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# ON LARGE COHOMOLOGICAL DIMENSION AND TAUTNESS

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**Abstract**

We prove that for a non-discrete space $X$, the inequality $\mathrm{Dim}_L(X) \geqslant \dim_L(X) + 1$ always holds if (i) $X$ is paracompact and each point is $G_\delta$, or (ii) $X$ is a completely paracompact Morita $k$-space. Consequently, if $X$ is a non-discrete completely paracompact space in which each point is a $G_\delta$-set or it is also a Morita $k$-space then, the equality $\mathrm{Dim}_L(X) = \dim_L(X) + 1$ always holds. We apply this equality to show that for such a space $X$ there exists a point $x \in X$ and a family $\varphi$ of supports on $X$ such that $\{x\}$ is not $\varphi$-taut with respect to sheaf cohomology. This generalizes a corresponding known result for $\mathbf{R}^n$. We also discuss the usual sum theorems for this large cohomological dimension; the finite sum theorem for closed sets is proved, and for all others, counter examples are given. Subject to a small modification, however, all of the sum theorems hold for a large class of spaces.

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## 1. Introduction

Suppose $L$ is any given ring, $\dim_L(X)$ and $\mathrm{Dim}_L(X)$ denote the *cohomological dimension* and *large cohomological dimension* of a space $X$ over $L$ respectively. It was proved in [3] that if $X$ is a $n$-manifold, $n \geqslant 1$, then $\mathrm{Dim}_L(X) = \dim_L(X) + 1$. As a matter of fact if $X$ is completely paracompact, then it was already proved in [1] that $\mathrm{Dim}_L(X) = \dim_L(X)$ or $\mathrm{Dim}_L(X) = \dim_L(X) + 1$, and the problem as to which one it is, was settled in [2] and [3] for manifolds. The general question as to whether or not the equality $\mathrm{Dim}_L(X) = \dim_L(X) + 1$ holds for any completely paracompact space, still remains open. One objective of this paper is to show that *if $X$ is a non-discrete locally paracompact space in which each point is a $G_\delta$-set, then*

$$\mathrm{Dim}_L(X) \geqslant \dim_L(X) + 1.$$

Under the assumption that $X$ is completely paracompact, first countable and non-discrete this inequality was first proved by Kuźminov-Švedov [11] using their concept of $\dim_{\mathcal{A}}(X)$ for a given sheaf $\mathcal{A}$ on $X$. In this generality their argument, however, makes essential use of completely paracompactness. We improve upon their technique and generalize the above inequality to non-discrete paracompact space in which points are $G_\delta$-sets. Consequently, if $X$ is non-discrete, completely paracompact and each point is $G_\delta$, then the equality $\mathrm{Dim}_L(X) = \dim_L(X) + 1$ always holds. We can completely disregard discrete spaces because it was proved in [5] that: an arbitrary locally paracompact space $X$ is discrete if and only if $\mathrm{Dim}_{\mathbf{Z}}(X) = \dim_{\mathbf{Z}}(X) = 0$. *We also observe that the above equality remains valid if $X$ is assumed to be a completely paracompact Morita $k$-space.* Since a paracompact locally compact space $X$ is always a Morita $k$-space, this further generalizes the result of [11]. All of this has been done in Section 3.

In Section 4, we discuss all forms of sum theorems for the large cohomological dimension $\mathrm{Dim}_L(X)$: this turns out to be interesting, because certain forms of sum theorems hold for $\mathrm{Dim}_L(X)$, whereas other forms don't-we present counter-examples for these.

Finally in Section 5, we apply the equality $\mathrm{Dim}_L(X) = \dim_L(X) + 1$ to show that *if $X$ is a locally completely paracompact space of finite cohomological dimension in which either points are $G_\delta$ or $X$ is also a Morita $k$-space, then there exists a point $x \in X$ and a family $\varphi$ of supports on $X$ such that $\{x\}$ is not $\varphi$-taut in $X$ with respect to sheaf cohomology.* This generalizes as well as specializes a result of [6], where it has been proved that if $X = \mathbf{R}^n$, $1 \leqslant n < \infty$, then for each point $x \in X$ there exists a family $\varphi$ of supports on $X$ such that $\{x\}$ is not $\varphi$-taut in $X$. This shows that, in general, a retract of such a space $X$ need not be $\varphi$-taut with respect to sheaf cohomology for an arbitrary family $\varphi$ of supports on $X$. In contrast to this, any neighbourhood retract of an arbitrary space is taut with respect to Alexander-Spanier cohomology [4].

## 2. Preliminaries

Unless otherwise stated, all our notations are standard [1]. By a space $X$ we mean a Hausdorff space. Let $X$ be a space, $\varphi$ be a family of supports and $L$ be a ring. The largest integer $n$ (or $\infty$) for which there exists a sheaf $\mathcal{Q}$ of $L$-modules on $X$ such that the Grothendieck cohomology group $H^n_{\varphi,L}(X; \mathcal{Q}) \neq 0$ is called the *cohomological* $\varphi$*-dimension* ($\dim_{\varphi,L}(X)$) of $X$ over the ring $L$. The extent $E(\varphi)$ of a family of supports $\varphi$ is defined to be the union of all members of $\varphi$. It is then known that if $\varphi$, $\psi$ are two paracompactifying families of supports on $X$ such that $E(\varphi) \subset E(\psi)$, then $\dim_{\varphi,L}(X) \leqslant \dim_{\psi,L}(X)$. It follows, therefore, that if $X$ admits a paracompactifying family $\varphi$ of supports such that $E(\varphi) = X$, then one can unambiguously define the *cohomological dimension* ($\dim_L(X)$) of $X$ over $L$ to be $\dim_{\varphi,L}(X)$. Locally paracompact space $X$ admits of such a family of supports and hence for such a space $X$, $\dim_L(X)$ is always defined.

