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A Complete Classification of AI Algebras with the Ideal Property

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Abstract. Let A be an AI algebra; that is, A is the C^*-algebra inductive limit of a sequence

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots \longrightarrow A_n \longrightarrow \cdots,$$

where $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_n^i))$, X_n^i are $[0,1]$, k_n, and $[n,i]$ are positive integers. Suppose that A has the ideal property: each closed two-sided ideal of A is generated by the projections inside the ideal, as a closed two-sided ideal. In this article, we give a complete classification of AI algebras with the ideal property.

1 Introduction

Remarkable classification theorems have been obtained for the AH algebras, the inductive limits of matrix algebras over metric spaces (with uniformly bounded dimensions), in two important special cases:

(i) AH algebras of real rank zero (see [1, 2, 5]) and
(ii) simple AH algebras (see [3, 4, 6, 7, 9, 11]).

To unify and generalize the classification of these two special cases, we will consider C^*-algebras with the ideal property: every closed proper two sided ideal is generated by its projections. Obviously, the class of C^*-algebras with the ideal property includes C^*-algebras of real rank zero and simple C^*-algebras as very special cases.

An *approximate interval algebra* (*AI algebra*) is a separable C^*-algebra that is the inductive limit of a sequence of finite direct sums of matrix algebras over $C[0,1]$, *i.e.,* $(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C[0,1]))$.

In 1991, George Elliott classified the simple unital approximate interval algebras using an invariant consisting of K_0 theory and tracial state data (see [2] or [13]). In other words,

$$A \cong B \Longleftrightarrow (K_0(A), \quad T(A)) \cong (K_0(B), T(B)).$$

In 1995, Kenneth H. Stevens proved a generalization of this result by permitting the algebras to be unital and to have the ideal property (see [13]). Furthermore, the algebra was also assumed to be approximately divisible. In these circumstances, he

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proved that $A \cong B$ if and only if, for any projection $e \in A$ with $\psi_0[e] = [f]$, there exist

$$\psi_0 \colon \mathrm{K}_0(A) \xrightarrow{\cong} \mathrm{K}_0(B) \quad \text{and} \quad \psi_T^{ef} \colon \mathrm{T}(fBf) \xrightarrow{\cong} \mathrm{T}(eAe)$$

such that the affine isomorphisms $\psi_T^{ef}, \psi_T^{e'f'}$ are compatible with one another for $e' < e$ and $f' < f$ with $\psi_0[e] = [f]$ and $\psi_0[e'] = [f']$, where compatibility means the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{T}(fBf) & \xrightarrow{\psi_T^{ef}} & \mathrm{T}(eAe) \\ \downarrow & & \downarrow \\ \mathrm{T}(f'Bf') & \xrightarrow{\psi_T^{e'f'}} & \mathrm{T}(e'Ae'). \end{array}$$

In this paper, our purpose is to generalize the Stevens result to classify all of the AI algebras with the ideal property; that is, both of the above restrictions (of being unital and being approximately divisible) will be removed.

Let us point out that our proof is completely different from Stevens' proof of his theorem. In his proof, Stevens introduced a lot of special concepts such as "ribbon structure", "n-curtain", "weighted n-curtain" , and "$\delta - n$ subribbon structure", which heavily depend on the condition that the spectrum is the interval [0,1], and do not have higher dimensional analogues.

In this paper, we will prove a dichotomy result (Theorem 4.2) that can be used to avoid all the technicalities of Stevens' paper. Let us point out that this dichotomy result can be generalized to higher dimensions (as will be shown in a joint work of the second author with others; see [8]). Once the dichotomy theorem is proved, many techniques of the simple case (see [6, 7, 10]) can be used in this new setting. We believe that this new approach will be very helpful for the future classification of AH algebras with higher dimensional spectrum. Besides this, we also need to overcome the difficulty of the lack of approximate divisibility. As in [6], we will use Li's refinement of Thomsen's theorem (see [9, 15]). But in our case, the partial homomorphism may not be large as in [6, 1.9]. Lemma 2.5 deals with this problem.

The paper is organized as follows. In Section 1, some notation and known results will be introduced. In Section 2, we will prove the existence theorem in the case that the first algebra has only one block. In Sections 3 and 4 we will introduce the uniqueness theorem and prove the dichotomy theorem. In Section 5, we will use the existence theorem and the results of Sections 3 and 4 to prove the main theorem. Since the partial maps may not be unital, we consider the minimal direct summands A_n^i of A_n and reduce to the case of unital maps by using the projections (the images of the unit of A_n^i under partial maps $\phi_{n,m}^{i,j}$) to cut down A_m. This technique can be used to avoid the assumption of unital maps and make the existence theorem and uniqueness theorem compatible. Then, combining with dichotomy theorem, we finish the classification of AI algebras with the ideal property.

We will first introduce some notation and known results. All the notation is adopted from [7, 10] (see [10, Section 1] and [7, §1.1 and §1.2]).

In the inductive system $(A_n, \phi_{n,m})$, we understand that $\phi_{n,m} = \phi_{m-1,m} \circ \phi_{m-2,m-1} \circ \cdots \circ \phi_{n,n+1}$, where all $\phi_{n,m} \colon A_n \to A_m$ are homomorphisms.

We shall assume that, for any summand A_n^i in the direct sum $A_n = \bigoplus_{i=1}^{k_n} A_n^i$, necessarily, $\phi_{n,n+1}(\mathbf{1}_{A_n^i}) \neq 0$; otherwise, we could simply delete A_n^i from A_n without changing the limit algebra.

If $A_n = \bigoplus_i A_n^i$ and $A_m = \bigoplus_j A_m^j$, we shall use $\phi_{n,m}^{i,j}$ to denote the partial map of $\phi_{n,m}$ from the i-th block A_n^i of A_n to the j-th block A_m^j of A_m.

For a unital C*-algebra A, let TA denote the space of tracial states of A, *i.e,* $\tau \in$ TA, if and only if τ is a positive linear map from A to the complex plane $\mathbb{C}$, with $\tau(xy) = \tau(yx)$ and $\tau(\mathbf{1}) = 1$. $\operatorname{Aff}\mathrm{T}\,A$ is the collection of all the affine maps from TA to $\mathbb{C}$. (In the most references, $\operatorname{Aff}\mathrm{T}\,A$ is defined to be the set of all the affine maps from TA to $\mathbb{R}$. Our $\operatorname{Aff}\mathrm{T}\,A$ is a complexification of the standard $\operatorname{Aff}\mathrm{T}\,A$.) An element $\mathbf{1} \in \operatorname{Aff}\mathrm{T}\,A$, defined by $\mathbf{1}(\tau) = 1$ for all $\tau \in TA$, will be called the unit of $\operatorname{Aff}\mathrm{T}\,A$. $\operatorname{Aff}\mathrm{T}\,A$, together with the positive cone $\operatorname{Aff}\mathrm{T}\,A_+$ and the unit element $\mathbf{1}$, form a scaled ordered complex Banach space. (Notice that for any element $x \in \operatorname{Aff}\mathrm{T}\,A$, there are $x_1, x_2, x_3, x_4 \in \operatorname{Aff}\mathrm{T}\,A_+$ such that $x = x_1 - x_2 + ix_3 - ix_4$.)

For a unital C*-algebra A, let $\bigvee(A)$ denote the collection of all Murray-von Neumann equivalence class of projections in $\bigcup_{n=1}^{\infty} M_n(A)$. Define

$$\mathrm{K}_0(A) = \{(a,b) : a \in \bigvee(A), b \in \bigvee(A)\}/\sim,$$

where $(a,b) \sim (a^{'}, b^{'})$ if and only if there is $c \in \bigvee(A)$ such that

$$a + b^{'} + c = a^{'} + b + c \in \bigvee(A).$$

Let $\mathrm{K}_0(A)_+ = \{[(a,0)] \in \mathrm{K}_0(A), a \in \bigvee(A)\}$ be the positive cone of $\mathrm{K}_0(A)$. If we further assume that A is stably finite, then $\mathrm{K}_0(A)$ has properties

$$\mathrm{K}_0(A)_+ - \mathrm{K}_0(A)_+ = \mathrm{K}_0(A) \quad \text{and} \quad \mathrm{K}_0(A)_+ \cap (-\mathrm{K}_0(A)_+) = 0.$$

To each C*-algebra A, define the *scale* of A to be the subset $\sum A \triangleq \{[p] |$ p is a projection of $A\}$. Every morphism $\Lambda \colon A \to B$ induces a homomorphism of scaled ordered groups $(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, \sum A) \to (\mathrm{K}_0(B), \mathrm{K}_0(B)_+, \sum B)$ in the sense that $\mathrm{K}_0(\Lambda)\mathrm{K}_0(A)_+ \subset \mathrm{K}_0(B)_+$, and $\mathrm{K}_0(\Lambda)\sum A \subset \sum B$.

