THE $\mathscr{H}$-EQUIVALENCE IN A COMPACT SEMIGROUP II

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In [1] we considered various aspects of the quotient semigroup $H \mathbin{.\cdot} H$[2] where H is an $\mathscr{H}$-class of a semigroup S. In particular, the action of the Schützenberger group of H upon SH was studied to obtain various results on the existence of subcontinua. Crucial in [1] was the notion of the (right handed) core of an $\mathscr{H}$-class which may be considered as a generalization of the notion of the core of a homogroup, [2].

The view taken here is the possible description of $H \mathbin{.\cdot} H$ as a fibre bundle, in the sense of Steenrod, in which the core and Schützenberger group play an appropriate rôle as fibre and structure group respectively. This is done somewhat in [2] in the very special case in which H is the kernel of S, and is an orbit of the maximal subgroup at the identity.

We recall now some basic definitions for the convenience of the reader. Let S be any semigroup. The Green equivalences are defined as follows

$$x = y(\mathscr{L}) \rightleftharpoons x \cup Sx = y \cup Sy$$
$$x = y(\mathscr{R}) \rightleftharpoons x \cup xS = y \cup yS$$
$$x = y(\mathscr{J}) \rightleftharpoons x \cup xS \cup Sx \cup SxS = y \cup yS \cup Sy \cup SyS$$
$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}$$
$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}.$$

For $A \subset S$ and $x \in S$, set

$$A \mathbin{.\cdot} x \equiv \{y \in S | xy \in A\}.$$

Here, our interest lies principally in the $\mathscr{H}$-equivalence.

We first establish a series of lemmas for later use.

LEMMA 1. *Let S be a semigroup, H an $\mathscr{H}$-class in S and $x \in H$. If $A \subset S$ with $xA = H$ then $yA = H$ for all $y \in H$ and pA meets $x \mathbin{.\cdot} x$ for all $p \in H \mathbin{.\cdot} x$.*

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[2] Let A and B be subsets of a semigroup S then $A \mathbin{.\cdot} B$ is defined as $\{x | Bx \subset A\}$.

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PROOF. Since $y \in H$, we have $y = q \cdot x$ for some $q \in S$. Thus $yA = (qx)A = q(xA) = qH$ and since qH meets H, $qH = H$. Now if $p \in H \mathbin{.\cdot} x$ then $x(pA) = (xp)A = H$ since $xp \in H$. Hence it follows that pA meets $x \mathbin{.\cdot} x$.

LEMMA 2. *Let S be a semigroup, H an $\mathscr{H}$-class in S and $x \in H$. If $T \subset S$ with $xT \subset H$ and if $e \in T$ is a right identity for T then*

$$(H \mathbin{.\cdot} x)T = (H \mathbin{.\cdot} x) \cap Se = (H \mathbin{.\cdot} x)e.$$

PROOF. First note that since $xT \subset H$, we have $T \subset H \mathbin{.\cdot} x$. Since $e \in T$ we have

$$(H \mathbin{.\cdot} x)e \subset (H \mathbin{.\cdot} x)T.$$

Also

$$(H \mathbin{.\cdot} x)T = (H \mathbin{.\cdot} x)(Te) = ((H \mathbin{.\cdot} x)T)e(H \mathbin{.\cdot} x)e$$

since $H \mathbin{.\cdot} x$ is a subsemigroup of S. It is clear that

$$(H \mathbin{.\cdot} x)Se = (H \mathbin{.\cdot} x)e.$$

LEMMA 3. *Let S be a semigroup, H an $\mathscr{H}$-class in S and $x \in H$. If G is a subgroup of S with identity e and if $xG = H$ then $(x \mathbin{.\cdot} x) \cdot G = (H \mathbin{.\cdot} x) \cap Se$ $(H \mathbin{.\cdot} x)e$.*

PROOF. By the preceding lemma we have $(x \mathbin{.\cdot} x)G \subset (H \mathbin{.\cdot} x) \cap Se$. If $p \in (H \mathbin{.\cdot} x) \cap Se$ then by lemma 1 we have $(x \mathbin{.\cdot} x)$ meets pG. Say $pg \in x \mathbin{.\cdot} x$ with $g \in G$. Then

$$p = pe = p(gg^{-1}) = (pg)g^{-1} \in (x \mathbin{.\cdot} x)G.$$

If A and B are subsets of the semigroup S, we say that A is *right* (left) *cancellative on* B if for any $a \in A$ and $b_1, b_2, \in B$, $b_1 a = b_2 a$ $(ab_1 = ab_2)$ implies $b_1 = b_2$.