Now the following is obvious:

LEMMA 2.1. *Suppose* $\varphi$ *is a family of supports on a locally paracompact space* $X$. *If* $\dim_L(X) = n$ *and there is a sheaf* $\mathcal{Q}$ *of L-modules on X such that* $H^p_\varphi(X; \mathcal{Q}) \neq 0$ *for some* $p > n$, *then* $\varphi$ *is not paracompactifying.*

The dimension function $\dim_L$ is monotone for locally closed subsets of $X$. Further, $\dim_L(X)$ is a local concept in the sense that [1, page 74]: $\dim_L(X) \leqslant n$ if and only if for each $x \in X$ there is a locally closed neighbourhood $N(x)$ of $x$ such that $\dim_L(N(x)) \leqslant n$. It follows that if $\{V_\alpha\}$ is a family of open sets covering $X$, then each $V_\alpha$ is locally paracompact and

$$\dim_L(X) = \sup_\alpha \{\dim_L(V_\alpha)\}.$$

The *large cohomological dimension* ($\operatorname{Dim}_L(X)$) of a space $X$ over the ring $L$ is defined to be [1, page 110] the supremum of all $\varphi$-dimensions of $X$ when $\varphi$ runs over all the families of supports on $X$, that is,

$$\operatorname{Dim}_L(X) = \sup_\varphi \{\dim_{\varphi,L}(X)\}.$$

It is easy to verify that, $\operatorname{Dim}_L(X)$ is also monotone for locally closed subsets of $X$. However, the local property for $\operatorname{Dim}_L(X)$ is slightly different [*ibid*]: $\operatorname{Dim}_L(X) \leqslant n$ if and only if for each $x \in X$ there exists an open neighbourhood $U(x)$ of $x$ such that $\operatorname{Dim}_L(U(x)) \leqslant n$. It is obvious that if $X$ is discrete then for all non-trivial rings $L$, $\operatorname{Dim}_L(X) = \dim_L(X) = 0$, and it was shown in [5] that if $X$ is non-discrete then $\operatorname{Dim}_{\mathbf{Z}}(X) > 0$.

Recall that, a space $X$ is said to have Morita weak topology with respect to a family $\{C_\alpha\}$ of closed subsets of $X$ if (i) $X = \bigcup C_\alpha$, and

(ii) for each subfamily $\{C_\beta\}$ of $\{C_\alpha\}$, $\bigcup C_\beta$ is closed in $X$, and a subset $A$ of $\bigcup C_\beta$ is closed in $\bigcup C_\beta$ if and only if $A \cap C_\beta$ is closed in $C_\beta$ for each $\beta$.

A space $X$ which has Morita weak topology with respect to a family $\{K_\alpha\}$ of compact subsets of $X$, will be called a *Morita k-space*.

Note that each $CW$-complex is a Morita $k$-space. A locally compact paracompact space is also a Morita $k$-space: the converse, however, is not in general true; for example, any $CW$-complex which is not locally finite—such a space is a Morita $k$-space but it can not be locally compact.

Suppose $d$ is a dimension function defined for topological spaces. The various forms of sum theorems for the dimension function $d$ [12] are the following:

COUNTABLE SUM THEOREM. *Suppose $\{F_i \mid i \in I\}$ is a countable family of closed subsets of a space X. Then*

$$d\Big(\bigcup_{i\in I} F_i\Big) = \sup\{d(F_i) \mid i \in I\}.$$

If the indexing set $I$ is finite then we shall call the above theorem as *Finite sum theorem*.

LOCALLY FINITE SUM THEOREM. *Suppose $\{F_\alpha \mid \alpha \in I\}$ is a locally finite family of closed subsets of a space X. Then*

$$d\Big(\bigcup_{\alpha\in I} F_\alpha\Big) = \sup\{d(F_\alpha) \mid \alpha \in I\}.$$

MORITA WEAK TOPOLOGY SUM THEOREM. *Suppose $\bigcup_{\alpha\in I} F_\alpha$ has the Morita weak topology defined by a family $\{F_\alpha \mid \alpha \in I\}$ of closed subsets of X. Then*

$$d\Big(\bigcup_{\alpha\in I} F_\alpha\Big) = \sup\{d(F_\alpha) \mid \alpha \in I\}.$$

DISJOINT SUM THEOREM. *Suppose F is a closed subset of a space X. If $d(F) \leqslant n$ and $d(A) \leqslant n$ for every closed subset A of X disjoint from F, then $d(X) \leqslant n$.*

COMPLEMENTARY SUM THEOREM. *Suppose F is a closed subset of X. Then*

$$d(X) = \max\{d(F), d(X\setminus F)\}.$$

The above sum theorems need not necessarily hold for a given dimension function $d$. It is obvious that if a dimension function $d$ satisfies the local property then, for such a $d$ the finite sum theorem and the locally finite sum theorem are equivalent. It is, however, easy to verify that the disjoint sum theorem and the complementary sum theorem are equivalent for $\mathrm{Dim}_L$ (and also for $\dim_L$), because of the local and monotone behaviour of these dimension functions. That all of the above forms of sum theorems hold for $\dim_L$ has been proved in [7].