Remark 1.1 The pairing $\langle \cdot , \cdot \rangle \colon \mathrm{T}A \times \mathrm{K}_0(A) \to \mathbb{R}$ is defined by

$$\langle \tau, x \rangle = \sum_{i=1}^{k} \tau(p_{ii}) - \sum_{i=1}^{k} \tau(q_{ii}), \quad \forall \tau \in \mathrm{T}A,$$

where $x = [p] - [q] \in \mathrm{K}_0(A)$ is represented by the formal difference of two projections $p, q \in M_k(A)$. Set $\tau(x) = \langle \tau, x \rangle$. Then τ induces a group homomorphism from $\mathrm{K}_0(A)$ to $\mathbb{R}$ by $x(\tau) \triangleq \tau(x)$. In this way, each element $x \in \mathrm{K}_0(A)$ induces an affine map from TA to $\mathbb{R}$, and therefore, defines an element of $\operatorname{Aff}\mathrm{T}\,A$. This gives us a map $\sigma \colon \mathrm{K}_0(A) \to \operatorname{Aff}\mathrm{T}\,A$.

Let $\alpha\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ be a scaled ordered group homomorphism, and let $\xi\colon \mathrm{T}B \to \mathrm{T}A$ be an affine map. Then, ξ induces a linear map $\xi^*\colon \operatorname{Aff}\mathrm{T}\,A \to \operatorname{Aff}\mathrm{T}\,B$ defined by $\xi^*(f)(\tau) = f(\xi(\tau))$ for all $f \in \operatorname{Aff}\mathrm{T}\,A$ and $\tau \in \mathrm{T}B$. It is obvious that

$$\xi^*(\operatorname{Aff}\mathrm{T}\,A_+) \subset \operatorname{Aff}\mathrm{T}\,B_+, \quad \xi^*(\mathbf{1}) = (\mathbf{1}).$$

Hence, ξ induces a positive unital linear map (or scaled ordered map) from $\operatorname{Aff}\mathrm{T}\,A$ to $\operatorname{Aff}\mathrm{T}\,B$.

We shall say that α and ξ are *compatible* if

$$\tau(\alpha(x)) = (\xi(\tau))(x), \quad \forall x \in \mathrm{K}_0(A), \quad \tau \in \mathrm{T}B.$$

It is evident that α and ξ are compatible if and only if the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}_0(A) & \xrightarrow{\ \sigma\ } & \operatorname{Aff}\mathrm{T}\,A \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\xi^*} \\ \mathrm{K}_0(B) & \xrightarrow{\ \sigma\ } & \operatorname{Aff}\mathrm{T}\,B. \end{array}$$

In the rest of this paper, we will only use the map from $\operatorname{Aff}\mathrm{T}\,A$ to $\operatorname{Aff}\mathrm{T}\,B$. So instead of ξ^*, we will use ξ to denote this map.

Remark 1.2 Any unital homomorphism $\phi\colon A \to B$ induces a unital positive linear map

$$\operatorname{Aff}\mathrm{T}\,\phi\colon \operatorname{Aff}\mathrm{T}\,A \to \operatorname{Aff}\mathrm{T}\,B.$$

Suppose that $P \in M_l(C(X))$ is a non-zero projection with constant rank . It is well known that

$$\operatorname{Aff}\mathrm{T}(PM_l(C(X))P) = \operatorname{Aff}\mathrm{T}(M_l(C(X))) = C(X).$$

If $\phi\colon C(X) \to M_l(C(Y))$ is a unital homomorphism, then $\operatorname{Aff}\mathrm{T}\,\phi\colon C(X) \to C(Y)$ is given by

$$\operatorname{Aff}\mathrm{T}\,\phi(f) = \frac{1}{l}\sum_{i=1}^{l} \phi(f)_{ii}, \quad \forall f \in C(X),$$

where $\phi(f)_{ii}$ denotes the entry of $\phi(f) \in M_l(C(Y))$ at the position (i, i).

Remark 1.3 Let $\phi_1\colon C(X) \to PM_{l_1}(C(Y))P$, $\phi_2\colon C(X) \to QM_{l_2}(C(Y))Q$ be two unital homomorphisms. Set

$$\phi = \operatorname{diag}(\phi_1, \phi_2)\colon C(X) \to (P \oplus Q)M_{l_1+l_2}(C(Y))(P \oplus Q).$$

Then by Remark 1.2,

$$\operatorname{Aff}\mathrm{T}\,\phi = \frac{k_1}{k_1 + k_2}\operatorname{Aff}\mathrm{T}\,\phi_1 + \frac{k_2}{k_1 + k_2}\operatorname{Aff}\mathrm{T}\,\phi_2,$$

where $k_1 = \operatorname{rank} P$ and $k_2 = \operatorname{rank} Q$. Also, if P and Q are orthogonal projections in $M_l(C(Y))$, then $\phi = \operatorname{diag}(\phi_1, \phi_2)$ can be considered to be a homomorphism from $C(X)$ to $(P + Q)M_l(C(Y))(P + Q)$, and the above equality still holds.

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Remark 1.4 Let $\phi\colon C(X) \to PM_{k_1}(C(Y))P$ be a unital homomorphism. For any given point $y \in Y$, there are points $x_1(y), x_2(y), \dots, x_k(y) \in X$, and a unitary $U_y \in M_{k_1}(C(Y))$ such that

$$\phi(f)(y) = P(y)U_y \begin{pmatrix} f(x_1(y)) & & & & & \\ & \ddots & & & & \\ & & f(x_k(y)) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} U_y^* P(y) \in P(y)M_{k_1}(C(Y))P(y)$$

for all $f \in C(X)$. Equivalently, there are k rank one orthogonal projections $p_1, p_2, \dots, p_k$ with $\sum_{i=1}^k p_i(y) = P(y)$ and $x_1(y), x_2(y), \dots, x_k(y) \in X$, such that

$$\phi(f)(y) = \sum_{i=1}^{k} f(x_i(y))p_i(y),\ \forall\, f \in C(X).$$

Let us denote the set $\{x_1(y), x_2(y), \dots, x_k(y)\}$, counting multiplicities, by $\mathrm{SP}\,\phi_y$. In other words, if a point is repeated in the diagonal of the above matrix, it is included with the same multiplicity in $\mathrm{SP}\,\phi_y$. We shall call $\mathrm{SP}\,\phi_y$ the spectrum of ϕ at the point y (see also [6]). Let us define the *spectrum of* ϕ, denoted by $\mathrm{SP}\,\phi$, to be the closed subset

$$\mathrm{SP}\,\phi := \overline{\bigcup_{y\in Y} \mathrm{SP}\,\phi_y} \subseteq X.$$

Alternatively, $\mathrm{SP}\,\phi$ is the complement of the spectrum of the kernel of ϕ, considered as a closed ideal of $C(X)$. The map ϕ can be factored as

$$C(X) \xrightarrow{i^*} C(\mathrm{SP}\,\phi) \xrightarrow{\phi_1} PM_{k_1}(C(Y))P$$

with ϕ_1 an injective homomorphism, where i denotes the inclusion $\mathrm{SP}\,\phi \hookrightarrow X$.

Also, if $A = PM_{k_1}(C(Y))P$, then we shall call the space Y the spectrum of algebra A and write $\mathrm{SP}\,A = Y (= \mathrm{SP}(\mathrm{id}))$.

Remark 1.5 In Remark 1.4, if we group together all the repeated points in $\{x_1(y), x_2(y), \dots, x_k(y)\}$, and sum their corresponding projections, we can write

$$\phi(f)(y) = \sum_{i=1}^{l} f(\lambda_i(y))P_i \quad (l \le k),$$

where $\{\lambda_1(y), \lambda_2(y), \dots, \lambda_l(y)\}$ is equal to $\{x_1(y), x_2(y), \dots, x_k(y)\}$ as a set, but $\lambda_i(y) \neq \lambda_j(y)$ if $i \neq j$; and each P_i is the sum of the projections corresponding to $\lambda_i(y)$. If $\lambda_i(y)$ has multiplicity m (*i.e.,* it appears m times in $\{x_1(y), x_2(y), \dots, x_k(y)\}$), then $\mathrm{rank}(P_i) = m$.