LEMMA 4. *Let S be a semigroup, H an $\mathscr{H}$-class in S and $x \in H$. If $B \subset S$ is right cancellative on $x \mathbin{.\cdot} x$ and x is left cancellative on B then the mapping $m : (x \mathbin{.\cdot} x) \times B \to (x \mathbin{.\cdot} x)B$ defined by $m(z, b) = zb$ is one-to-one. If $zb = bz$ for all $z \in x \mathbin{.\cdot} x$ and $b \in B$ then m is an isomorphism. If, further, S is compact and B is closed then m is a homeomorphism.*

PROOF. If $zb = wc$ where $z, w \in x \mathbin{.\cdot} x$ and $b, c \in B$ then $xb = (xz)b = x(zb) = x(wc) = (xw)c = xc$ hence $b = c$ whence it follows that $z = w$. The remaining conclusions are easily established. We have now established

THEOREM 1. *Let S be a semigroup, H an $\mathscr{H}$-class in S and $x \in H$. If G is a subgroup of S with identity e and if the mapping $\lambda_x : G \to H$ defined by $\lambda_x(g) = xg$ is one-to-one and onto then the mapping $m : (x \mathbin{.\cdot} x) \cap Se \times G \to (x \mathbin{.\cdot} x)G = (H \mathbin{.\cdot} x) \cap Se = (H \mathbin{.\cdot} x)e$ defined by $m(z, g) = zg$ is one-to-one and onto. If $zg = gz$ for all $z \in x \mathbin{.\cdot} x$ and $g \in G$ then m is an isomorphism. If, further, S is compact and G is closed then m is a homeomorphism.*

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We note that the usual factorization (e.g. see [2]) of an isogroup is included in the above theorem.

Recall, [1], that if H is an $\mathscr{H}$-class in S and $h \in H$ then $H \, . \cdot h$ is a subsemigroup of S. The relation $\mathscr{S}_h$ defined on $H \, . \cdot h$ by

$$x \equiv y(\mathscr{S}_h) \rightleftharpoons hx = hy$$

is a congruence and

$$(H \, . \cdot h)/\mathscr{S}_h \equiv \mathscr{S}\mathscr{I}_h$$

is a group. In case S is compact, $\mathscr{S}\mathscr{I}_h$ is a compact group homeomorphic with H. If H contains an idempotent then the subgroup H is isomorphic with $\mathscr{S}\mathscr{I}_h$.

Throughout the following we will assume S is a compact homogroup, i.e. the kernel, K, of S is a group. Since the adjunction of an identity does not disturb the $\mathscr{H}$-class structure of S, we will assume S has an identity. Letting e be the identity of K we note that $ex = xe$ for all $x \in S$, hence the function $\lambda_e : S \to K$ defined by $\lambda_e(x) = ex$ is a homomorphism. We recall from [1] that if $x, y \in S$ with $xH_y = H_x$ then $H_y \, . \cdot y \subset H_x \, . \cdot x$ and the identity homomorphism $i : H_y \, . \cdot y \to H_x \, . \cdot x$ induces an epimorphism $\varphi_y : \mathscr{S}\mathscr{I}_y \to \mathscr{S}\mathscr{I}_x$. Moreover, $\mathscr{S}\mathscr{I}_y$ acts as a group of homomorphisms on $(H_x \, . \cdot x) \cap Sy$, and Ker (φ_y) is an effective transformation group on $x \, . \cdot x \cap Sy$. Also note that if $z \in H_x \, . \cdot x \cap Sy$ and $\alpha \in \mathscr{S}\mathscr{I}_y$ then $x(z\alpha) = (xz)\varphi_y(\alpha)$. We now assume that for some $a \in S$, $eH_a = H_e = K$. Let G be the kernel of φ_a where $\varphi_a : \mathscr{S}\mathscr{I}_a \to \mathscr{S}\mathscr{I}_e = K$ is as above. Observe that $\alpha \in G$ if and only if α acts as the identity on K, i.e. $k\alpha = k$ for all $k \in K$. We now assume that G has a local cross section in $\mathscr{S}\mathscr{I}_a$ (as is the case, for example, if $\mathscr{S}\mathscr{I}_a$ is finite dimensional), i.e. there is an open set $V = V^{-1}$ contained in $K = \mathscr{S}\mathscr{I}_a/G$ and containing e and there is a continuous function $\gamma : V^* \to \mathscr{S}\mathscr{I}_a$ such that $\varphi_a(\gamma(x)) = x$ for all $x \in V^*$. This being the case, then Sa is a fibre bundle in the sense of the Steenrod [3]. The base space is K, the projection mapping is $\lambda_e|Sa = p$. The fibre is given by $F = (e \, . \cdot e) \cap Sa$. Observe that if Sa is connected then by a result in [1], the fibre F is also connected. For the coordinate neighborhoods we take the family of all $V_\alpha = V \cdot \alpha$ for $\alpha \in \mathscr{S}\mathscr{I}_a$. Now we define the coordinate functions