One can also establish the following:

LEMMA 2.2. *Let* $\{F_\alpha \mid \alpha \in \Lambda\}$ *be a locally finite family of disjoint closed sets of a space* $X$ *and for each* $\alpha$, *let* $\varphi_\alpha$ *be a family of supports on* $F_\alpha$. *If we define*

$$\Phi = \left\{ G \subset X \middle| G = \bigcup_{\alpha \in \Lambda} A_\alpha \text{ such that } A_\alpha \in \varphi_\alpha \text{ for each } \alpha \right\},$$

*then* $\Phi$ *is a family of supports on* $X$ *and for each sheaf* $\mathcal{Q}$ *of L-modules on* $\bigcup F_\alpha$, *there is a natural isomorphism*

$$H^*_\Phi\left( \bigcup F_\alpha; \mathcal{Q} \right) \approx \prod_{\alpha \in \Lambda} H^*_{\varphi_\alpha}(F_\alpha; \mathcal{Q}).$$

We shall also require the following well-known results:

($R_1$). If $F \subset X$ is a closed, then there is a natural isomorphism

$$H^*_\varphi(X, F; \mathcal{Q}) \approx H^*_\varphi(X; \mathcal{Q}_{X \setminus F}),$$

valid for any sheaf $\mathcal{Q}$ on $X$ and any family $\varphi$ of supports on $X$.

($R_2$). Suppose $\varphi$ is an arbitrary family of supports on $X$ and $A \subset X$ is closed. Then there is a natural isomorphism

$$H^*_\varphi(X; \mathcal{Q}_A) \approx H^*_{\varphi|A}(A; \mathcal{Q})$$

of functors of sheaves on $X$.

($R_3$). If $A$ is a $\varphi$-taut subspace of $X$, then there is a natural isomorphism

$$H^*_\varphi(X, A; \mathcal{Q}) \approx H^*_{\varphi|X \setminus A}(X; \mathcal{Q}).$$

We shall also be using the Theorem 10.5 [1, page 52] about $\varphi$-tautness of a subspace.

## 3. Large cohomological dimension is greater than cohomological dimension at least by one

It is obvious that if $X$ is a discrete space, then for any non-trivial ring $L$, $\mathrm{Dim}_L(X) = \dim_L(X)$ always holds. If $X$ is non-discrete, then the general feeling is that $\mathrm{Dim}_L(X)$ always exceeds $\dim_L(X)$ for any space $X$ (still open). This was first proved in [2] for real line $\mathbf{R}$, and then for $n$-manifolds in [3]. Prompted by [2], Kuźminov-Švedov have proved this result for any completely paracompact first countable space was well as for any completely paracompact and locally compact space [11], which is more general and precise than in [3]: it is precise in the sense of $\dim_{\mathcal{Q}}$ for any sheaf $\mathcal{Q}$ on $X$. In this section we extend this result for any paracompact space in which each point is $G_\delta$, and also for any completely paracompact Morita $k$-space. Bredon has already proved [1] that for any completely paracompact space $X$, $\mathrm{Dim}_L(X) \leqslant \dim_L(X) + 1$. Consequently, for a completely paracompact space in which each point is $G_\delta$ or completely paracompact Morita $k$-space, the equality

$$\mathrm{Dim}_L(X) = \dim_L(X) + 1$$

always holds. First we prove

THEOREM 3.1. *Suppose $X$ is a locally paracompact space of which each point is a $G_\delta$-set. Then, unless $X$ is discrete*

$$\mathrm{Dim}_L(X) \geqslant \dim_L(X) + 1.$$

PROOF. Suppose $\dim_L(X) = n$. By the local property of $\dim_L(X)$, there is a point $x \in X$ such that $\dim_L(V) = n$ for every neighbourhood $V$ of $x$. Then by the complementary sum theorem [7], we have $\dim_L(V \backslash x) = n$ for any such neighbourhood $V$ of $x$. It is easy to see that for $n > 0$ the point $x$ is a non-isolated point of $X$. In case when $n = 0$, we choose any point $x \in X$ which is not isolated. The non-discreteness of the space $X$ guarantees the existence of such a point. Further, we can as well assume that $X$ is paracompact. Since $X$ is normal and the point $x \in X$ is a $G_\delta$-set, there exists a Urysohn function $f: X \to [0, 1]$ such that $f^{-1}(0) = \{x\}$. Defining $U_i = f^{-1}([0, 1/i))$ for each $i = 1, 2, 3, \ldots$, we find that

$$\overline{U}_{i+1} \subset f^{-1}\left(\left[0, \frac{1}{i+1}\right]\right) \subset f^{-1}\left(\left[0, \frac{1}{i}\right)\right) = U_i,$$

