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Maximal sum-free sets in finite abelian groups

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Maximal sum-free sets in elementary abelian 3-groups and groups $G = Z_3 \oplus Z_3 \oplus Z_p$ where p is a prime congruent to 1 modulo 3 are completely characterized.

Let G be an additive group. If S and T are non-empty subsets of G, we write $S \pm T$ for $\{s \pm t;\ s \in S,\ t \in T\}$ respectively, $|S|$ for the cardinality of S and $\bar{S}$ for the complement of S in G. We say that S is sum-free in G if S and $S + S$ have no common element and that S is maximal sum-free in G if S is sum-free in G and $|S| \geq |T|$ for every T sum-free in G. We denote by $\lambda(G)$ the cardinality of a maximal sum-free set in G.

The numbers $\lambda(G)$ for abelian groups G were determined except when every prime divisor of $|G|$ is congruent to 1 modulo 3. In this exceptional case,

$$|G|(m-1)/3m \leq \lambda(G) \leq \big(|G|-1\big)/3$$

where m is the exponent of G [1]. If G is an elementary abelian p-group of order p^n, where $p = 3k + 1$, then $\lambda(G) = kp^{n-1}$ [3].

The structure of maximal sum-free sets in the following groups were completely characterized:

(i) G is an abelian group such that $|G|$ has a prime divisor congruent to 2 modulo 3 [1, 5];

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(ii) $G = Z_p$ where p is a prime congruent to 1 modulo 3 [6, 3];

(iii) G (abelian and non-abelian) is of order $3p$, where p is a prime congruent to 1 modulo 3 [7];

(iv) G is an elementary abelian p-group where $p = 3k + 1$ [4];

(v) in a recent letter to A.H. Rhemtulla, Anne Penfold Street mentioned that she is able to characterize maximal sum-free sets completely in $G = Z_{p^2}$ where p is a prime congruent to 1 modulo 3.

In this note, we shall completely characterize maximal sum-free sets in the following groups:

(i) G is an elementary abelian 3-group;

(ii) $G = Z_3 \oplus Z_3 \oplus Z_p$ where p is a prime congruent to 1 modulo 3.

We shall apply Theorem 4 of [1] and Theorem 1 of [7], which are restated respectively as Theorems 1 and 2 here, to prove Theorems 3 and 4.

THEOREM 1. *Let G be a finite abelian group. Suppose $|G|$ has no prime factor congruent to 2 modulo 3 but has 3 as a factor. If S is a maximal sum-free set in G, then S is a union of cosets of a subgroup H, of order $|G|/3m$ $(3m \mid |G|)$, of G, such that one of the following holds:*

(i) $|S+S| = 2|S| - |H|$,

(ii) $|S+S| = 2|S|$ *and* $S \cup (S+S) = G$.

THEOREM 2. *Let S be a maximal sum-free set in $G = Z_{3p}$ such that S is not a coset of H, $H = \{0, 3, 6, \ldots, 3(p-1)\}$; then there exists an automorphism θ of G for which $S = S'\theta$ where $S' = \{p, p+1, \ldots, 2p-1\}$.*

THEOREM 3. *Let G be an elementary abelian 3-group. If S is a maximal sum-free set in G, then S is a coset of a subgroup H, of order $|G|/3$, of G.*

THEOREM 4. *Let* S *be a maximal sum-free set in* $G = Z_3 \oplus Z_3 \oplus Z_p$. *Then either* S *is a union of cosets of* Z_p *and* S/Z_p *is a maximal sum-free set in* G/Z_p *or there exists an automorphism* ϕ *of* G *such that* $S = S'\phi$ *where* S' *is a union of cosets of a subgroup* K *, of order* 3 *, of* G *for which* S'/K *is a maximal sum-free set in* G/K .

Theorem 4 together with Theorems 2 and 3 completely characterize maximal sum-free sets in $G = Z_3 \oplus Z_3 \oplus Z_p$.

Proof of Theorem 3. Let $|G| = 3^n$, $n \geq 2$.