$$\varphi_a : V_\alpha \times F \to p^{-1}(V_\alpha) \text{ by } \varphi_\alpha(z, w) = w(\gamma(z\alpha^{-1})\alpha)$$

for $z \in V_\alpha$ and $w \in F$. It is easily seen that φ_α is well defined and continuous. We now show that φ_α is a homeomorphism. Suppose $z, z' \in V_\alpha$, $w, w' \in F$ and $\varphi_\alpha(z, w) = \varphi_\alpha(z', w')$. Then $(\varphi_\alpha(z, w)) = ew(\gamma(z\alpha^{-1})\alpha) = (e\gamma(z\alpha^{-1})\alpha) = (z\alpha^{-1})\alpha = z$ and $e(\varphi(z', w')) = ew'(\gamma(z'\alpha^{-1})\alpha) = (e\alpha(z'\gamma^{-1})\alpha) = (z'\alpha^{-1})\alpha = z'$ hence $z = z'$. Since $\mathscr{S}\mathscr{I}_a$ acts as a group of permutations on Sa, it follows from the fact that $w(\gamma(z\alpha^{-1})\alpha) = w'(\gamma(z\alpha^{-1})\alpha)$ that $w = w'$. Thus φ is one-to-one. To see that φ is onto, choose $x \in p^{-1}(V_\alpha)$ so that $x \in Sa$ and

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$ex \in V_\alpha$. Then $ex\alpha^{-1} \in V_\alpha$ and so $\gamma(ex\alpha^{-1}) \in \mathscr{S}\mathscr{I}_a$. Now set $z = ex$ and $w = x\alpha^{-1}(\gamma(ex\alpha^{-1})^{-1})$ then

$$\varphi(z, w) = [x^{-1}(\gamma(ex\alpha^{-1})^{-1})][\gamma(ex\alpha^{-1})\alpha] = xe_a = x,$$

where e_a is the identity of $\mathscr{S}\mathscr{I}_a$. To see that $w \in F$ first note that $w \in Sa$ and $ew = e[x\alpha^{-1}(\gamma(ex\alpha^{-1})^{-1})] = (ex)\varphi_a(\alpha^{-1}(\gamma(ex\alpha^{-1}))) = (ex)\varphi_a(\alpha^{-1})\varphi_a(\gamma(ex\alpha^{-1})^{-1}) = (ex\alpha^{-1})(ex\alpha^{-1}) = (ex\alpha^{-1}(\alpha(ex)^{-1})) = e$ and so $w \in e$. e. For $\alpha \in \mathscr{S}\mathscr{I}_a$ and $z \in V_\alpha$ define $\varphi_{\alpha,z} : F \to p^{-1}(z)$ by $\varphi_{\alpha,z}(w) = \varphi_\alpha(z, w)$. Then for $z \in V_\alpha \cap V_\beta$ and $w \in F$, we have $\varphi^{-1}_{\alpha,z}\varphi_{\beta,z}(w) = w(\gamma(z\beta^{-1})\beta)(\alpha^{-1}\gamma(z\alpha^{-1})^{-1})$.