and

$$\bigcap_{i \geqslant 1} U_i = \{x\}.$$

Since $U_i \backslash x = \bigcup_{k \geqslant i}(U_i \backslash \overline{U}_k)$, we have for some $k$, $\dim_L(U_i \backslash \overline{U}_k) = n$. Consequently, we may assume that $\dim_L(U_i \backslash \overline{U}_{i+1}) = n$ for each $i$. Again, applying the local property of $\dim_L$ to the locally paracompact space $U_i \backslash \overline{U}_{i+1}$, we obtain a closed set $F_i$ of $X$ such that $F_i \subset U_i \backslash \overline{U}_{i+1}$ and $\dim_L(F_i) = n$. Therefore, for each $i$, there is a sheaf $\mathcal{Q}_i$ of $L$-modules on $F_i$ such that $H^n_{\mathrm{cl}\, d \cap F_i}(F_i; \mathcal{Q}_i) \neq 0$. Let $F = \bigcup_{i \geqslant 1} F_i$ and $Y = F \cup \{x\}$. Because $\{F_i\}$ is a family of mutually disjoint closed subsets of $X$ which is discrete in $X \backslash x$, $F$ is closed in $X \backslash x$. Hence, $\overline{F} = F \cup \{x\}$ and $Y$ is a closed subset of $X$. Consequently, it suffices to prove that $\mathrm{Dim}_L(Y) \geqslant n + 1$.

Now $U_i \cap Y = (\bigcup_{k \geqslant i} F_i) \cup \{x\} = V_i$, say. Then $\{V_i\}$ is a family of clopen sets in $Y$ such that $x = \bigcap_{i \geqslant 1} V_i$ and $\{F_i\}$ is a discrete family of clopen sets in $Y \backslash x$. Notice that there is a unique sheaf $\mathcal{Q}$ of $L$-modules on $F$ such that $\mathcal{Q} | F_i = \mathcal{Q}_i$.

Because $F$ is open in $Y$, there is a unique sheaf $\mathcal{Q}^Y$ on $Y$ such that $\mathcal{Q}^Y | F_i = \mathcal{Q}_i$ and $\mathcal{Q}^Y | \{x\} = 0$. Suppose $\varphi_0$ is the family of all closed subsets of $\{x\}$. Put $\varphi_i = \text{cl } d \cap F_i$. Define

$$\Phi = \left\{ K \subset Y \middle| K = \bigcup_{i \geq 0} \begin{array}{l} \text{where } A_i \in \varphi_i \text{ for each } i \text{ and either} \\ A_i \text{ is empty for all except finitely} \\ \text{many } i \text{ or else } A_0 = \{x\} \end{array} \right\}.$$

Then $\Phi$ is a family of supports on $Y$ and each member of $\Phi$ is paracompact. Also, it is easy to check that $\Phi$ is a paracompactifying family of supports on $Y$. To prove $\text{Dim}_L(Y) \geq n + 1$, it suffices to prove that the restriction homomorphism

$$i^*: H^n_\Phi(Y; \mathcal{Q}^Y) \to H^n_{\Phi \cap (Y \setminus x)}(Y \setminus x; \mathcal{Q}^Y)$$

is not onto. Because then, from exact cohomology sequence of the pair $(Y, Y \setminus x)$ for the sheaf $\mathcal{Q}^Y$ and the family $\Phi$ of supports, we find that $H^{n+1}_\Phi(Y, Y \setminus x; \mathcal{Q}^Y) \neq 0$. This by $(R_3)$ implies that $H^{n+1}_{\Phi | \{x\}}(Y; \mathcal{Q}^Y) \neq 0$. Consequently, $\text{Dim}_L(Y) \geq n + 1$.

To prove that $i^*$ is not onto, we compute $H^p_\Phi(Y; \mathcal{Q}^Y)$: consider the exact cohomology sequence of the pair $(Y, V_i)$ for the family $\Phi$ of supports and the sheaf $\mathcal{Q}^Y$ (supressed)

$$\cdots \to H^{p-1}_{\Phi \cap V_i}(V_i) \to H^p_\Phi(Y, V_i) \to H^p_\Phi(Y) \to H^p_{\Phi \cap V_i}(V_i) \to \cdots.$$

Because $\Phi$ is paracompactifying, each $V_i$ as well as $\{x\}$ being closed in $Y$, are $\Phi$-taut in $Y$. Thus, $\{V_i\}$ is a family of $\Phi$-taut neighbourhoods of $x$ in $Y$ and for each $K \in \Phi | Y \setminus x$ there is $V_i \subset Y \setminus K$, we have by [1, Theorem 10.5, p. 52]

$$\varinjlim_i H^*_{\Phi \cap V_i}(V_i; \mathcal{Q}^Y) \approx H^*_{\Phi \cap \{x\}}(\{x\}; \mathcal{Q}^Y) = 0.$$

Taking the direct limit of above exact sequences over $i$ and noting the fact that direct limit of exact sequences is exact, we obtain the isomorphism

$$H^p_\Phi(Y; \mathcal{Q}^Y) \approx \varinjlim_i H^p_\Phi(Y, V_i; \mathcal{Q}^Y)$$

for each $p$. But for each $p$, we have

$$\begin{aligned} H^p_\Phi(Y, V_i; \mathcal{Q}^Y) &\approx H^p\left(Y; \left(\mathcal{Q}^Y | Y \setminus V_i\right)^Y\right) \quad (\text{by } R_1) \\ &\approx H^p_{\Phi | \bigcup_{k<i} F_k}\left(\bigcup_{k<i} F_k; \bigoplus_{k<i} (\mathcal{Q}^Y)_{F_k} \middle| \bigcup_{k<i} F_k\right) \quad (\text{by } R_2) \\ &\approx \bigoplus_{k<i} H^p_{\text{cl } d \cap F_k}(F_k; \mathcal{Q}_k), \end{aligned}$$

since $\{F_k\}_{k<i}$ is mutually disjoint family of clopen sets in $Y$ and $\Phi | F_k = \mathrm{cl}\, d \cap F_k = \varphi_k$. Thus, we have the isomorphism

$$H^p_\Phi(Y; \mathscr{Q}^Y) \approx \bigoplus_{i \geq 1} H^p_{\mathrm{cl}\, d\cap F_i}(F_i; \mathscr{Q}_i).$$