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Definition 1.6 We shall call the projection P_i in Remark 1.5 the *spectral projection of ϕ at y with respect to the spectral element $\lambda_i(y)$*. If $X_1 \subset X$ is a subset of X, we shall call $\sum_{\lambda_i(y)\in X_1} P_i$ the *spectral projection of ϕ at y corresponding to the subset X_1 (or with respect to the subset X_1).*

Let $\phi\colon M_k(C(X)) \to PM_l(C(Y))P$ be a unital homomorphism. Set $\phi(e_{11}) = p$, where e_{11} is the canonical matrix unit corresponding to the upper left corner. Set

$$\phi_1 = \phi|_{e_{11}M_k(C(X))e_{11}}\colon C(X) \longrightarrow pM_l(C(Y))p.$$

Then $PM_l(C(Y))P$ can be identified with $pM_l(C(Y))p \otimes M_k$ in such a way that $\phi = \phi_1 \otimes \mathrm{id}_k$. Let us define

$$\mathrm{SP}\,\phi_y := \mathrm{SP}(\phi_1)_y, \quad \mathrm{SP}\,\phi := \mathrm{SP}\,\phi_1.$$

The following fact will be frequently used: For homomorphisms ϕ and ϕ_1 with rank $p = k$,

$$\mathrm{AffT}\,\phi_1(f)(y) = \frac{1}{k}\sum_{x_i(y)\in\mathrm{SP}(\phi_1)_y} f(x_i(y)) \quad \text{and} \quad \mathrm{AffT}\,\phi = \mathrm{AffT}\,\phi_1.$$

Let $\phi\colon M_k(C(X)) \to PM_l(C(Y))P$ be a (not necessary unital) homomorphism, where X and Y are connected finite simplicial complexes. Then

$$\#(\mathrm{SP}\,\phi_y) = \frac{\mathrm{rank}\,\phi(1_k)}{\mathrm{rank}(1_k)}, \quad \text{for any } y \in Y,$$

where $\#(\,\cdot\,)$ denotes the number of elements in the set counting multiplicity. It is also true that for any nonzero projection

$$p \in M_k(C(X)), \quad \#(\mathrm{SP}\,\phi_y) = \frac{\mathrm{rank}\,\phi(p)}{\mathrm{rank}(p)}.$$

Let

$$\phi\colon A = \bigoplus_{i=1}^{q} M_{k_i}(C(X^i)) \to B = \bigoplus_{j=1}^{t} P_j M_{l_j}(C(Y^j))P_j$$

be a homomorphism and denote by Y the disjoint union $\coprod Y^j$ of the spaces $\{Y^j\}_{j=1}^t$. For each $y \in Y$, $y \in Y^j$ for some j. The spectrum of the homomorphism ϕ at the point $y \in Y$ is defined by

$$\mathrm{SP}\,\phi_y = \bigcup_{i=1}^{q} \mathrm{SP}(\phi^{i,j})_y,$$

where the homomorphism

$$\phi^{i,j}\colon A^i = M_{k_i}(C(X^i)) \to \phi^{i,j}(\mathbf{1}_{A^i})P_jM_{l_j}(C(Y^j))P_j\phi^{i,j}(\mathbf{1}_{A^i})$$

is the partial map of ϕ corresponding to i, j. Note that

$$\mathrm{SP}\,\phi_y = \bigcup_{i=1}^{q} \mathrm{SP}(\phi^{i,j})_y \subset X := \coprod X_i.$$

For any $f \in \mathrm{Aff\,T}\,A^i = C(X^i)$,

$$\mathrm{Aff\,T}\,\phi^{i,j}(f) = \frac{\mathrm{rank}\,P_j}{\mathrm{rank}(\phi^{i,j}(\mathbf{1}_{A_i}))}(\mathrm{Aff\,T}\,\phi(f))_j,$$

where the AffT map on the left hand side is taken by regarding the homomorphism $\phi^{i,j}$ as a map from A^i to $\phi^{i,j}(\mathbf{1}_{A^i})B\phi^{i,j}(\mathbf{1}_{A^i})$, and the AffT map on the right hand side is taken by regarding the homomorphism ϕ as map from A to B^j, the j-th summand of B.

Remark 1.7 For any $\eta > 0$, $\delta > 0$, a unital homomorphism

$$\phi\colon C(X) \to QM_k(C(Y))Q$$

is said to have the property $\mathrm{sdp}(\eta, \delta)$ (*spectral distribution property with respect to* η *and* δ), if for any η-ball

$$B_\eta(x) := \{x' \in X;\ \mathrm{dist}(x', x) < \eta\} \subset X$$

and any point $y \in Y$,

$$\#(\mathrm{SP}\,\phi_y \cap B_\eta(x)) \ \geq\ \delta\#(\mathrm{SP}\,\phi_y),$$

counting multiplicity.

For a unital homomorphism $\phi\colon PM_k(C(X))P \to QM_l(C(Y))Q$, we shall say that ϕ has the property $\mathrm{sdp}(\,\cdot\,,\,\cdot\,)$ if

$$\phi|_{pM_k(C(X))p}\colon C(X)\ (\cong pM_l(C(X))p)\ \to \phi(p)M_l(C(Y))\phi(p)$$

has the property $\mathrm{sdp}(\,\cdot\,,\,\cdot\,)$, where P and Q are non-zero projections and p is a rank 1 subprojection of P.

The following lemma is well known. (See [10]).

Lemma 1.8 *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$ be unital AI algebras, and let $\alpha\colon K_0A \to K_0B$ be a scaled ordered group isomorphism. Then there are subsequences $A_{n_1}, A_{n_2}, \dots, A_{n_i}, \dots$ and $B_{m_1}, B_{m_2}, \dots, B_{m_i}, \dots$ and scaled ordered K_0 maps $\alpha_i\colon K_0A_{n_i} \to K_0B_{m_i}$ and $\beta_i\colon K_0B_{m_i} \to K_0A_{n_{i+1}}$ such that*

$$\beta_i \circ \alpha_i = K_0\phi_{n_i,n_{i+1}},\ \ \alpha_{i+1} \circ \beta_i = K_0\psi_{m_i,m_{i+1}},$$
$$\alpha \circ K_0\phi_{n_i,\infty} = K_0\psi_{m_i,\infty} \circ \alpha_i,\ \ \alpha^{-1} \circ K_0\psi_{m_i,\infty} = K_0\phi_{n_{i+1},\infty} \circ \beta_i.$$

For convenience, from now on, we will assume that $n_i = i$ and $m_i = i$.

Remark 1.9 For scaled ordered K_0 maps $\alpha_i\colon K_0A_i \to K_0B_i$, $\beta_i\colon K_0B_i \to K_0A_{i+1}$ in Lemma 1.8, by [16, Lemma 12.1.2], there exist homomorphisms $\widetilde{\Lambda}_i\colon A_i \to B_i$, $\widetilde{\mathcal{M}}_i\colon B_i \to A_{i+1}$ such that $K_0(\widetilde{\Lambda}_i) = \alpha_i$, $K_0(\widetilde{\mathcal{M}}_i) = \beta_i$, where

$$A_i = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_n^i)), \quad B_i = \bigoplus_{j=1}^{l_m} M_{\{m,j\}}(C(Y_m^j)) \quad \text{and} \quad X_n^i, Y_m^j$$

are all intervals.

Remark 1.10 Let A be a unital C*-algebra, and let $q \in A$ be a non-zero projection. If $k[q] = l[\mathbf{1}_A]$ in $K_0(A)$, then

$$\operatorname{Aff}\mathrm{T}\, i(f) = \tfrac{l}{k} f, \quad \forall f \in \operatorname{Aff}\mathrm{T}\, qAq,$$

where $\mathbf{1}_A$ is the unit of A and $i\colon qAq \to A$ is the embedding map. In particular, for the interval algebra $A = M_n(C(X)), X = [0,1]$, let $q \in A$ be a non-zero projection, then we have

$$\operatorname{Aff}\mathrm{T}\, i(g) = \frac{\operatorname{rank} q}{n} g, \quad \forall g \in \operatorname{Aff}\mathrm{T}\, qM_n(C(X))q.$$

Remark 1.11 Let $A = M_n(C(X))$ be an interval algebra, and let $q \in A$ be a non-zero projection. For convenience of description, we need to use the notation $qM_n(C(X))q$ to denote the subalgebra of A that is constructed by using the projection q to cut down the original algebra. Since $qM_n(C(X))q \cong M_{\operatorname{rank} q}(C(X))$, the subalgebra $qM_n(C(X))q$ is still an interval algebra.

