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A CHARACTERIZATION OF SPACES HAVING BASES OF COUNTABLE ORDER IN TERMS OF PRIMITIVE BASES

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1. Introduction. The main theorem of this paper characterizes the class $\mathscr{B}$ of essentially T_1 spaces having bases of countable order as those spaces of the class $\mathscr{P}$ of essentially T_1 spaces having primitive bases in which closed sets are sets of interior condensation. In addition we deduce some corollaries of this theorem, derive some other characterizations, and prove a lemma concerning primitive sequences which is a key to the proof of the main theorem and has other applications.

The class $\mathscr{B}$ has, in the past decade, been perceived to be a fundamental class of spaces. Some initial reasons for this perception may be found in the following two theorems.

1.1. THEOREM [Arhangel'skiĭ, **1**]. *A T_2 paracompact space is metrizable if and only if it has a base of countable order.*

In [**20**] it was shown that if essentially T_1 is added to base of countable order, then the presence of such a base can be expressed by means of a sequence of covers. This brought out clearly the resemblance to and distinction from developable spaces and showed the anticipation of the concept by N. Aronszajn [**2**] in an axiom (which incorporated some completeness) formulated for the express purpose of proving an arc theorem. A precise connection with developable spaces is given next.

1.2. THEOREM [**20**]. *A space is developable if and only if it is essentially T_1, θ-refinable, and has a base of countable order.*

We think a particularly significant aspect of 1.1 and 1.2 is that there are spaces having bases of countable order which are not even weakly θ-refinable, e.g., the space Ω_0 of countable ordinals with the order topology. Thus 1.1 achieves a factorization of metrizability in which a covering property and a base property are isolated with little obvious overlap; this is in contrast to factorizations involving developability which carries the covering property of subparacompactness with it.

Since [**20**], the theory has undergone generalization to non-first-countable cases and the underlying techniques have been systematized [**7–13**; **19**]. In rough terms, what has been shown is that concepts such as developable space, p-space, and $w\Delta$-space, which involve sequences $\mathscr{G}$ of open covers and arbitrary

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representatives G of $\mathscr{G}$ with nonempty intersection, permit generalization to concepts involving monotonically contracting sequences $\mathscr{G}$ of open covers and their monotone decreasing representatives, i.e., there is a transition from finite intersection type properties to monotone ones. The possibility of such a transition involves techniques illustrated below. The resulting "monotone" theories are more general and, in a sense, more harmonious. The results that have been obtained for $\mathscr{B}$ are typical: invariance, for the regular case (subsequently, pararegular [**14**]), under open, continuous, uniformly monotonically complete mappings [**10**], characterization of the T_1 elements of $\mathscr{B}$ as images of metrizable spaces under such mappings [**10**], a theory of complete T_2 members of $\mathscr{B}$ [**12**], invariance under perfect mappings [**25**], hereditarity, countable productivity, and "local implies global". Many of these theorems have been generalized to non-first-countable cases in [**7**; **8**; **11**; **13**; **22**].

The class $\mathscr{P}$ properly includes $\mathscr{B}$ and has a theory of comparable richness [**15**; **16**; **17**; **23**; **24**]. The fundamental technique of base of countable order theory takes a place in its very definition and an even more general theory is obtained which also has a non-first-countable counterpart [**18**]. The main purpose of this paper is to characterize those members of $\mathscr{P}$ which are also in $\mathscr{B}$. A notable aspect of $\mathscr{P}$ is that it includes both $\mathscr{B}$ and the class of essentially T_1 quasi-developable spaces, or, what is equivalent, by [**4**], the class of essentially T_1 spaces having θ-bases [**20**].

We here analyze the property of membership in $\mathscr{B}$ into two factors:

(1) having a primitive base, and
(2) closed sets are sets of interior condensation.

This may be compared with the two (equivalent) factorizations of developable into (1) quasi-developable, (2) closed sets are G_δ's [**3**], or into (1′) θ-base, and (2) [**20**].

Section 2 provides information on primitive bases and sets of interior condensation, respectively. We prove a key lemma in Section 3 and the main theorem in Section 4. Section 5 contains another point of view concerning the distinction between developable spaces and the members of $\mathscr{B}$ and $\mathscr{P}$.