We also compute $H^p_{\Phi\cap(Y\setminus x)}(Y\setminus x; \mathscr{Q}^Y)$: Since $\{F_i\}$ is a discrete family of clopen sets of $Y\setminus x$, it follows from Lemma 2.2, $(\mathrm{R}_1)$ and the definition of $\Phi$ that for each $p$

$$H^p_{\Phi\cap(Y\setminus x)}(Y\setminus x; \mathscr{Q}^Y) \approx \prod_{i\geq 1} H^p_{\mathrm{cl}\, d\cap F_i}(F_i; \mathscr{Q}_i).$$

Using some cardinal arithmetic one can easily show that if $G_i \neq 0$ for each $i \in I$, then no mapping from $\oplus_{i\in I} G_i \to \prod_{i\in I} G_i$ can be onto provided the indexing set is infinite. Now we take the case $p = n$. Since for each $i$, $H^n_{\mathrm{cl}\, d\cap F_i}(F_i; \mathscr{Q}_i) \neq 0$, no mapping from

$$\bigoplus_{i\geq 1} H^n_{\mathrm{cl}\, d\cap F_i}(F_i; \mathscr{Q}_i)$$

to

$$\prod_{i\geq 1} H^n_{\mathrm{cl}\, d\cap F_i}(F_i; \mathscr{Q}_i)$$

can be onto, and this atonce implies that the restriction homomorphism $i^*$ is not onto. This proves the theorem.

REMARK. The above theorem improves the result of Kuźminov-Švedov [11] in two different ways. First, since in any first countable paracompact space each point is a $G_\delta$-set, our result holds for a much larger class of spaces. Secondly and more importantly, our result shows that complete paracompactness of the space $X$ in the proof of the inequality $\mathrm{Dim}_L(X) \geq \dim_L(X) + 1$ is superfluous. In their treatment [*ibid*] complete paracompactness of $X$ was essential because of their $\dim_{\mathscr{Q}}$ function.

Using the weak topology sum theorem [7] for $\dim_L$, we generalize the theorem of Kuźminov-Švedov [11] for Morita $k$-spaces as follows:

THEOREM 3.2. *Suppose $X$ is a locally completely paracompact and Morita k-space. Then, unless $X$ is discrete*, $\mathrm{Dim}_L(X) \geq \dim_L(X) + 1$.

PROOF. Suppose $\dim_L(X) = n$. We can assume that $X$ is completely paracompact and Morita $k$-space. By definition, $X$ has the Morita weak topology generated by a family say $\{K_\alpha\}$ of compact subsets. Hence, by the Morita weak topology sum theorem [7],

$$n = \dim_L(X) = \sup_\alpha \{\dim_L(K_\alpha)\}.$$

Therefore, there is an $\alpha$ such that $\dim_L(K_\alpha) = n$. We claim that, if $X$ is locally compact and completely paracompact then also, $\operatorname{Dim}_L(X) \geqslant \dim_L(X) + 1$. Thus, we find that $\operatorname{Dim}_L(K_\alpha) \geqslant n + 1$; which, in turn, implies that $\operatorname{Dim}_L(X) \geqslant n + 1$. Now, prove our claim: In the light of the proof of Theorem 3.1, it suffices to find a point $x \in X$ and sets $F_\beta \subset X \backslash x$ and sheaves $\mathcal{Q}_\beta$ of $L$-modules on $F_\beta$ such that $\{F_\beta\}$ is a family of mutually disjoint closed sets which is discrete in $X \backslash x$ and $H^n_{\mathrm{cl}\, d \cap F_\beta}(F_\beta; \mathcal{Q}_\beta) \neq 0$. The family $\{F_\beta\}$ is found in the following way: choose the point $x \in X$ as in the proof of Theorem 3.1. Obviously, we can suppose that $X$ is compact. Because $X \backslash x$ is paracompact and locally compact, we can select a locally finite open cover $\{V_\beta\}$ of $X \backslash x$ such that $\overline{V}_\beta$ is compact. Now, $X$ being the one-point compactification of $X \backslash x$, infinitely many $V_\beta$ satisfy the equality $\dim_L(V_\beta) = \dim_L(X) = n$. In each of these $V_\beta$, choose a point $x_\beta$ such that $\dim_L(U) = n$ for each neighbourhood $U$ of $x_\beta$. Omitting repetitions, we can assume that $x_\beta$ are distinct. The space $X \backslash x$ is collectionwise normal. Hence there exist open neighbourhoods $U_\beta$ of the points $x_\beta$ satisfying the following two conditions: 1) the sets $U_\beta$ are disjoint. 2) $x_\beta \in U_\beta \subset V_\beta$. Since $\dim_L(U_\beta) = n$, there is a set $F_\beta \subset U_\beta$ closed in $X$ such that $\dim_L(F_\beta) = n$. Then there is a sheaf $\mathcal{Q}_\beta$ on $F_\beta$ such that $H^n_{\mathrm{cl}\, d \cap F_\beta}(F_\beta; \mathcal{Q}_\beta) \neq 0$. The family $\{F_\beta\}$ and sheaves $\mathcal{Q}_\beta$ are as required.