In this paper, for the AI algebras with the ideal property A and B, we will use K_0 groups and the ordered vector spaces $\operatorname{Aff}\mathrm{T}(eAe), \operatorname{Aff}\mathrm{T}(fBf)$ as the invariants of the classification, where $eAe := \{eae \mid a \in A\}$, $fBf := \{fbf \mid b \in B\}$, and e, f are certain projections in A and B, respectively (see Theorem 5.1).

Now let us discuss the question of the compatibility of these invariants. In Theorem 5.1, we need the projections $e \in A$ and $f \in B$ to satisfy that $\alpha[e] = [f]$, where $\alpha\colon K_0(A) \to K_0(B)$ is a scaled ordered group isomorphism. And if we let $\xi^{e,f}$ denote the isomorphism from $\operatorname{Aff}\mathrm{T}(eAe)$ to $\operatorname{Aff}\mathrm{T}(fBf)$, then we require the following conditions in Theorem 5.1:

(i) α and $\xi^{e,f}$ are compatible (See Remark 1.1);
(ii) $\xi^{e,f}$ and $\xi^{e',f'}$ are compatible ($\forall e' < e, f' < f$), *i.e.,* the diagram

$$\begin{array}{ccc} \operatorname{Aff}\mathrm{T}(eAe) & \xrightarrow{\ \xi^{e,f}\ } & \operatorname{Aff}\mathrm{T}(fBf) \\ \uparrow & & \uparrow \\ \operatorname{Aff}\mathrm{T}(e'Ae') & \xrightarrow{\ \xi^{e',f'}\ } & \operatorname{Aff}\mathrm{T}(f'Bf') \end{array}$$

is commutative.

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In fact, we can deduce condition (i) from condition (ii). First, we have the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{K}_0(eAe) & \xrightarrow{\sigma} & \mathrm{AffT}(eAe) \\ \uparrow & & \uparrow \\ \mathrm{K}_0(e'Ae') & \xrightarrow{\sigma_1} & \mathrm{AffT}(e'Ae') \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{K}_0(fBf) & \xrightarrow{\sigma'} & \mathrm{AffT}(fBf) \\ \uparrow & & \uparrow \\ \mathrm{K}_0(f'Bf') & \xrightarrow{\sigma_2} & \mathrm{AffT}(f'Bf'). \end{array}$$

If we choose $[e'] \in \mathrm{K}_0(eAe)$, where $e' \in eAe$ is a non-zero projection ($e' < e$), then $\sigma([e'])$ is just the unit of $\mathrm{AffT}(e'Ae')$. Since $\xi^{e',f'}$ is an isomorphism, we have

$$\xi^{e',f'}(\sigma_1([e'])) = \mathbf{1}_{\mathrm{AffT}(f'Bf')} = \sigma_2([f']) = \sigma_2(\alpha[e']),$$

where $\alpha[e'] = [f']$, and

$$\sigma_1\colon \mathrm{K}_0(e'Ae') \to \mathrm{AffT}(e'Ae'), \quad \sigma_2\colon \mathrm{K}_0(f'Bf') \to \mathrm{AffT}(f'Bf')$$

are the imbedding maps (see Remark 1.1). By condition (ii), the compability of $\xi^{e,f}$ and $\xi^{e',f'}$, and the two diagrams above, we know that

$$\xi^{e,f}(\sigma[e']) = \xi^{e',f'}(\sigma_1([e'])) = \sigma_2(\alpha[e']) = \sigma'(\alpha[e']), \quad \forall [e'] \in \mathrm{K}_0(eAe),$$

and the following diagram

$$\begin{array}{ccc} \mathrm{K}_0(eAe) & \xrightarrow{\sigma} & \mathrm{AffT}(eAe) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\xi^{e,f}} \\ \mathrm{K}_0(fBf) & \xrightarrow{\sigma'} & \mathrm{AffT}(fBf) \end{array}$$

is commutative, then we get condition (i) naturally. So we do not list condition (i) in the main theorem of this paper (Theorem 5.1).

In this paper, we will denote by $\mathcal{P}(A)$ the set of all projections in the algebra A. For convenience, we will use the symbol $\bullet$ to denote every possible positive integer.

2 Existence Theorem

Let A, B be two AI algebras with the ideal property,

$$A = \lim_{n\to\infty}(A_n, \phi_{n,m}), \quad B = \lim_{n\to\infty}(B_n, \psi_{n,m}),$$

$$A_n = \bigoplus_{i=1}^{k_n} A_n^i = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_n^i)), \quad B_n = \bigoplus_{j=1}^{l_n} B_n^j = \bigoplus_{j=1}^{l_n} M_{\{n,j\}}(C(Y_n^j)).$$

Let $\alpha\colon K_0A \to K_0B$ be a scaled ordered group isomorphism, with inverse α^{-1}, and let $\xi\colon \operatorname{Aff}\mathrm{T}\,A \to \operatorname{Aff}\mathrm{T}\,B$ be an isomorphism of ordered complex Banach spaces, with inverse ξ^{-1}. Assume that α and ξ are compatible. In this section, we will lift the two maps to finite stages of the sequences, that is, define maps $\alpha_n\colon K_0A_n \to \mathrm{K}_0B_m$ and $\xi_n\colon \operatorname{Aff}\mathrm{T}\,A_n \to \operatorname{Aff}\mathrm{T}\,B_m$ with certain properties, and find a homomorphism $\Lambda_n\colon A_n \to B_m$ such that $\mathrm{K}_0\Lambda_n = \alpha_n$, and $\operatorname{Aff}\mathrm{T}\,\Lambda_n$ is equal to ξ_n approximately. This is called the "existence theorem" in Elliott's framework of the classification theory [10].

To prove the existence theorem, we need to introduce some lemmas, some of which are well known.

Lemma 2.1 ([10]) *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$ be unital AI algebras as in Lemma 1.8. Let $\alpha\colon K_0A \to K_0B$ be a scaled ordered group isomorphism, and let $\xi\colon \operatorname{Aff}\mathrm{T}\,A \to \operatorname{Aff}\mathrm{T}\,B$ be an isomorphism of scaled ordered complete Banach spaces compatible with α. For any A_n, any given finite set $F \subseteq \operatorname{Aff}\mathrm{T}\,A_n$, and any $\varepsilon > 0$, there exists $m > n$ and a map $\xi_n\colon \operatorname{Aff}\mathrm{T}\,A_n \to \operatorname{Aff}\mathrm{T}\,B_m$ such that, for all $f \in F$,*

$$\|(\operatorname{Aff}\mathrm{T}\,\psi_{m,\infty} \circ \xi_n)(f) - (\xi \circ \operatorname{Aff}\mathrm{T}\,\phi_{n,\infty})(f)\| < \varepsilon.$$

In particular, ξ_n can be chosen to be compatible with $\mathrm{K}_0\psi_{n,m} \circ \alpha_n$, where α_n is as described in Lemma 1.8.

For Lemma 2.1, although the condition simple was indirectly mentioned in Li's paper, we think the proof does not require it after checking the whole proof step by step.

Lemma 2.2 ([9]) *For any connected compact metric space X, finite subset $F \subset C(X)$ and $\varepsilon > 0$, there is an positive number $N \geq 0$ such that, if $P \in M_r(C(Y))$ is a trivial projection with $\operatorname{rank} P \geq N$, and $\xi\colon \operatorname{Aff}\mathrm{T}(C(X)) \to \operatorname{Aff}\mathrm{T}(PM_r(C(Y))P) = C(Y)$ is a unital positive linear map, where Y is an arbitrary compact metrizable space, then there is a unital homomorphism*

$$\phi\colon C(X) \to PM_r(C(Y))P$$

such that

$$\|\operatorname{Aff}\mathrm{T}\,\phi(f) - \xi(f)\| < \varepsilon, \quad \forall f \in F.$$

Lemma 2.3 ([12]) *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$, with*

$$A_n = \bigoplus_{i=1}^{k_n} A_n^i, A_n^i = P_n^i M_{[n,i]}(C(X_n^i))P_n^i,$$

where X_n^i are finite, connected CW complexes and $P_n^i \in M_{[n,i]}(C(X_n^i))$ are non-zero projections. Suppose that any ideal of A is generated by projections, i.e., *A has the ideal property. Then, for any n, any finite subset $F_n^i \subset A_n^i \subset A_n$, any positive integer N and any $\varepsilon > 0$, there is $m_0 > n$ such that any partial map $\phi_{n,m}^{i,j}$ with $m \geq m_0$ satisfies either*

(a) $\operatorname{rank}(\phi_{n,m}^{i,j}(P_n^i)) \geq N \cdot \operatorname{rank}(P_n^i)$, *or*

(b) *there exists $\psi^{i,j}_{n,m}$, a homomorphism with finite dimensional range, such that*

$$\phi^{i,j}_{n,m}(P^i_n) = \psi^{i,j}_{n,m}(P^i_n), \quad \text{and} \quad \|\phi^{i,j}_{n.m}(f) - \psi^{i,j}_{n,m}(f)\| < \varepsilon, \quad \forall f \in F^i_n,$$

and $K_0\phi^{i,j}_{n,m} = K_0\psi^{i,j}_{n,m}$.