Our set-theoretic usage is close to that of [**5**]. The letter N denotes the set of natural numbers. We use $<$ and $\leqq$ ambiguously to denote orderings whose fields are contextually clear. We frequently use single letters to denote sequences. Given a sequence Γ, by a *representative of* Γ we mean a sequence Δ such that $\Delta_n \in \Gamma_n$ for all $n \in N$. A *decreasing representative* Δ is one such that $\Delta_n \supseteq \Delta_{n+1}$ for all $n \in N$. A space is *essentially* T_1 [**20**] if and only if for all $x, y \in X$, $x \in \{\bar{y}\}$ implies $y \in \{\bar{x}\}$. For other concepts not defined here see [**7**; **12**; **20**; **22**].

2. Primitive base, primitive sequences, sets of interior condensation.

In this section we define *primitive base*, some terminology of primitive sequence theory, and *sets of interior condensation*. We also state several theorems involving these concepts, some of which we apply in the proof of the main theorem.

2.1. *Definition* [**21**]. A space X has a *primitive base* if and only if there exists a sequence $\mathscr{W}$ of well-ordered collections of open sets such that for each $x \in X$, if U is open and $x \in U$, then there exist $k \in N$ and $n \in N$ such that x belongs to at least n elements of $\mathscr{W}_k$ and the n-th such element is a subset of U.

2.2. THEOREM [**17**]. *Spaces having bases of countable order and quasi-developable spaces (equivalently* [**4**], *spaces with θ-bases) have primitive bases.*

2.3. *Example* [**17**]. The topological sum and topological product of the Michael line [**6**, p. 90], and the space of countable ordinals with the order topology are spaces having a primitive base but neither a base of countable order nor a θ-base.

2.4. *Definition* [**17**]. Suppose $(\mathscr{Z}, \leqq)$ is a well-ordered collection of sets. For each $W \in \mathscr{Z}$ let

$$p(W, \mathscr{Z}) \text{ denote } \{x \in W\text{: if } W' \in \mathscr{Z} \text{ and } W' < W, \text{ then } x \notin W'\}.$$

2.5 *Definition* [**11**]. Suppose X is a set and $M \subseteq X$. A *primitive sequence of M in X* is a sequence $\mathscr{H}$ of well-ordered collections of subsets of X which cover M such that for each $n \in N$:

(a) For all $H \in \mathscr{H}_n$, $M \cap p(H, \mathscr{H}_n) \neq \emptyset$.
(b) If $j < n$ and $M \cap p(H, \mathscr{H}_n) \cap p(H', \mathscr{H}_j) \neq \emptyset$, then $H \subseteq H'$.

In case $M = X$, $\mathscr{H}$ is called a *primitive sequence* of X.

2.6 *Definition* [**17**]. If (X, τ) is a space and $M \subseteq X$, then an *open primitive sequence of M in X* is a primitive sequence $\mathscr{H}$ of M in X such that each $\mathscr{H}_n \subseteq \tau$.

2.7 *Definition.* Let $\mathscr{H}$ be a primitive sequence of M in X. A *primitive representative* of $\mathscr{H}$ is a sequence H such that for all $n \in N$,

$$p(H_n, \mathscr{H}_n) \cap p(H_{n+1}, \mathscr{H}_{n+1}) \cap M \neq \emptyset.$$

Notation. If $\mathscr{H}$ is a primitive sequence of M in X, $\mathrm{PR}(\mathscr{H})$ denotes

$$\{H\text{: } H \text{ is a primitive representative of } \mathscr{H}\}.$$

2.8 *Definition* [**17**]. If $\mathscr{H}$ is a primitive sequence, then for all $H \in \mathrm{PR}(\mathscr{H})$, $\mathrm{pc}(H)$ denotes $\bigcap_{n \in N} p(H_n, \mathscr{H}_n)$. This set is called the *primitive core* of $\mathscr{H}$.

2.9 THEOREM [**17**]. *A topological space is essentially T_1 and has a primitive base if and only if it has an open primitive sequence $\mathscr{H}$ such that for all $H \in \mathrm{PR}(\mathscr{H})$, if $\mathrm{pc}(H) \neq \emptyset$, then $\{H_n\text{: } n \in N\}$ is a base at each element of $\bigcap_{n \in N} H_n$.*

2.10 *Definition.* A sequence $\mathscr{H}$ related to a space X as in 2.9, will be called a *primitive sequence (of X) of basic type.*

2.11 THEOREM [**20**]. *A space* $X \in \mathscr{B}$ *if and only if there exists an open primitive sequence* $\mathscr{H}$ *of* X *such that for all* $H \in \mathrm{PR}(\mathscr{H})$, *if* $\bigcap_{n \in N} H_n \neq \emptyset$, *then* $\{H_n : n \in N\}$ *is a base at each point of* $\bigcap_{n \in N} H_n$.