For a non-discrete locally completely paracompact space $X$, $\operatorname{Dim}_L(X) \leqslant \dim_L(X) + 1$ follows from [1, page 74]. Also it is known that $\operatorname{Dim}_L(X) = 0$ if and only if $X$ is discrete [5]. Then, using Theorem 3.1 and Theorem 3.2, the following are immediate.

COROLLARY 3.3. *Suppose $X$ is a locally completely paracompact space in which each point is a $G_\delta$-set. Then* $\operatorname{Dim}_L(X) = \dim_L(X) + 1$ *if $X$ is not discrete, and* $\operatorname{Dim}_L(X) = \dim_L(X) = 0$ *if $X$ is discrete.*

COROLLARY 3.4. *Suppose $X$ is a locally completely paracompact and Morita $k$-space. Then*, $\operatorname{Dim}_L(X) = \dim_L(X) + 1$ *if $X$ is not discrete, and* $\operatorname{Dim}_L(X) = \dim_L(X) = 0$ *if $X$ is discrete.*

## 4. Sum theorems for $\operatorname{Dim}_L$

It has been proved in [7] that all forms of sum theorems are valid for cohomological dimension $\dim_L(X)$, for any locally paracompact space $X$ and any ring $L$. It is, however, curious to observe that all forms of sum theorems fail for large cohomological dimension $\operatorname{Dim}_L(X)$, even for metric spaces! The only sum theorem which remains valid for any space $X$ is the finite sum theorem for closed sets, which is, of course, equivalent to locally finite sum theorem for closed sets. We have

THEOREM 4.1. *Let $F_1$, $F_2$ be closed subsets of $X$ such that $X = F_1 \cup F_2$. Then*

$$\mathrm{Dim}_L(X) = \max\{\mathrm{Dim}_L(F_1), \mathrm{Dim}_L(F_2)\}.$$

PROOF. It suffices to show that if $\mathrm{Dim}_L(F_1) \leqslant n$ and $\mathrm{Dim}_L(F_2) \leqslant n$, then $\mathrm{Dim}_L(X) \leqslant n$. Let $\mathcal{A}$ be a sheaf of $L$-modules on $X$ and $\varphi$ be a family of supports on $X$. Since

$$H^{n+1}_{\varphi\cap F_1}(F_1, F_1 \cap F_2; \mathcal{A}) \approx H^{n+1}_{\varphi\cap F_1}\left(F_1; \mathcal{A}_{F_1\setminus(F_1\cap F_2)}\right) = 0,$$

we find from the long exact sequence (sheaf $\mathcal{A}$ supressed)

$$\cdots \to H^n_{\varphi\cap F_1}(F_1, F_1 \cap F_2) \to H^n_{\varphi\cap F_1}(F_1)$$
$$\to H^n_{\varphi\cap F_1\cap F_2}(F_1 \cap F_2) \to H^{n+1}_{\varphi\cap F_1}(F_1, F_1 \cap F_2) \to \cdots$$

for the pair $(F_1, F_1 \cap F_2)$, that

$$j_1 \colon H^n_{\varphi\cap F_1}(F_1; \mathcal{A}) \to H^n_{\varphi\cap F_1\cap F_2}(F_1 \cap F_2; \mathcal{A})$$

is onto. Now, since $\mathrm{Dim}_L(F_i) \leqslant n$, $i = 1, 2$, and the pair $(F_1, F_2)$ is $\varphi$-excisive, the Mayer-Vietoris sequence (sheaf $\mathcal{A}$ supressed) becomes

$$\cdots \to H^n_\varphi(X) \to H^n_{\varphi\cap F_1}(F_1) \oplus H^n_{\varphi\cap F_2}(F_2)$$
$$\xrightarrow{j_1 - j_2} H^n_{\varphi\cap F_1\cap F_2}(F_1 \cap F_2) \to H^{n+1}_\varphi(X) \to \cdots$$

in which $j_1 - j_2$ is onto. This implies that $H^p_\varphi(X; \mathcal{A}) = 0$ for each $p > n$. Hence $\mathrm{Dim}_L(X) \leqslant n$.

Now, let us consider the following:

EXAMPLE 4.2. Let $X = \{0\} \cup \{1/n \mid n \in \mathbf{N}\}$ be the subspace of real line. Then $X$ is a metric space of covering dimension zero, and hence $\dim_{\mathbf{Z}}(X) = 0$. Also because $X$ is completely paracompact, first countable and non-discrete, $\mathrm{Dim}_{\mathbf{Z}}(X) = \dim_{\mathbf{Z}}(X) + 1 = 1$. We have

(a) If we let $F_n = \{1/n\}$, $F_0 = \{0\}$, then $\mathrm{Dim}_{\mathbf{Z}}(F_n) = 0$ for each $n = 0, 1, 2, \ldots,$ but $\mathrm{Dim}_{\mathbf{Z}}(\bigcup_{n\geqslant 0} F_n) = \mathrm{Dim}_{\mathbf{Z}}(X) = 1$. This shows that the countable sum theorem for closed sets does not hold for $\mathrm{Dim}_L$.