Also

$$e(\gamma(z\beta^{-1})\beta)(\alpha^{-1}\gamma(z\alpha^{-1})^{-1}) = [(z\beta^{-1})\beta][\alpha^{-1}(\alpha z^{-1})] = e$$

so that

$$(\gamma(z\beta^{-1})\beta)(\alpha^{-1}\gamma(z\alpha^{-1})^{-1}) \in G.$$

Clearly the mapping

$$g_{\alpha,\beta} : V_\alpha \cap V_\beta \to G \text{ defined by } g_{\alpha,\beta}(z) = (\gamma(z\beta^{-1})\beta)\alpha^{-1}(\gamma(z\alpha^{-1})^{-1})$$

is continuous.

We summarize the foregoing discussion in

THEOREM 2. *Let S be a compact semigroup with identity whose kernel K is a group. If $a \in S$ such that $kH_a = K$ for some $k \in K$ and if the kernel of the homeomorphism*

$\varphi_a : \mathscr{S}\mathscr{I}_a \to K$ has a local cross section then Sa has the structure of a fibre bundle. Moreover if Sa is connected then the fibre of the bundle is also connected.

For S a compact semigroup there is associated with each $x \in S$ a homogroup [1] namely, the homogroup

$$H_x \,.\, \cdot\, x/\mathscr{S}'_x$$

where $\mathscr{S}'_x$ is the restriction of $\mathscr{S}_x$ to $\ker(H_x \,.\, \cdot\, x)$ (i.e. $\mathscr{S}'_x = (\mathscr{S}_x \cap \operatorname{Ker}(H_x \,.\, \cdot\, x) \times \operatorname{Ker}(H_x \,.\, \cdot\, x)) \cup \Delta)$. Thus if S is finite dimensional and $x, y \in S$ with $H_x = xH_y$ then the homogroup

$$(H_x \,.\, \cdot\, x) \cap Sy/\mathscr{S}'_{x,y}$$

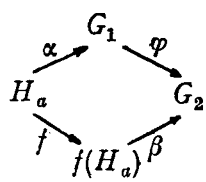
(where $\mathscr{S}'_{x,y}$ is the restriction of $\mathscr{S}'_x$ to Sy) associated with the pair (x, y) has the structure of a fibre bundle.

COROLLARY. *Let S be a compact connected homogroup such that $S = SE$. If for some pair of points x and y we have $xH_y = K$ then K is an orbit of some H-class H_f, where $f^2 = f$, and the core contains a non-degenerate continuum at e. Moreover, if S is n-dimensional and K is $n-1$ dimensional then K is arcwise accessible. Indeed, in this case there exist an idempotent f and a standard thread from e, the identity of K, to f.*

From [1] we know that there is available an idempotent f such that $yH_f \supseteqq H_y$. It follows that $eH_f = K$. It is immediate that H_f is $n-1$ dimensional and so, the homomorphism $t \to et$ defined upon H_f has a local cross section. Since this is an epimorphism the bundle has connected fibres. Now fSf is at most n-dimensional, K is $n-1$-dimensional and fSf is locally the product of K and $e \,.\!\cdot\, e \cap fSf$. Since the dimension of K can be defined in terms of the largest call it contains it follows that $\dim(e \,.\!\cdot\, e \cap fSf) + \dim K = \dim(e \,.\!\cdot\, e \cap fSf \times K)$. So that $e \,.\!\cdot\, e \cap fSf$ is precisely one dimensional. It follows now from [6] that $e \,.\!\cdot\, e \cap fSf$ contains thread $[e, f]$.

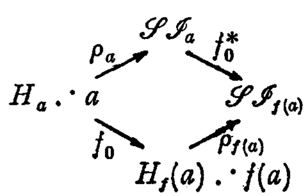
As we have seen [1], an $\mathscr{H}$-class H behaves in a way remarkably similar to a group. For example xH meets yH implies $xH = yH$. Of course, H is the underlying space of the Schützenberger group. In the following we note another such phenomenon.

THEOREM 3. *Let S be a compact semigroup with identity. If $f : S \to T$ is a continuous epimorphism then for each $a \in S$, $f|H_a$ is equivalant to a group epimorphism in the sense that there exists groups G_1 and G_2 such that the diagram*



is commutative and α, β are homeomorphisms and φ is an epimorphism.