2.12 *Definition* [**7**; **11**]. A subset M of a space X is a *set of interior condensation in* X if and only if there exists a sequence $\mathscr{A}$ of collections of open sets covering M such that:

(1) For all $n \in N$ and $x \in M$, if $x \in A \in \mathscr{A}_n$, then there exists $A' \in \mathscr{A}_{n+1}$ such that $x \in A' \subseteq A$, and

(2) if for each $n \in N$, $A_n \in \mathscr{A}_n$ and $A_{n+1} \subseteq A_n$, then $\bigcap_{n \in N} A_n \subseteq M$.

The next theorem is proved in [**19**]. A proof may be constructed using the proof of Theorem 1 of [**20**] as a guide.

2.13 THEOREM. *Suppose* M *is a subspace of a space* X *such that each* $x \in M$ *is in an open set* U *such that* $U \cap M$ *is a set of interior condensation in* X. *Then* M *is a set of interior condensation in* X, *i.e., sets of interior condensation locally are sets of interior condensation globally.*

2.14 THEOREM. *A subspace* M *of a space* X *is a set of interior condensation in* X *if and only if there exists an open primitive sequence* $\mathscr{H}$ *of* M *in* X *such that for all* $H \in \mathrm{PR}(\mathscr{H})$, $\bigcap_{n \in N} H_n \subseteq M$.

Proof. By Lemma 2.1 of [**12**] and Definition 2.12, there exists an $\mathscr{H}$ with the property described. On the other hand, if such an $\mathscr{H}$ exists, apply Lemma 2.3 of [**12**] to obtain an $\mathscr{A}$ satisfying the conditions of 2.12.

2.15 THEOREM. *If* $\langle A_n : n \in N \rangle$ *is a sequence of sets of interior condensation in a space* X, *then* $\bigcap_{n \in N} A_n$ *is a set of interior condensation in* X.

Proof. See 3.3.

3. A key lemma. The lemma we prove here is basic to the proof of 4.1. The result needed for 4.1 is Corollary 3.2. Another application is 3.3.

3.1. LEMMA. *Suppose* $\mathscr{W}$ *is a primitive sequence of* M *in* X. *Suppose that* $\langle f_n : n \in N \rangle$ *is a sequence of functions such that for each* $n \in N$:

(1) *the domain of* f_n *is* $\mathscr{W}_n$ *and its range is the power set of* X,

(2) *for each* $W \in \mathscr{W}_n$, $M \cap p(W, \mathscr{W}_n) \subseteq f_n(W) \subseteq p(W, \mathscr{W}_n)$, *and*

(3) *for each* $W \in \mathscr{W}_n$, *there exists a primitive sequence* $\mathscr{V}$ *of* $f_n(W)$ *in* W *such that for all* $V \in \mathrm{PR}(\mathscr{V})$, $\bigcap \{V_j : j \in N\} \subseteq f_n(W)$.

Then there exists a primitive sequence $\mathscr{H}$ *of* M *in* X *such that for all* $H \in \mathrm{PR}(\mathscr{H})$ *there is a* $W \in \mathrm{PR}(\mathscr{W})$ *such that for each* $n \in N$,

$$H_n \subseteq W_n \text{ and } \bigcap \{H_n : n \in N\} \subseteq \bigcap \{f_n(W_n) : n \in N\}.$$

If X is a space and the primitive sequences involved in the above hypothesis are open in X, then $\mathscr{H}$ is also open in X.