If $x \in S$, then $-x = 2x \notin S$. Thus $-S \cap S = \emptyset$. Also $S \cap (S-S) = \emptyset$ and $-S \cap (S-S) = \emptyset$ imply that $|S-S| \leq 3^n - 2\cdot 3^{n-1} = 3^{n-1}$. But $|S-S| \geq |S| = 3^{n-1}$. Hence $|S-S| = 3^{n-1}$.

By Kneser's Theorem [2, Theorem 1.5], there exists a subgroup H of G such that

$$S - S + H = S - S \quad \text{and} \quad |S-S| \geq |S+H| + |-S+H| - |H| .$$

It is clear that H is a proper subgroup of G .

Suppose that $|H| = 3^m$, $n-2 \geq m \geq 0$. Then $|S-S| \geq 2|S| - |H| > 3^{n-1}$ which is impossible. Consequently, S is a coset of a subgroup H , of order 3^{n-1} , of G .

Proof of Theorem 4. Let $H = H_0$ be a subgroup, of order $3p$, of G . Let $x_1, x_2 \in G$ be such that $x_1 + x_2 = 0$, $2x_1 = x_2$ and $G = H_0 \cup H_1 \cup H_2$, where $H_i = x_i + H$.

Let $x_0 = 0$, $x_i + S_i = S \cap H_i$, $i = 0, 1, 2$. (This method is due to Rhemtulla and Street [3].)

If S is a coset of H , then there is nothing to prove. We assume that $S \neq H_1$ and $S_1 \neq \emptyset$. We know that $|S_0| \leq p$ and we assume that $|S_1| \geq |S_2|$.

Case 1. Suppose that $0 \leq |S_0| < p$.

We first consider the case that $|S_0| > 0$, $|S_2| < p$. From $(S_1-S_1) \cap S_0 = \emptyset$ and $(S_1-S_1) \cup S_0 \subseteq H$, we have $3p \geq |S_0| + |S_1-S_1|$.

By Kneser's Theorem, there exists a subgroup K of G such that $S_1 - S_1 + K = S_1 - S_1$ and $|S_1-S_1| \geq |S_1+K| + |-S_1+K| - |K|$.

It is clear that K is a proper subgroup of H .

If $|K| = p^j$, $j = 0$ or 1 , then

$$3p \geq |S_0| + |S_1| + |S_1+K| - |K|$$
$$\geq |S_0| + |S_1| + \left(\frac{p+v}{p^j}\right]p^j - p^j \quad (|S_1| = p+v > p)$$

where $(x]$ denotes the smallest positive integer $\geq x$. Thus

$$3p \geq |S_0| + |S_1| + p > |S_0| + |S_1| + |S_2| = 3p$$

which is impossible.

If $|K| = 3$, then

$$\begin{aligned} 3p &\geq |S_0| + 2\left(\frac{p+v}{3}\right]3 - 3 && (v = 2p-|S_0|-|S_2|) \\ &= |S_0| + 2(k+(t+1))3 - 3 && (p = 3k+1,\ 3t < v+1 \leq 3(t+1)) \\ &\geq 6p - |S_0| - 2|S_2| - 3 \\ &\geq 6p - 3(p-1) - 3 = 3p . \end{aligned}$$

Hence equality holds good for each of all the above steps. We then have

$$|S_0| = p - 1 = |S_2| , \quad |S_1| = p + 2 ,$$

and

$$S_1 , \quad S_0 = \overline{S_1-S_1} , \quad S_2 = \overline{S_1+S_1}$$

are unions of cosets of K in H . Applying Vosper's Theorem [2, Theorem 1.3] to $S_1/K + S_1/K$ in H/K , we can prove that $S_2 = \overline{S_1+S_1}$.

Suppose that $S_0 = \cup(\alpha_i+K)$, $S_1 = \cup(\beta_i+K)$, $S_2 = \cup(\gamma_i+K)$, $\alpha_i, \beta_i, \gamma_i \in H$. Then

$$S = \left(\cup\alpha_i \cup (x_1+\cup\beta_i) \cup (x_2+\cup\gamma_i)\right) + K$$

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which is what we intend to prove.