(b) If we let $F = \{0\}$ and $U = \{1/n \mid n \in \mathbf{N}\}$, then both $F$ and $U$ are locally closed in $X$ and $X = F \cup U$. Obviously, $\mathrm{Dim}_{\mathbf{Z}}(F) = 0$, $\mathrm{Dim}_{\mathbf{Z}}(U) = 0$, but

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$\mathrm{Dim}_{\mathbf{Z}}(X) = 1$ and hence the finite sum theorem does not hold for locally closed sets; it does hold for closed sets (Theorem 4.1).

(c) It is also obvious that for $F = \{0\}$, the disjoint sum theorem and the complementary sum theorem do not hold.

We know that all forms of sum theorem hold for $\dim_L$. One can easily prove the following

PROPOSITION 4.3. *If $X$ is either* (i) *completely paracompact in which each point is $G_\delta$ or* (ii) *$X$ is completely paracompact Morita k-space, then for* $\mathrm{Dim}_L$

(a) *the countable sum theorem holds for a family $\{F_i\}$ of closed sets for which there exists an $i$ such that $F_i$ is non-discrete* (*that is*, $\mathrm{Dim}_L(F_i) > 0$),

(b) *the locally finite sum theorem holds for a family $\{F_\alpha\}$ of closed sets for which there exists a $F_\alpha$ such that $F_\alpha$ is non-discrete*,

(c) *the disjoint sum theorem holds for a closed set $F$ for which either $F$ is non-discrete or there exists a closed set $A$ of $X$ disjoint from $F$ such that $A$ is non-discrete*,

(d) *the complementary sum theorem holds for a closed set $F$ for which either $F$ or $X \setminus F$ is non-discrete.*

All the above results follow from Corollary 3.3 and Corollary 3.4. Also in view of Example 4.2, they are best possible. *We conjecture that the sum theorems in above form hold for any paracompact space $X$.*

## 5. An application

It is well known that the concept of cohomological dimension is basic not only in the proof of generalized Vietoris-Begle mapping theorems via spectral sequences, but also in the cohomological theory of topological transformation groups [10, page 39]. On the other hand the concept of $\varphi$-tautness is also fundamental in a systematic development of relative sheaf cohomology and in proving continuity property and so on. It is interesting to note that these two concepts themselves are related. The first result of this type was proved in [6], namely: If $X$ is a $n$-manifold ($1 \leqslant n < \infty$), then for any $x \in X$, there exists a family $\varphi$ of supports on $X$ such that $\{x\}$ is not $\varphi$-taut in $X$ with respect to sheaf cohomology. This is in contrast with the fact that, for Alexander-Spanier cohomology, every neighbourhood retract is known to be taut [4]. Of course, if $\varphi$ is paracompactifying then any neighbourhood retract in a normal space $X$ is always $\varphi$-taut. Here we are going to apply our result of Section 3 to show that there is nothing special in manifolds for points being not $\varphi$-taut. In fact, it is the equality $\mathrm{Dim}_L(X) = \dim_L(X) + 1$ which is crucial for such a result. In discrete spaces ($\mathrm{Dim}_L(X) = 0 = \dim_L(X)$) each point is $\varphi$-taut for any $\varphi$, and the following theorem shows that this is indeed rare.

THEOREM 5.1. *Suppose $X$ is a non-discrete locally completely paracompact space whose points are $G_\delta$-sets and $\dim_L(X) = n \geqslant 0$. If each point is $\varphi$-taut for any family $\varphi$ of supports on $X$, then $\mathrm{Dim}_L(X) = \infty$.*

PROOF. It is obvious that $\mathrm{Dim}_{\mathbf{Z}}(X) \geqslant n$. Suppose $\mathrm{Dim}_L(X) = n + k = m$, say, where $k < \infty$. The case $n = 0 = k$ is impossible, since $X$ is non-discrete [5]. Hence we can assume that $m > 0$. By the local property of $\mathrm{Dim}_{\mathbf{Z}}(X)$, there is a point $x \in X$ such that $\mathrm{Dim}_{\mathbf{Z}}(V) = m$ for every open neighbourhood $V$ of $x$. Obviously, $V$ is non-discrete [*ibid*] and hence by Proposition 4.3(d), $\mathrm{Dim}_{\mathbf{Z}}(V \backslash x) = m$ for any such neighbourhood of $x$. Since $m > 0$, $x$ can not be an isolated point in $X$. Using the same arguments as given in the proof of Theorem 3.1, we can find a countable family $\{U_i\}$ of open sets such that $x = \bigcap_{i \geqslant 1} U_i$, $\overline{U}_{i+1} \subset U_i$ and $\mathrm{Dim}_{\mathbf{Z}}(U_i \backslash \overline{U}_{i+1}) = m$ for each $i$. Choose a closed set $F_i$ of $X$ such that $F_i \subset U_i \backslash \overline{U}_{i+1}$ and $\mathrm{Dim}_{\mathbf{Z}}(F_i) = m$. Hence, there is a family $\varphi_i$ of supports and a sheaf $\mathcal{Q}_i$ on $F_i$ such that $H^m_{\varphi_i}(F_i; \mathcal{Q}_i) \neq 0$. Let $F = \bigcup_{i \geqslant 1} F_i$ and $Y = F \cup \{x\}$. As before, $F$ is closed in $X \backslash x$ and $Y$ is a closed subset of $X$. We prove that $\mathrm{Dim}_{\mathbf{Z}}(Y) > m$—a contradiction to the fact that $\mathrm{Dim}_{\mathbf{Z}}(X) = m$.