Proof. We introduce some notation: If $(A_1, \ldots, A_j)$ is a j-tuple of sets let $\alpha(A_1, \ldots, A_j)$ denote $\bigcap \{A_i: 1 \leqq i \leqq j\}$. If each $A_i \in \mathscr{L}_i$, where $\mathscr{L}_i$ is well-ordered, let $\alpha\bar{p}(A_1, \ldots, A_j)$ denote

$$\alpha(p(A_1, \mathscr{L}_1), \ldots, p(A_j, \mathscr{L}_j))$$

(in $\alpha\bar{p}(A_1, \ldots, A_j)$ the appropriate $\mathscr{L}_i$'s are contextually understood). For each $W \in \mathscr{W}_n$, let $\langle \mathscr{H}(W, n, i): i \in N\rangle$ denote a primitive sequence of $f_n(W)$ in W with the property of (3) of 3.1. Let $n \in N$. Define $\mathscr{C}_n$ as

$$\{(W_1, \ldots, W_n): \text{for all } i \leqq n, W_i \in \mathscr{W}_i \text{ and } \alpha\bar{p}(W_1, \ldots, W_n) \cap M \neq \emptyset\}.$$

Well-order $\mathscr{C}_n$ by the lexicographic order on $\mathscr{W}_1 \times \ldots \times \mathscr{W}_n$. For each $(W_1, \ldots, W_n) \in \mathscr{C}_n$, let

$$\mathscr{D}(W_1, \ldots, W_n) = \{(W_{1,n}, \ldots, W_{n,1}): W_{ij} \in \mathscr{H}(W_i, i, j) \text{ for } 1 \leqq i, j \leqq n \text{ and } i + j = n + 1 \text{ such that } \alpha\bar{p}(W_{1,n}, \ldots, W_{n,1}) \cap M \neq \emptyset\}.$$

Well-order each $\mathscr{D}(W_1, \ldots, W_n)$ by the lexicographic order on $\mathscr{H}(W_1, 1, n) \times \ldots \times \mathscr{H}(W_n, n, 1)$. Let $\mathscr{E}_n = \{(A, B): A \in \mathscr{C}_n \text{ and } B \in \mathscr{D}(A)\}$, again well-ordered lexicographically.

The function $\alpha|\mathscr{E}_n$ is an injection on $\mathscr{E}_n$. For suppose $\alpha(A_1, B_1) = \alpha(A_2, B_2)$, where $A_k = (W_1^k, \ldots, W_n^k)$ and $B_k = (W_{1,n}^k, \ldots, W_{n,1}^k)$ for $k = 1, 2$. Then $\alpha(A_k, B_k) \supseteq \alpha\bar{p}(A_j, B_j)$ for $(k, j) = (1, 2)$ and $(k, j) = (2, 1)$. Thus each $W_i^1 = W_i^2$ and each $W_{ij}^1 = W_{ij}^2$.

For each $n \in N$ we define $\mathscr{H}_n$ as the range of $\alpha|\mathscr{E}_n$, well-ordered by $H \leqq H'$ if and only if $\alpha^{-1}(H) \leqq \alpha^{-1}(H')$. If $x \in M$, there exists $(A, B) \in \mathscr{E}_n$ such that $x \in \alpha\bar{p}(A, B)$. For if $x \in M$, there is, for each $k \leqq n$, a $W_k \in \mathscr{W}_k$ with $x \in \alpha\bar{p}(W_1, \ldots, W_n) \cap M$. Thus, $x \in f_k(W_k)$ for all $k \leqq n$. Hence there is a $B \in \mathscr{D}(W_1, \ldots, W_n)$ such that $x \in \alpha\bar{p}(A, B)$. Therefore $x \in \alpha(A, B) \in \mathscr{H}_n$. If $x \in \alpha(A_1, B_1)$, then $\alpha\bar{p}(A, B) \cap \alpha(A_1, B_1) \neq \emptyset$. Hence $(A, B) \leqq (A_1, B_1)$. Thus $x \in p(\alpha(A, B), \mathscr{H}_n)$. If $H \in \mathscr{H}_n$, there exists $(A, B) \in \mathscr{E}_n$ such that $H = \alpha(A, B)$. Since $\alpha\bar{p}(A, B) \neq \emptyset$, condition (a) of 2.5 is satisfied. Note that:

(*) $p(\alpha(A, B), \mathscr{H}_n) \cap M = \alpha\bar{p}(A, B) \cap M$ for all $(A, B) \in \mathscr{E}_n$.