For either the case $|S_0| = 0$ and $0 < |S_2| < p$ or the case $|S_2| \geq p$, using $(S_1+S_1) \cap S_2 = \emptyset$, $(S_1+S_1) \cup S_2 \subseteq H$ and applying arguments similar to that given above, we will get a contradiction.

Case 2. When $|S_0| = p$, S_0 is a maximal sum-free set in H. We now write $H = \{0, 1, 2, \ldots, 3p-1\}$.

If S_0 is a coset of $H' = \{0, 3, 6, \ldots, 3(p-1)\}$, then since $|S_1| \geq |S_2|$, we have $|S_1| \geq p$. If $|S_1| > p$, then $|S_0+S_1| \geq 2p$ which contradicts $(S_0+S_1) \cap S_1 = \emptyset$. Hence $|S_1| = p = |S_2|$.

By simple arguments, using $(S_0+S_1) \cap S_1 = \emptyset$, we can show that S_1 is a coset of H'. Similarly, S_2 is also a coset of H'.

Let $S_i = \alpha_i + H'$, $\alpha_i \in H$, $i = 0, 1, 2$. Then

$$S = (\alpha_0+H') \cup (x_1+\alpha_1+H') \cup (x_2+\alpha_2+H')$$
$$= (\alpha_0 \cup (x_1+\alpha_1) \cup (x_2+\alpha_2)) + H',$$

which shows that S is a union of cosets of Z_p and S/Z_p is sum-free in G/Z_p.

If S_0 is not a coset of H', then by Theorem 2, S_0 is isomorphic to $\{p, p+1, \ldots, 2p-1\}$ under the automorphism θ of H given in [7]. We now extend θ to an automorphism ϕ of G by means of the following mapping:

For each $x \in H$, we define

$$(x_i+x)\phi = x_i + x\theta, \quad i = 0, 1, 2.$$

Thus, up to isomorphism, we can write

$$S_0 = \{p, p+1, \ldots, 2p-1\}.$$

Now, if $|S_1| > p$, then applying Kneser's Theorem to $S_1 - S_1$ in H, we will get a contradiction. Hence $|S_1| = |S_2| = p$.

Let $S_1 = \{s_1, s_2, \ldots, s_p\}$, $0 \le s_1 < s_2 < \ldots < s_p \le 3p-1$.

We first consider the case that $s_p - s_1 > p - 1$.

Suppose that $s_{i+1} - s_i \le p$ for every $i = 1, 2, \ldots, p-1$; then from $S_1 \subseteq \overline{S_0+S_1}$, we will get a contradiction. Otherwise, for at most one $i = 1, 2, \ldots, p-1$, $s_{i+1} - s_i > p$, and again from $S_1 \subseteq \overline{S_0+S_1}$, we get another contradiction.

We now consider the case that $s_p - s_1 = p - 1$. We have

$$\left.\begin{array}{l} S_1 = \{\alpha, \alpha+1, \ldots, \alpha+p-1\} \\ S_2 = \{\beta, \beta+1, \ldots, \beta+p-1\} \end{array}\right\} , \quad 0 \le \alpha \le \beta .$$

We can prove that $\alpha + \beta = 2p$ or $\alpha + \beta = 2p + 1$.

The case that $\alpha + \beta = 2p + 1$ cannot occur.

The case that $\alpha + \beta = 2p$ will yield $\alpha = 0$ or $\alpha = p$.

The final results are

(i) $S_2 = S_1 = S_0$ and

(ii) $S_1 = \{0, 1, \ldots, p-1\}$, $S_2 = \{2p, 2p+1, \ldots, 3p-1\}$.

The first case shows that S/Z_3 is a maximal sum-free set in G/Z_3

In the second case, if we write $H = \{0, 1, \ldots, 3p-1\}$ as $\{(0, 0, 0), (0, 0, 1), \ldots, (0, 2, p-1)\}$ and take $x_1 = (1, 0, 0)$, $x_2 = (2, 0, 0)$, then $S = \{(0, 1, 0), (1, 0, 0), (2, 2, 0)\} + K$, $K = \{(0, 0, 0), (0, 0, 1), \ldots, (0, 0, p-1)\}$.

The proof of Theorem 4 is now complete.

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