In the statement of the original theorem in [12], ϕ and ψ also satisfy that $\phi^{i,j}_{n,m} \overset{h}{\sim} \psi^{i,j}_{n,m}$. But we do not need this fact; we only need $\mathrm{K}_0\phi^{i,j}_{n,m} = \mathrm{K}_0\psi^{i,j}_{n,m}$. This always holds here (at least if the sets F^i_n are large enough).

Remark 2.4 By the proof of Lemma 2.3, we can see the following result is also true:

$$\|\operatorname{AffT}\phi^{i,j}_{n.m}(f) - \operatorname{AffT}\psi^{i,j}_{n,m}(f)\| < \varepsilon, \quad \forall f \in e_{11}F^i_n e_{11},$$

where $e_{11}F^i_n e_{11} \subset \operatorname{AffT} M_{[n,i]}(C(X^i_n)) = C(X^i_n)$.

Lemma 2.5 *Let A_1, A_2, A_3 be C^*-algebras expressed as $P^s M_{n_s}(C(X_s))P^s$, where P^s is a non-zero projection in $M_n(C(X_s))$, $X_s = [0,1]$, $s = 1,2,3$.*

Let $\phi\colon A_1 \to A_2$ be a unital homomorphism. Let $\xi\colon \operatorname{AffT} A_2 \to \operatorname{AffT} A_3$ be a unital positive linear map, and let $\widetilde{\Lambda}\colon A_2 \to A_3$ be a unital homomorphism such that $K_0(\widetilde{\Lambda})$ and ξ are compatible. Let $\varepsilon > 0$ be a fixed number, and let $E \subseteq \operatorname{AffT} A_1$ be a finite set. The following statement is true:

If there is a homomorphism $\psi\colon A_1 \to A_2$ defined by point valuations at points $x_1, x_2, \dots, x_n \in X_1$ such that $\psi(f) = \sum_{i=1}^n f(x_i) \otimes p_i$, $\sum_{i=1}^n P_i = \mathbf{1}_{A_2}$, $P_i = \bigoplus_1^l p_i$, $P_iP_j = 0, i \neq j$, $p_i \in \mathcal{P}(A_2)$, $l = \operatorname{rank} A_1$, and

$$\|\operatorname{AffT}\phi(f) - \operatorname{AffT}\psi(f)\| < \varepsilon, \quad \forall f \in E,$$

$K_0(\phi) = K_0(\psi)$, then there is a homomorphism $\Lambda\colon A_1 \to A_3$ such that

(i) $K_0(\Lambda) = K_0(\widetilde{\Lambda}) \circ K_0(\phi)$, $\operatorname{AffT}\Lambda(f) = \xi \circ \operatorname{AffT}\psi(f)$, $\forall f \in E$, *and*
(ii) $\|\operatorname{AffT}\Lambda(f) - \xi \circ \operatorname{AffT}\phi(f)\| \le \varepsilon$, $\forall f \in E$.

Proof Without loss of generality, we may assume that

$$A_1 = M_l(C(X_1)) = M_l(C([0,1])),\ A_2 = pM_r(C([0,1]))p,\ A_3 = qM_k(C([0,1]))q,$$

where p, q are projections in $M_r(C([0,1])$ and $M_k(C([0,1])$, respectively (see Remark 1.11). For this given ε, by the condition of the lemma, there exists $\psi(f) = \sum_{i=1}^n f(x_i) \otimes p_i$, $p_i \in \mathcal{P}(A_2)$, satisfying

$$\|\operatorname{AffT}\phi(f) - \operatorname{AffT}\psi(f)\| < \varepsilon, \quad \forall f \in E.$$

Define $\Lambda\colon A_1 \to A_3$, $\Lambda(f) = \sum_{i=1}^n f(x_i) \otimes \widetilde{\Lambda_{1,i}}(p_i)$, where we set

$$\widetilde{\Lambda} = \widetilde{\Lambda_1} \otimes \mathbf{1}_{\operatorname{rank} p}, \quad \widetilde{\Lambda_{1,i}} = \widetilde{\Lambda_1} \otimes \mathbf{1}_{\operatorname{rank} p_i}.$$

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Set $\operatorname{rank}(\widetilde{\Lambda}(p_i)) = r_i'$, $\operatorname{rank}(p_i) = r_i$. By the definition of AffT, for any $f \in C([0,1])$, we have that

$$\operatorname{AffT}\Lambda(f) = \frac{l}{\operatorname{rank} q}\sum_{i=1}^{n} r_i' f(x_i), \quad \operatorname{AffT}\psi(f) = \frac{l}{\operatorname{rank} p}\sum_{i=1}^{n} r_i f(x_i),$$

where $\operatorname{rank} p = \sum_{i=1}^{n} lr_i$, $\operatorname{rank} q = \sum_{i=1}^{n} lr_i'$. Since ξ and $K_0(\widetilde{\Lambda})$ are compatible, we have

$$\xi\Big(\frac{lr_i}{\operatorname{rank} p}\Big) = \frac{lr_i'}{\operatorname{rank} q}, \quad \forall i = 1, 2, \dots, n.$$

So $\operatorname{AffT}\Lambda(f) = \xi \circ \operatorname{AffT}\psi(f)$.

Then for any $f \in E$, we have

$$\begin{aligned} &\|\operatorname{AffT}\Lambda(f) - \xi \circ \operatorname{AffT}\phi(f)\| \\ &\quad\le \|\operatorname{AffT}\Lambda(f) - \xi \circ \operatorname{AffT}\psi(f)\| + \|\xi \circ \operatorname{AffT}\phi(f) - \xi \circ \operatorname{AffT}\psi(f)\| \\ &\quad= \|\xi \circ \operatorname{AffT}\phi(f) - \xi \circ \operatorname{AffT}\psi(f)\| \le \varepsilon. \end{aligned}$$

So $\|\operatorname{AffT}\Lambda(f) - \xi \circ \operatorname{AffT}\phi(f)\| \le \varepsilon, \forall f \in E$.

Notice that $\mathrm{K}_0(\phi) = \mathrm{K}_0(\psi)$. By the definition of Λ,

$$\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\widetilde{\Lambda} \circ \psi) = \mathrm{K}_0(\widetilde{\Lambda}) \circ \mathrm{K}_0(\phi).$$

This completes the proof. ■

Theorem 2.6 (Existence Theorem) *Let*

$$A = \lim_{n\to\infty}(A_n, \phi_{n,m}) \quad \textit{and} \quad B = \lim_{n\to\infty}(B_n, \psi_{n,m})$$

be unital AI algebras with the ideal property, where $\phi_{n,m}$, $\psi_{n,m}$ are both unital homomorphisms,

$$A_n = \bigoplus_{i=1}^{k_n} A_n^i, \quad B_m = \bigoplus_{j=1}^{l_m} B_m^j, \quad A_n^i = P_n^i M_{[n,i]}(C(X_n^i))P_n^i,$$
$$B_m^j = Q_m^j M_{\{m,j\}}(C(Y_m^j))Q_m^j \quad \textit{and} \quad X_n^i = Y_m^j = [0,1].$$

Let us assume that A_1 has only one block, i.e., $k_1 = 1$. *Suppose that there exists an isomorphism $\xi\colon \operatorname{AffT} A \to \operatorname{AffT} B$ and an ordered group isomorphism $\alpha\colon K_0A \to K_0B$, such that ξ and α are compatible. It follows that for any $\varepsilon > 0$, and any finite set $E \subset \operatorname{AffT} A_1$, there exists a map $\Lambda\colon A_1 \to B_m$ (m large) such that*

(i) $\|\operatorname{AffT}\psi_{m,\infty} \circ \operatorname{AffT}\Lambda(f) - \xi \circ \operatorname{AffT}\phi_{1,\infty}(f)\| < \varepsilon, \forall f \in E$, *and*
(ii) $K_0\Lambda = K_0\psi_{1,m} \circ \alpha_1$.