Suppose

$$x \in M \cap p(H, \mathscr{H}_n) \cap p(H', \mathscr{H}_{n+1}), \text{ where } H = \alpha((W_1, \ldots, W_n), (W_{1,n}, \ldots, W_{n,1})) \text{ and } H' = \alpha((W_1', \ldots, W_n', W_{n+1}), (W_{1,n+1}', \ldots, W_{n+1,1}')).$$

By (*), $W_i' = W_i$ for $1 \leqq i \leqq n$ and $W_{jk} \supseteq W_{j,k+1}'$ for $1 \leqq j, k \leqq n$ and $j + k = n + 1$. Thus $H' \subseteq H$. Therefore $\mathscr{H}$ is a primitive sequence of M in X.

Suppose $H \in \mathrm{PR}(\mathscr{H})$, where for each $n \in N$,

$$H_n = \alpha((W_1^n, \ldots, W_n^n), (W_{1,n}^n, \ldots, W_{n,1}^n))$$

and each $W_{ij}^n \in \mathscr{H}(W_i^n, i, j)$. Thus

$$M \cap p(W_n^n, \mathscr{W}_n) \cap p(W_{n+1}^{n+1}, \mathscr{W}_{n+1}) \neq \emptyset$$

by the definition of $\mathscr{H}_{n+1}$. As was seen in the preceding paragraph, $W_k^n = W_k^{n+1}$ for all $1 \leqq k \leqq n$ and $n \in N$. Let W_n denote W_n^n. Then $W_i^n = W_i$ for all $i \leqq n$, and we have, for each $n \in N$, $W_{ij}^n \in \mathscr{H}(W_i, i, j)$ for $1 \leqq i, j \leqq n$ and $i + j = n + 1$. Moreover, $H \in \mathrm{PR}(\mathscr{H})$ and (*) imply that

$$p(W_{ij}^n, \mathscr{H}(W_i, i, j)) \cap p(W_{i,j+1}^{n+1}, \mathscr{H}(W_i, i, j+1)) \neq \emptyset.$$

Thus $\langle W_{i,j}^{i+j-1}: j \in N\rangle \in \mathrm{PR}(\langle \mathscr{H}(W_i, i, j): j \in N\rangle)$. Thus

$$x \in \bigcap \{H_n: n \in N\}$$

implies that

$$x \in \bigcap \{W_{i,j}^{i+j-1}: j \in N\} \subseteq f_i(W_i) \text{ for all } i \in N.$$

Therefore $\bigcap \{H_n: n \in N\} \subseteq \bigcap \{f_n(W_n): n \in N\}$. Since the elements of each $\mathscr{H}_n$ are finite intersections of elements of primitive sequences, the last statement of the conclusion is valid.

3.2. COROLLARY. *Suppose X is a space and $\mathscr{W}$ is an open primitive sequence of X such that for each $n \in N$ and $W \in \mathscr{W}_n$, $p(W, \mathscr{W}_n)$ is a set of interior condensation in X. Then there exists an open primitive sequence $\mathscr{H}$ in X such that for each $H \in \mathrm{PR}(\mathscr{H})$ there is a $W \in \mathrm{PR}(\mathscr{W})$ such that $H_n \subseteq W_n$ for all $n \in N$ and $\bigcap \{H_n: n \in N\} \subseteq \mathrm{pc}(W)$.*

Proof. In the hypothesis of Lemma 3.1, let $f_n(W) = p(W, \mathscr{W}_n)$ for all $W \in \mathscr{W}_n$. By 2.14, each $f_n(W)$ has an associated open primitive sequence in W. Thus an open primitive sequence $\mathscr{H}$ of X exists such that if $H \in \mathrm{PR}(\mathscr{H})$, there is $W \in \mathrm{PR}(\mathscr{W})$ such that each $H_n \subseteq W_n$ and

$$\bigcap \{H_n: n \in N\} \subseteq \bigcap \{p(W_n, \mathscr{W}_n): n \in N\} = \mathrm{pc}(W).$$

3.3. *Proof of* 2.15. Let $M = \bigcap \{A_n: n \in N\}$. For each $n \in N$ let $\mathscr{W}_n = \{X\}$ considered as a well-ordered, one-element set. Let $f_n(X) = A_n$ for each $n \in N$. There exists an open primitive sequence of $f_n(X)$ in X by 2.14. By 3.1 there is an open primitive sequence $\mathscr{H}$ of M in X satisfying the conclusions of 3.1. The only member of $\mathrm{PR}(\mathscr{W})$ is W where $W_n = X$ for all $n \in N$. If $H \in \mathrm{PR}(\mathscr{H})$, then

$$\bigcap \{H_n: n \in N\} \subseteq \bigcap \{f_n(W_n): n \in N\} = M.$$

Thus M is a set of interior condensation in X by 2.14.

