becomes $I_\lambda f(t)=ct^{\lambda-1}$ for almost all t satisfying $0<t<1$. Taking the $(1-\lambda)$th integral gives

$$I_1 f(t)=\frac{c}{\Gamma(1-\lambda)}\int_0^t (t-u)^{-\lambda}u^{\lambda-1}\,du=c\Gamma(\lambda),\quad\text{for}\quad 0<t<1.\tag{9}$$

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Differentiating (9) gives $f(t)=0$ for almost all t satisfying $0<t<1$, whence $h(u)=0$ for almost all $u>1$, and $c=0$.

Case (b): Suppose $\lambda=1$. Equation (4) becomes $\int_u^\infty h(v)\,dv=c$, for almost all $u>1$, and hence for all $u>1$ by (absolute) continuity. The result follows by differentiating.

Case (c): Suppose $\lambda>1$. Differentiating equation (4), we obtain

$$(10)\qquad -\frac{1}{\Gamma(\lambda-1)}\int_u^\infty (v-u)^{\lambda-2}h(v)\,dv=0,\quad \text{for almost all}\quad u>1,$$

and the result follows by induction from cases (a) and (b).

This completes the proof of Theorem 2.

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