Proof By Lemma 1.8, there exists an intertwining of K_0 level,

$$\begin{array}{ccccccccc}
\mathrm{K}_0A_1 & \longrightarrow & \mathrm{K}_0A_2 & \longrightarrow & \mathrm{K}_0A_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0A \\
\downarrow{\scriptstyle\alpha_1} & \nearrow{\scriptstyle\beta_1} & \downarrow{\scriptstyle\alpha_2} & \nearrow{\scriptstyle\beta_2} & \downarrow{\scriptstyle\alpha_3} & \nearrow{\scriptstyle\beta_3} & & & \downarrow{\scriptstyle\alpha} \\
\mathrm{K}_0B_1 & \longrightarrow & \mathrm{K}_0B_2 & \longrightarrow & \mathrm{K}_0B_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0B
\end{array}$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{K}_0(A) & \xrightarrow{\sigma} & \mathrm{Aff}\,\mathrm{T}\,A \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\xi} \\
\mathrm{K}_0(B) & \xrightarrow{\sigma} & \mathrm{Aff}\,\mathrm{T}\,B,
\end{array}$$

where α_i, β_i, are scaled ordered homomorphisms, and there exist homomorphisms $\widetilde{\Lambda}_i \colon A_i \to B_i$, $\widetilde{\mathcal{M}}_i \colon B_i \to A_{i+1}$ such that $\mathrm{K}_0\widetilde{\Lambda}_i \circ \mathrm{K}_0\widetilde{\mathcal{M}}_i = \mathrm{K}_0\phi_{i,i+1}$.

For $E \subset \mathrm{Aff}\,\mathrm{T}\,A_1$, we can find a finite set $F \subset A_1$ such that $E \subset e_{11}Fe_{11}$. For arbitrary given $\varepsilon > 0$, we can find $N > 0$ to satisfy the conditions of Lemma 2.5. Then, for the given $\varepsilon > 0$, $N > 0$ and finite set F, applying Lemma 2.3 and Remark 2.4, we obtain $n_1 > 0$ such that for any $n' \geq n_1$, the partial map $\phi^{1,i'}_{1,n'}$ satisfies either one of the conditions (recall that A_1 only has one block A_1^1)

(a) $\operatorname{rank}(\phi^{1,i'}_{1,n'}(P_1^1)) \geq N \cdot \operatorname{rank}(P_1^1)$ or

(b) $\phi^{1,i'}_{1,n'}(P_1^1) = \psi^{1,i'}_{1,n'}(P_1^1)$, $\psi^{1,i'}_{1,n'}$ is a homomorphism with finite dimensional range, and

$$\left\| \phi^{1,i'}_{1,n'}(f) - \psi^{1,i'}_{1,n'}(f) \right\| < \tfrac{\varepsilon}{2}, \quad \forall f \in F,$$

$$\left\| \mathrm{Aff}\,\mathrm{T}\,\phi^{1,i'}_{1,n'}(f) - \mathrm{Aff}\,\mathrm{T}\,\psi^{1,i'}_{1,n'}(f) \right\| < \tfrac{\varepsilon}{2}, \quad \forall f \in e_{11}Fe_{11} \subseteq \mathrm{Aff}\,\mathrm{T}\,A_1^i.$$

For n', applying Lemma 2.1, we obtain an integer $m > n'$ such that for all $f \in E$, the following diagram is approximately commutative to within $\frac{\varepsilon}{2}$:

$$\begin{array}{ccccc}
\mathrm{Aff}\,\mathrm{T}\,A_1 & \longrightarrow & \mathrm{Aff}\,\mathrm{T}\,A_{n'} & \longrightarrow & \mathrm{Aff}\,\mathrm{T}\,A \\
 & \searrow{\scriptstyle\xi_1} & \downarrow{\scriptstyle\xi_n'} & & \downarrow{\scriptstyle\xi} \\
 & & \mathrm{Aff}\,\mathrm{T}\,B_m & \longrightarrow & \mathrm{Aff}\,\mathrm{T}\,B.
\end{array}$$

Set $\xi_1 = \xi_n' \circ \mathrm{Aff}\,\mathrm{T}\,\phi_{1,n'}$. Then

$$\| \mathrm{Aff}\,\mathrm{T}\,\psi_{m,\infty} \circ \xi_1(f) - \xi \circ \mathrm{Aff}\,\mathrm{T}\,\phi_{1,\infty}(f) \| < \tfrac{\varepsilon}{2}, \quad \forall f \in e_{11}Fe_{11}.$$

By Lemma 2.1, ξ_n' and $\mathrm{K}_0\psi_{n',m}\circ\alpha_{n'}$ are compatible. Set

$$p_{i',j}=(\psi_{n',m}\circ\widetilde{\Lambda}_{n'})_{i',j}\circ\phi_{1,n'}^{1,i'}(\mathbf{1}_{A_1}),\quad P_j=\bigoplus_{i'}p_{i',j}.$$

Then

$$\frac{\mathrm{size}B_m^j}{\mathrm{rank}\,p_{i',j}}(\xi_n')_{i',j}\circ(\mathrm{AffT}\,\phi_{1,n'})_{1,i'}\colon\ \mathrm{AffT}\,A_1\to\mathrm{AffT}(p_{i',j}B_m^jp_{i',j})$$

is unital, provided that $\mathrm{rank}(p_{i',j})\neq 0$.

(1) If $\phi_{1,n'}^{1,i'}$ satisfies condition (a), and $(\xi_n')_{i',j}$ is non-zero, then

$$\frac{\mathrm{rank}\,p_{i',j}}{\mathrm{rank}\,\mathbf{1}_{A_1}}\geq\frac{\mathrm{rank}\,\phi_{1,n'}^{1,i'}(\mathbf{1}_{A_1})}{\mathrm{rank}\,\mathbf{1}_{A_1}}\geq N\quad(\forall i',j).$$

By Lemma 2.2, there exists a unital homomorphism $\Lambda_{i',j}\colon A_1\to p_{i',j}B_m^jp_{i',j}$ such that for any $f\in e_{11}Fe_{11}$,

$$\Big\|\mathrm{AffT}\,\Lambda_{i',j}(f)-\frac{\mathrm{size}B_m^j}{\mathrm{rank}\,p_{i',j}}(\xi_n')_{i',j}\circ(\mathrm{AffT}\,\phi_{1,n'})_{1,i'}(f)\Big\|<\frac{\varepsilon}{2}.$$

(2) If $\phi_{1,n'}^{1,i'}$ satisfies condition (b), and $(\xi_n')_{i',j}$ is non-zero, set

$$A_1=A_1,\quad A_2=\phi_{1,n'}^{1,i'}(\mathbf{1}_{A_1})A_{n'}^{i'}\phi_{1,n'}^{1,i'}(\mathbf{1}_{A_1}),\quad A_3=p_{i',j}B_m^jp_{i',j}.$$

Applying Lemma 2.5, we can get a unital homomorphism $\Lambda_{i',j}\colon A_1\to p_{i',j}B_m^jp_{i',j}$ such that

$$\mathrm{AffT}\,\Lambda_{i',j}(f)=\frac{\mathrm{size}B_m^j}{\mathrm{rank}\,p_{i',j}}(\xi_n')_{i',j}\circ(\mathrm{AffT}\,\psi_{1,n'}^{1,i'})(f).$$

Since $k_1=1$ and $\phi_{1,n'}^{1,i'},\psi_{1,n'}^{1,i'}$ are both unital, we have

$$(\mathrm{AffT}\,\psi_{1,n'})_{1,i'}=\mathrm{AffT}\,\psi_{1,n'}^{1,i'},\quad(\mathrm{AffT}\,\phi_{1,n'})_{1,i'}=\mathrm{AffT}\,\phi_{1,n'}^{1,i'}.$$

So

$$\mathrm{AffT}\,\Lambda_{i',j}(f)=\frac{\mathrm{size}B_m^j}{\mathrm{rank}\,p_{i',j}}(\xi_n')_{i',j}\circ(\mathrm{AffT}\,\psi_{1,n'})_{1,i'}(f).$$

By Remark 2.4, we have

$$\|\,\mathrm{AffT}\,\phi_{1,n'}^{1,i'}(f)-\mathrm{AffT}\,\psi_{1,n'}^{1,i'}(f)\|<\tfrac{\varepsilon}{2}.$$

Then, as in the proof of Lemma 2.5, we also can get

$$\Big\|\mathrm{AffT}\,\Lambda_{i',j}(f)-\frac{\mathrm{size}B_m^j}{\mathrm{rank}\,p_{i',j}}(\xi_n')_{i',j}\circ(\mathrm{AffT}\,\phi_{1,n'}^{1,i'})(f)\Big\|<\frac{\varepsilon}{2}.$$