4. The main theorem.

4.1. THEOREM. *A space is essentially T_1 and has a base of countable order if and only if it has a primitive base and closed sets are sets of interior condensation locally.*

Proof. Necessity. If $X \in \mathscr{B}$, then X has a primitive base by 2.1 or 2.2. It is easy to establish that closed sets are sets of interior condensation.

Sufficiency. Let X satisfy the condition. Closed sets in X are sets of interior condensation by 2.12. Suppose $x, y \in X$ and $x \in \{\bar{y}\}$. Since $\{\bar{x}\}$ is a set of interior condensation, there exists a sequence $\mathscr{A}$ related to $\{\bar{x}\} = M$ as in 2.11. There exists a decreasing representative A of $\mathscr{A}$ such that $x \in \bigcap \{A_n: n \in N\}$. Since $x \in \{\bar{y}\}$, $y \in \bigcap \{A_n: n \in N\}$. Thus $y \in \{\bar{x}\}$. Therefore X is essentially T_1. By Theorem 2.9, there exists an open primitive sequence $\mathscr{W}$ of basic type for X. Suppose $W \in \mathscr{W}_n$. Then

$$p(W, \mathscr{W}_n) = W \backslash \bigcup \{W' \in W_n: W' < W\}.$$

Hence $p(W, \mathscr{W}_n)$ is the intersection of an open set and a closed set. Since open sets are obviously sets of interior condensation, $p(W, \mathscr{W}_n)$ is also, by 2.15. Therefore there exists an open primitive sequence $\mathscr{H}$ of X related to $\mathscr{W}$ as in 3.2. Suppose $H \in \mathrm{PR}(\mathscr{H})$ and $x \in \bigcap \{H_n: n \in N\}$. There exists $W \in \mathrm{PR}(\mathscr{W})$ such that $H_n \subseteq W_n$ for all $n \in N$ and $x \in \mathrm{pc}(W)$. Thus $\{W_n: n \in N\}$ is a base at x and, therefore $\{H_n: n \in N\}$ is also. By 2.11, $X \in \mathscr{B}$.

4.2. THEOREM. *A space is essentially T_1 and has a base of countable order if and only if it has a primitive sequence $\mathscr{W}$ of basic type such that for all $n \in N$ and $W \in \mathscr{W}_n$, $p(W, \mathscr{W}_n)$ is a set of interior condensation in W.*

Proof. This follows from 2.10 and the proof of 4.1.

4.3. THEOREM. *A space is developable if and only if it is essentially T_1, θ-refinable, has a primitive base, and closed sets are sets of interior condensation locally.*

Proof. This follows from 4.1 and 1.2.

4.4. THEOREM. *A space is metrizable if and only if it is T_2 paracompact, has a primitive base, and closed sets are sets of interior condensation locally.*

Proof. This follows from 4.1 and 1.1.

In view of 4.2, it seems natural to ask whether a space having a primitive sequence $\mathscr{W}$ of basic type has a base of countable order if for all $W \in \mathrm{PR}(\mathscr{W})$, $\mathrm{pc}(W)$ is a set of interior condensation. The Michael line [**6**, p. 90] provides a counterexample, since singletons are G_δ's and the space has a primitive sequence $\mathscr{W}$ of basic type such that for all $W \in \mathrm{PR}(\mathscr{W})$, $\mathrm{pc}(W) \neq \emptyset$ implies that $\mathrm{pc}(W)$ is a singleton.

5. Sets of interior condensation uniformly. In this section we define uniform notions of a space having closed sets G_δ's or sets of interior condensation in order to view a different aspect of the base of countable order-developable distinction. We also define primitive set of interior condensation to bring bring out a distinction from primitive base. Similar approaches may be made to other concepts as in the final note.

5.1. *Definition*. Suppose M is a set. If $\mathscr{C}$ is a collection of sets we define $\mathscr{C}^M = \{A \in \mathscr{C}: A \cap M\} \neq \emptyset$.