Now $U_i \cap Y = (\bigcup_{k \geqslant i} F_k) \cup \{x\} = V_i$, say. Then $\{V_i\}$ is a family of clopen sets in $Y$ such that $x = \bigcap_{i \geqslant 1} V_i$ and $\{F_i\}$ is a discrete family of clopen sets in $Y \backslash x$. Notice there is a unique sheaf $\mathcal{Q}$ of $\mathbf{Z}$-modules on $F$ such that $\mathcal{Q} | F_i = \mathcal{Q}_i$. Because $F$ is open in $Y$, there is a unique sheaf $\mathcal{Q}^Y$ on $Y$ such that $\mathcal{Q}^Y | F = \mathcal{Q}$ and $\mathcal{Q}^Y | \{x\} = 0$. Next, we remark that the family $\bigcup_{i \geqslant 1} \varphi_i$ of closed subsets of $Y$ is never a family of supports on $Y$, simply because it is not closed under finite unions. But if we let $\varphi_0$ to be the family of all closed subsets of $\{x\}$ and define

$$\Phi = \left\{ K \subset Y \middle| K = \bigcup_{i \geqslant 0} A_i \quad \begin{array}{l} \text{where } A_i \in \varphi_i \text{ and either } A_i \text{ is} \\ \text{empty for all except finitely many} \\ i \text{ or else } A_0 = \{x\} \end{array} \right\},$$

then one can easily see that $\Phi$ is indeed a family of supports on $Y$. We also point out that $\Phi$ is not the family of supports generated by $\bigcup_{i \geqslant 0} \varphi_i$ (that is, the smallest family of supports containing the family $\bigcup_{i \geqslant 0} \varphi_i$: such a family exists because the intersection of any family of supports is again a family of supports) but is strictly larger than that. Further, although $\Phi$ is a family of supports on $X$ yet, in case $k > 0$, it is not paracompactifying. This follows from the fact that $\Phi | F_i = \varphi_i$ and $\varphi_i$ is not paracompactifying in view of Lemma 2.1.

To prove that $\mathrm{Dim}_{\mathbf{Z}}(Y) > m$, it suffices to prove that

$$i^*: H^m_\Phi(Y; \mathcal{Q}^Y) \to H^m_{\Phi\cap(Y\setminus x)}(Y\setminus x; \mathcal{Q}^Y)$$

is not onto: in that case, we find from the exact cohomology sequence of the pair $(Y, Y\setminus x)$ for the sheaf $\mathcal{Q}^Y$ and the family $\Phi$ of supports that $H^{m+1}_\Phi(Y, Y\setminus x; \mathcal{Q}^Y) \neq 0$; this by $(\mathrm{R}_3)$ implies that $H^{m+1}_{\Phi|\{x\}}(Y; \mathcal{Q}^Y) \neq 0$ and consequently, $\mathrm{Dim}_{\mathbf{Z}}(Y) > m$. First, by our hypothesis, one can observe that $\{x\}$ is $\Phi$-taut even in $Y$. Then, just as in Theorem 3.1, we compute

$$H^p_\Phi(Y; \mathcal{Q}^Y) \approx \bigoplus_{i\geqslant 1} H^p_{\varphi_i}(F_i; \mathcal{Q}_i)$$

and

$$H^p_{\Phi\cap(Y\setminus x)}(Y\setminus x; \mathcal{Q}^Y) \approx \prod_{i\geqslant 1} H^p_{\varphi_i}(F_i; \mathcal{Q}_i).$$

Now taking the case $p = m$ and noticing the fact that for each $i$, $H^m_{\varphi_i}(F_i; \mathcal{Q}_i) \neq 0$, no mapping from $\bigoplus_{i\geqslant 1} H^m_{\varphi_i}(F_i; \mathcal{Q}_i)$ to $\prod_{i\geqslant 1} H^m_{\varphi_i}(F_i; \mathcal{Q}_i)$ can be onto. Thus $i^*$ is not onto. This completes the proof.

REMARK 5.2. In the light of the technique of the proof of the Theorem 3.2 for finding closed sets $F_i$, sheaves $\mathcal{Q}_i$ and families of supports $\varphi_i$, the above result remains valid if $X$ is completely paracompact Morita $k$-space.

It is clear from Corollary 3.3 and Corollary 3.4 that, for finite-dimensional (cohomological) non-discrete completely paracompact space $X$ which is either a Morita $k$-space or whose points are $G_\delta$-sets, the large cohomological dimension $\mathrm{Dim}_{\mathbf{Z}}(X)$ is also finite. Thus, the following improvements of the result of [6] immediately follow.

COROLLARY 5.3. *Suppose $X$ is a non-discrete locally completely paracompact space in which points are $G_\delta$-sets (or $X$ is a Morita $k$-space). Then, if* $\dim_{\mathbf{Z}}(X) < \infty$ *then there is a point $x \in X$ and a family $\varphi$ of supports on $X$ such that $\{x\}$ is not $\varphi$-taut in $X$ with respect to sheaf cohomology (such a $\varphi$ cannot be paracompactifying).*

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