PURE-SEMISIMPLICITY IS PRESERVED UNDER ELEMENTARY EQUIVALENCE

by MAHER ZAYED

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Introduction. In the present note, Σ_r denotes the class of all right pure semisimple rings (= right pure global dimension zero). It is known that if $R \in \Sigma_r$, then R is right artinian and every indecomposable right R-module is finitely generated. The class Σ_r is not closed under ultraproducts [**4**]. While Σ_r is closed under elementary descent (i.e. if $S \in \Sigma_r$ and R is an elementary subring of S then $R \in \Sigma_r$) [**4**], it is an open question whether right pure-semisimplicity is preserved under the passage to ultrapowers [**4**, Prob. 11.16]. In this note, this question is answered in the affirmative.

Main result. Let L be the first order language of rings. Two rings r and S are called elementarily equivalent (notation: $R \equiv S$) if R and S satisfy the same first order sentences in L. A class Γ of rings is called elementarily closed if $R \equiv S$, $S \in \Gamma$ implies $R \in \Gamma$.

PROPOSITION 1. *Let R be any ring. Then $R \in \Sigma_r$ if and only if for each ultrafilter pair $\langle I, F\rangle$ the ultrapower ring $R^I/F = R^* \in \Sigma_r$.*

Proof. (i) The "if" part: suppose that the ultrapower R^* is right pure semisimple ring, so the ring R is right artinian [**3**; 1.2]. We have to show that for each sequence of nonisomorphisms

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \dots$$

between finitely generated indecomposable right R-modules there exists an integer n with

$$f_n \dots f_2 f_1 = 0.$$

(This is a well-known characterization of right pure semisimple rings, see [**2** or **6**]). Since the ultrapower modules M_t^I/F, $t = 1, 2, 3, \dots$, are finitely generated and indecomposable over the ultrapower ring R^* [**3**; 1.4], the assertion follows from the fact that the following sequence of R^*-modules

$$M_1^I/F \xrightarrow{f_1^*} M_2^I/F \xrightarrow{f_2^*} M_3^I/F \xrightarrow{f_3^*} \dots$$

where $f_t^* = f_t^I/F$, $t = 1, 2, 3, \dots$, admits this property [**3**; 1.1(ii)].

(ii) The "only if" part; Let $R \in \Sigma_r$. Suppose that there exists an ultrafilter pair $\langle I, F\rangle$ such that $R^* \notin \Sigma_r$. Thus there is a sequence of nonisomorphisms

$$M_I^* \xrightarrow{f_1} M_2^* \xrightarrow{f_2} M_3^* \xrightarrow{f_3} \dots$$

between finitely generated indecomposable right R^*-modules such that for any $p > 0$, $f_p \dots f_1\ f_2 \neq 0$. Observe that R^* is right artinian and each M_t^* is isomorphic to an ultraproduct of a family $(M_{ti})_{i \in I}$ of indecomposable right R-modules of the same length [**3**; 1.4]. Since R admits only finitely many isomorphism types of indecomposable right modules of length n for each positive integer n (see [5] or [7]), thus one can deduce that each M_t^* is isomorphic to an ultrapower of a member of the family $(M_{ti})_{i \in I}$, say N_t. So, by [3; 1.1(ii)], we can obtain an infinite chain

$$N_1 \to N_2 \to N_3 \to \dots$$

of indecomposable right A-modules, where each composition is nonzero, contradicting the fact that R is right pure semisimple. Therefore $R^* \in \Sigma_r$. This completes the proof of the proposition.

Proposition 2. *The class Σ_r is elementary closed.*

Proof. The assertion follows from the Keisler–Shelah ultrapower theorem [1], and the preceding proposition.

Remark 3. Similarly, one can show that the class Σ_l of left pure semisimple rings is elementarily closed. Therefore the class $\Sigma = \Sigma_r \cup \Sigma_l$ of right or left pure semisimple rings is closed under elementary equivalence.

Corollary 4. Let $\langle R_\lambda : \lambda < \alpha \rangle$ be an elementary chain of elements of the class Σ (i.e. R_μ is an elementary subring of R_λ whenever $\mu \le \lambda < \alpha$). Then its union belongs to Σ. In particular, Σ is closed under the formation of ultralimits.

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Department of Mathematics
University of Bahrain
P.O. Box 32038
Isa Town, State of Bahrain