5.2. *Definition*. If X is a space, and $M \subseteq X$, a G_δ*-sequence for* M *in* X is a sequence $\mathscr{D}$ of X-open covers of M such that if D is a representative of $\mathscr{D}$, then $\bigcap \{D_n: n \in N\} \subseteq M$.

An SIC-*sequence for* M *in* X is a sequence $\mathscr{A}$ related to M as in 2.12.

We note that a set is a G_δ in a space if and only if it has a G_δ-sequence in the space.

5.3. *Definition*. If X is a space, then *closed sets are* G_δ *sets uniformly in* X if and only if there exists a sequence $\mathscr{G}$ of open covers of X such that for all closed $M \subseteq X$ the sequence $\langle \mathscr{G}_n{}^M: n \in N\rangle$ is a G_δ-sequence for M.

Closed sets are sets of interior condensation uniformly in X if and only if there exists a monotonically contracting sequence $\mathscr{A}$ of open covers of X such that for all closed $M \subseteq X$, $\langle \mathscr{A}_n{}^M: n \in N\rangle$ is an SIC-sequence of M in X. (A monotonically contracting, [**7**; **22**], sequence $\mathscr{A}$ is one such that $x \in A \in \mathscr{A}_n$ implies the existence of $A' \in \mathscr{A}_{n+1}$ with $x \in A' \subseteq A$ for all $n \in N$.)

5.4. THEOREM. *A space is developable if and only if closed sets are* G_δ *sets uniformly in the space.*

Proof. The necessity may be proved by using a development $\mathscr{G}$. Suppose $\mathscr{G}$ is a sequence for a space X as in 5.3. If $x \in X$ is in an open set U, then $M = X \backslash U$ has a G_δ-sequence $\langle \mathscr{G}_n{}^M: n \in N\rangle$. For some $n \in N$, $\operatorname{st}(x, \mathscr{G}_n) \subseteq U$. Otherwise there is a representative G of $\langle \mathscr{G}_n{}^M: n \in N\}$ such that $x \in \bigcap \{G_n: n \in N\}$ and thus $x \notin U$.

5.5. THEOREM. *A space is essentially* T_1 *and has a base of countable order if and only if closed sets are sets of interior condensation uniformly in the space.*

Proof. Similar to the proof of 5.4.

5.6. *Definition*. If X is a space and $M \subseteq X$, then M is a *primitive set of interior condensation* in X if and only if there is an open primitive sequence $\mathscr{W}$ of M in X such that if $W \in \mathrm{PR}(\mathscr{W})$ and $\mathrm{pc}(W) \neq \emptyset$, then $\bigcap \{W_n: n \in N\} \subseteq M$.

5.7. *Definition*. *Closed sets are primitive sets of interior condensation uniformly in a space* X if and only if there exists an open primitive sequence $\mathscr{H}$ of X such that for all closed $M \subseteq X$, $\langle \mathscr{H}_n{}^M: n \in N\rangle$ is related to M as $\mathscr{A}$ is in 5.6.

5.8. THEOREM. *A space is essentially* T_1 *and has a primitive base if and only if closed sets are primitive sets of condensation uniformly in the space.*

Proof. The necessity follows from the existence of a sequence for the space as in 2.9. Suppose there is a sequence $\mathscr{H}$ for a space X as in 5.7. Suppose $H \in \mathrm{PR}(\mathscr{H})$, $\mathrm{pc}(H) \neq \emptyset$, and $x \in \bigcap \{H_n: n \in N\}$. If U is open and $x \in U$, let $M = X \backslash U$. If $H_n \cap M \neq \emptyset$ for all $n \in N$, then $H \in \mathrm{PR}(\langle \mathscr{H}_n{}^M: n \in N\rangle)$.

Since $\mathrm{pc}(H) \neq \emptyset$, it follows that $\bigcap \{H_n: n \in N\} \subseteq M$, a contradiction. Thus some $H_n \subseteq U$. By 2.9, the proof is complete.

Note. If we replace closed sets by points in the preceding discussion we obtain analogous characterizations of the concepts of G_δ-diagonal, diagonal a set of interior condensation [**9**], and diagonal a primitive set of interior condensation. Extensions may be made to non first countable analogues of developable spaces and spaces having base of countable order as well. We describe the foundations of a theory of primitive structure for non first countable spaces in [**18**].

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