In case $(\xi'_n)_{i',j} = 0$, let $\Lambda_{i',j} = 0$. Let $\Lambda_j = \bigoplus_{i'} \Lambda_{i',j}$, then Λ_j is a unital homomorphism. Let $\Lambda\colon A_1 \to B_m$ be the map whose partial maps consist of Λ_j $(j = 1, 2, \dots l_m)$. Since $\operatorname{rank} \Lambda_{i',j}(\mathbf{1}_{A_1}) = \operatorname{rank} p_{i',j}$, then by Remark 1.3 we have

$$\begin{aligned}
(\operatorname{AffT}\Lambda)_j &= \frac{\operatorname{rank}\Lambda_j(\mathbf{1}_{A_1})}{\operatorname{size}B_m^j}\operatorname{AffT}\Lambda_j\\
&= \frac{\sum_{i'}\operatorname{rank}\Lambda_{i',j}(\mathbf{1}_{A_1})}{\operatorname{size}B_m^j}\operatorname{AffT}\Big(\bigoplus_{i'}\Lambda_{i',j}\Big)\\
&= \frac{\sum_{i'}\operatorname{rank}\Lambda_{i',j}(\mathbf{1}_{A_1})}{\operatorname{size}B_m^j}\left(\sum_{i'}\left(\frac{\operatorname{rank}p_{i',j}}{\sum_{i'}\operatorname{rank}p_{i',j}}\right)\operatorname{AffT}\Lambda_{i',j}\right)\\
&= \sum_{i'}\frac{\operatorname{rank}p_{i',j}}{\operatorname{size}B_m^j}\operatorname{AffT}\Lambda_{i',j}.
\end{aligned}$$

For $\xi_1\colon \operatorname{AffT}A_1 \to \operatorname{AffT}B_m$, the partial map $(\xi_1)_j = \sum_{i'}(\xi'_n)_{i',j}\circ(\operatorname{AffT}\phi_{1,n'})_{1,i'}$. When $\operatorname{rank} p_{i',j} \neq 0$, we have

$$\begin{aligned}
&\left\|\frac{\operatorname{rank}p_{i',j}}{\operatorname{size}B_m^j}\operatorname{AffT}\Lambda_{i',j}(f) - (\xi'_n)_{i',j}\circ(\operatorname{AffT}\phi_{1,n'})_{1,i'}(f)\right\|\\
&= \frac{\operatorname{rank}p_{i',j}}{\operatorname{size}B_m^j}\left\|\operatorname{AffT}\Lambda_{i',j}(f) - \frac{\operatorname{size}B_m^j}{\operatorname{rank}p_{i',j}}(\xi'_n)_{i',j}\circ(\operatorname{AffT}\phi_{1,n'})_{1,i'}(f)\right\|\\
&\le \frac{\operatorname{rank}p_{i',j}}{\operatorname{size}B_m^j}\frac{\varepsilon}{2}.
\end{aligned}$$

Then, for any $f \in E$, we have that

$$\|(\operatorname{AffT}\Lambda)_j(f) - (\xi_1)_j(f)\| < \frac{\operatorname{rank}\Lambda(\mathbf{1}_{A_1})}{\operatorname{size}B_m^j}\frac{\varepsilon}{2} < \frac{\varepsilon}{2}.$$

Thus,

$$\|\operatorname{AffT}\Lambda(f) - (\xi_1)(f)\| < \tfrac{\varepsilon}{2},$$

for all $f \in E$, and

$$\|\operatorname{AffT}\psi_{m,\infty}\circ\operatorname{AffT}\Lambda(f) - \xi\circ\operatorname{AffT}\phi_{1,\infty}(f)\| < \varepsilon,\ \ \forall f \in E.$$

By the progress of construction of Λ and Lemma 2.3, we have $\mathrm{K}_0\Lambda = \mathrm{K}_0\psi_{1,m}\circ\alpha_1$. This completes the proof. ∎

Remark 2.7 For the sake of simplicity, in this existence theorem, we assume that A_1 has only one block. In the future, when we apply the existence theorem to each block A_n^i, we will apply the theorem to the cut down algebra of A_m by the projection $\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i})$, which will correspond to a unital inductive limit with the first algebra A_n having only block A_n^i. In other words, we only need the existence theorem in the case that A_1 (or A_n, with n fixed) has only one block.

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3 Uniqueness Theorem

First we define the "test functions" introduced in [14].

Suppose that X is a path-connected compact metric space, T is a closed subset of X, and $M > 1$ is a positive number. Then $\chi_{T,M}$, called the test function associated with T, M, is defined as follows:

$$\chi_{T,M} = \begin{cases} 1, & x \in T, \\ 1 - M\mathrm{dist}(x, T), & \mathrm{dist}(x, T) \le \frac{1}{M}, \\ 0, & \mathrm{dist}(x, T) \ge \frac{1}{M}. \end{cases}$$

Lemma 3.1 ([9]) *Suppose that X is a path-connected compact metric space, and $\eta, \delta > 0$. There is a finite set $H \subset \mathrm{AffT}(C(X)) = C(X)$ such that the following statement is true. Let Y be a compact metric space, and let two unital homomorphisms $\phi, \psi \colon C(X) \to PM_k(C(Y))P$ satisfy the following two conditions:*

(i) *For any $x \in X$ and $\frac{\eta}{8}$ ball $B_{\frac{\eta}{8}}(x) = \{x^{'} \in X | dist(x, x^{'}) < \frac{\eta}{8}\}$ of x,*

$$\#\,\mathrm{SP}\,\phi_y \cap B_{\frac{\eta}{8}}(x) \ge \delta \#\,\mathrm{SP}\,\phi_y,$$

for all $y \in Y$ (notice that $\#\,\mathrm{SP}\,\phi_y = \mathrm{rank}(P)$);

(ii) $\|\,\mathrm{AffT}\,\phi(h) - \mathrm{AffT}\,\psi(h)\| < \frac{\delta}{4}$*, for any $h \in H$.*

Then $\mathrm{SP}\,\phi_y$ and $\mathrm{SP}\,\psi_y$ can be paired to within distance η for each $y \in Y$. That is, one may write

$$\mathrm{SP}\,\phi_y = \{x_1, x_2, \dots, x_n\} \quad \textit{and} \quad \mathrm{SP}\,\psi_y = \{x_1^{'}, x_2^{'}, \dots, x_n^{'}\}$$

(where $n = \mathrm{rank}(P)$) such that $dist(x_i, x^{'}_{\sigma(i)}) < \eta$ for each i.

Lemma 3.2 ([10]) *For each $\varepsilon > 0$, $X = [0, 1]$, there exists $\delta > 0$ such that, if unital homomorphisms $\phi, \psi \colon C(X) \to M_n(C(Y))$ $(Y = [0, 1])$ satisfy conditions: for each $y \in Y$, $\mathrm{SP}\,\phi_y$ and $\mathrm{SP}\,\psi_y$ can be paired within δ. Then there is a unitary $u \in M_n(C(Y))$ satisfying:*

$$\|\phi(h) - Adu \circ \psi(h)\| < \varepsilon,$$

where h is the generator of $C(X)$ with $h(x) = x$.

In fact, for any given finite set $F \subset C(X)$ (instead of $h(x) = x$), we also can find the corresponding number δ to make the statement of Lemma 3.2 hold for $h(x)$ and δ is the generator of $C(X)$.

Combining Lemmas 3.1 and 3.2 in a way similar to the proof of the uniqueness theorem in [10] (Theorem 5.14), we can easily obtain the following result.

Corollary 3.3 *Let $A = C(X)$, with $X = [0, 1]$, $F \subset A$ be a finite set. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $\delta > 0$, there is finite set $H(\eta, \delta, X) \subset \mathrm{AffT}(C(X))$ such that the following statement holds.*

If two unital homomorphisms

$$\phi, \psi \colon A \to B = \bigoplus_{j=1}^{m} M_{\{m,j\}}(C(Y_j)),$$

$Y_j = [0, 1]$, satisfy the conditions:

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(i) ϕ *or* ψ *has property* $\mathrm{sdp}(\eta, \delta)$,
(ii) $\| \mathrm{AffT}\, \phi(h) - \mathrm{AffT}\, \psi(h)\| < \delta$, $\forall h \in H(\eta, \delta, X)$, *and*
(iii) $K_0\phi = K_0\psi$,

then there exists a unitary $U \in B$ *such that*

$$\|\phi(f) - U\psi(f)U^*\| < \varepsilon, \quad \forall f \in F.$$

Remark 3.4 In the proof of Lemma 3.1, the finite set $H(\eta, \delta, X)$ is constructed by the following procedure. First choose $H_1 = \{\chi_{T, \frac{8}{\eta}} | T \subset X \text{ is closed set}\}$; since H_1 is a family of equi-continuous functions, there is a finite set $H \subset H_1$ such that $\mathrm{dist}(h, H_1) < \frac{\delta}{8}$, for any $h \in H$, let us denote this by $H(\eta, \delta, X)$. Notice that for any connected closed subset $X^{'}$ of X, if we consider the finite set

$$H(\eta, \delta, X^{'}) = \{f|_{X'} : f \in H(\eta, \delta, X)\} = \pi(H(\eta, \delta, X)),$$

where $\pi(f) = f|_{X'}, \forall f \in C(X)$, then the conclusion of Corollary 3.3 is also true when we consider $C(X^{'})$ instead of $C(X)$. Thus, we have the following corollary at once.