PROOF. We first note that $f|(H_a \,.\!\cdot\, a) = f_0 : H_a \,.\!\cdot\, a \to H_{f(a)} \,.\!\cdot\, f(a)$ is a homomorphism and $f_0(\mathscr{S}\mathscr{I}_a) \subset \mathscr{S}\mathscr{I}_{f(a)}$. Thus it follows from the induced homomorphism theorem ([4] or [5]) that there is a homomorphism f_0^* such that the diagram



is commutative. Now let $G_1 = \mathscr{S}\mathscr{I}_a$, $G_2 = f_0^*(\mathscr{S}\mathscr{I}_a)$ and $\varphi = f_0^*$. We now defined $\alpha : H_a \to G_1$ by $\alpha(y) = g$ if and only if $a \cdot g = y$. Then α is a homeomorphism (see [1]). We will now show that G_2 is a simply transitive group of homeomorphisms on $f(H_a)$. Choose $x \in H_a$, $g \in G_2$ and pick $s \in \rho_a^{-1}\varphi^{-1}(y)$. Now $xs \in H_a$ so $f(x)f(s) = f(xs) \in f(H_a)$, and by definition, $f(x) \cdot g = f(x)f(s)$, since $\rho_{f(a)}f(s) = g$, thus $f(H_a) \cdot g \subset f(H_a)$. Since $\mathscr{S}\mathscr{I}_{f(a)}$ is a simply transitive group of homeomorphisms on $H_{f(a)}$ it follows that G_2 is effective on $f(H_a)$. To see G_2 is transitive on $f(H_a)$ we choose $x, y, \in H_a$ and $s \in H_a \cdot\!\cdot\, a$ such that $xs = y$. Then $f(x)(\varphi(\rho_a(s))) = f(x)(\rho_{f(a)}(f_0(s))) = f(x)f(s) = f(xs) = f(y)$, thus the mapping $\beta : f(H_a) \to G_2$ defined by $\beta(y) = g$ if, and only if, $f(a)g = y$ is a homeomorphism and clearly

$$\varphi(\alpha(x)) = \beta(f(x)) \text{ for each } x \in H_a.$$

It is of interest to remark that the above result shows that $f|_H$ cannot be dimension raising for we must have in fact, for any $h \in H$

$$\dim f(H) + \dim[f^{-1}f(h)] = \dim H.$$

Also, if T is a group then $\mathscr{S}\mathscr{I}_{f(x)} = T$ and $f(H_x) = f(x \cdot \mathscr{S}\mathscr{I}_x) = f(x) \cdot f_0^*(\mathscr{S}\mathscr{I}_x)$, hence, $f(H_x)$ is a coset in T.

THEOREM 4. *Let S be a compact semigroup with identity. If for each $x \in S$ there is a positive integer $n(= n(x))$ such that $x^n \in E$ then the natural mapping $\varphi : S \to S/H$ is light.*

PROOF. Let H be an $\mathscr{H}$-class in S and $x \in H$. If $\alpha \in \mathscr{S}\mathscr{I}_x$, choose $y \in H \cdot\!\cdot\, x$ such that $\rho_x(y) = \alpha$. Then $y^n \in E \cap H \cdot\!\cdot\, x$ for some n, hence α has finite order since ρ_x is a homomorphism. Now if $\mathscr{S}\mathscr{I}_x$ is not totally disconnected then the component containing the identity contains a non-trivial homomorphic image of the real numbers, contradicting the fact that every element of $\mathscr{S}\mathscr{I}_x$ has finite order. Since H_x is homeomorphic with $\mathscr{S}\mathscr{I}_x$, it follows that H_x is also totally disconnected.

Observe that if each $\mathscr{H}$-class in a compact connected semigroup with identity is totally disconnected then an $\mathscr{H}$-class in the kernel reduces to a single point. Thus it follows that the semigroup is acyclic, since $H^n(S) \cong H^n(H)$ where $H \subset K$.

COROLLARY. *If S is a compact connected semigroup with identity having the property that some power of every element is idempotent then S is acyclic.*

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