Theorem 3.5 (Uniqueness Theorem) *Let* $A = C(X)$, *with* $X = [0, 1]$, *and let a finite set* $F \subset A$ *be given. For any* $\varepsilon > 0$, *there exists* $\eta > 0$ *such that for any* $\delta > 0$, *the following statement holds:*

For any connected subset $X_s \subset [0, 1]$, *if two unital homomorphisms*

$$\phi_s, \psi_s \colon C(X_s) \to B = \bigoplus_{l=1}^{m} M_{ml}(C(Y_l)), \quad Y_l = [0, 1],$$

satisfy the conditions:

(i) ϕ_s *or* ψ_s *have property* $\mathrm{sdp}(\eta, \delta)$,
(ii) $\| \mathrm{AffT}\, \phi_s(h) - \mathrm{AffT}\, \psi_s(h)\| < \delta$, $\forall h \in H(\eta, \delta, X_s) = \pi_s(H(\eta, \delta, X))$, *and*
(iii) $K_0\phi_s = K_0\psi_s$, *then there exists a unitary* $U \in B$ *such that*

$$\|\phi_s(f) - U\psi_s(f)U^*\| < \varepsilon, \quad \forall f \in \pi_s(F),$$

where $\pi_s(f) = f|_{X_s}$ *for any* $f \in C(X)$.

4 Dichotomy Theorem

When we try to prove the isomorphism of C^*-algebras $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$, it is necessary to consider whether or not the nonzero partial maps $\phi^{i,j}_{n,m}, \psi^{i,j}_{n,m}$ have the spectrum distribution property ($\mathrm{sdp}(\eta, \delta)$; see Remark 1.7). This is an important condition in the uniqueness theorem, which is one of the key components of the intertwining argument used to prove the isomorphism of the inductive limit C*-algebra; therefore, it is important to be able to ensure that the partial maps have the spectrum distribution property.

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In this section, we will solve this problem by creating a technique to ensure that the partial maps have the spectrum distribution property. As mentioned in the introduction, this technique can also be generalized to the case of higher dimensional spectrum.

We need to make the following preparations.

Lemma 4.1 ([12, Lemma 2.9]) *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ be an AI algebra with the ideal property, with $A_n = \bigoplus_{i=1}^{k_n} A_n^i$. For any fixed n, i, and $\delta > 0$, there is $m_0 > n$ such that the following statement is true.*

For any $F = \overline{F} \subset X_n^i$, and any $m > m_0$, we have that any partial map $\phi_{n,m}^{i,j}$ satisfies either

$$\mathrm{SP}(\phi_{n,m}^{i,j})_y \cap F = \varnothing, \forall y \in X_m^j \quad or \quad \mathrm{SP}(\phi_{n,m}^{i,j})_y \cap B_\delta(F) \neq \varnothing, \quad \forall y \in X_m^j.$$

Now for any fixed $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X^i))$ and for any $\eta > 0$, apply Lemma 4.1 with $\delta = \frac{\eta}{4}$ to obtain $m_0 > n$ satisfying the conclusion of Lemma 4.1 for all $i = 1, 2, \ldots, k_n$. Considering the partial map $\phi_{n,m}^{i,j}$, by the first isomorphism theorem, there exists an injective map

$$\phi_{n,m}^{'i,j} \colon A_n^i / \ker\phi_{n,m}^{i,j} \to A_m^j.$$

Denote by $X_i^{'j}$ the closed subset of X^i such that, in the natural way,

$$A_n^i / \ker\phi_{n,m}^{i,j} \cong M_{[n,i]}(C(X_i^{'j})).$$

Set $\pi_{i,j}^{'}(f) = f|_{X_i^{'j}}$ and $\pi = \bigoplus_{i,j} \pi_{i,j}^{'}$. Then $\phi_{n,m}$ can be written as

$$A_n \xrightarrow{\pi} \widetilde{B} = \bigoplus_i \bigoplus_j M_{[n,i]}(C(X_i^{'j})) \xrightarrow{\phi} A_m,$$

where $\phi = \bigoplus_i \bigoplus_j \phi_{n,m}^{'i,j}$. Notice that $X_i^{'j}$ is not necessarily the finite disjoint union of finite intervals; we wish to enlarge $X_i^{'j}$ in ordered to turn it into a finite disjoint union of intervals. In addition, we also notice that for all $y \in X_m^j$,

$$\mathrm{SP}(\phi_{n,m}^{i,j})_y = \mathrm{SP}(\phi_{n,m}^{'i,j})_y.$$

Set

$$F_j = \{x \in X_i^{'j} | B_{\frac{\eta}{4}}(x) \cap \mathrm{SP}(\phi_{n,m}^{i,j})_y \neq \varnothing, \forall y \in X_m^j\};$$

we will prove that $X_i^{'j} = F_j$. In fact, for all $y_0 \in X_m^j, x_0 \in \mathrm{SP}(\phi_{n,m}^{'i,j})_{y_0} = \mathrm{SP}(\phi_{n,m}^{i,j})_{y_0}$, we naturally have that

$$\mathrm{SP}(\phi_{n,m}^{i,j})_{y_0} \cap \{x_0\} \neq \varnothing.$$

By Lemma 4.1,

$$\mathrm{SP}(\phi_{n,m}^{i,j})_y \cap B_{\frac{\eta}{4}}(x_0) \neq \varnothing, \forall y \in X_m^j.$$

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It means that for all $y \in X_m^j$, $\mathrm{SP}(\phi'^{i,j}_{n,m})_y \subseteq F_j$, then $\bigcup_{y\in X_m^j} \mathrm{SP}(\phi'^{i,j}_{n,m})_y \subseteq F_j$. Since $\phi'^{i,j}_{n,m}$ is injective, then

$$X_i'^{j} = \bigcup_{y\in X_m^j} \mathrm{SP}(\phi'^{i,j}_{n,m})_y = F_j.$$

And for all $x \in X_i'^{j}$, $B_{\frac{\eta}{4}}(x) \cap \mathrm{SP}(\phi^{i,j}_{n,m})_y \neq \varnothing$, for all $y \in X_m^j$.

Since $X_i'^{j}$ is a closed set in $[0,1]$, there exist $\{x_k\}_{k=1}^L$, $x_k \in X_i'^{j}$ with $X_i'^{j} \subseteq \bigcup_{k=1}^L B_{\frac{\eta}{4}}(x_k)$. By the discussion above, we have

$$B_{\frac{\eta}{4}}(x_k) \subset B_{\frac{\eta}{2}}(a), \quad B_{\frac{\eta}{2}}(a) \cap \mathrm{SP}(\phi^{i,j}_{n,m})_y \neq \varnothing,$$

for all $y \in X_m^j$, $a \in B_{\frac{\eta}{4}}(x_k)$, $k = 1, 2, \dots, L$.

Let $Y_i^{j,1}, Y_i^{j,2}, \dots, Y_i^{j,\bullet}$, $(j = 1, 2, \dots l_m)$ denote all the connected components of $\bigcup_{k=1}^L B_{\frac{\eta}{4}}(x_k) \subset [0,1]$.

Then we claim that these finite disjoint intervals

$$Y_i^{1,1}, Y_i^{1,2}, \dots, Y_i^{1,\bullet}, Y_i^{2,1}, \dots, Y_i^{j,s}, \dots, Y_i^{l_m,\bullet}$$

satisfying the following properties.

Property 1 If $\widetilde{B} = \bigoplus_{i=1}^{k_n} \bigoplus_{j=1}^{l_m} \bigoplus_s M_{[n,i]} C(Y_i^{j,s})$, then $\phi_{n,m}$ can be written as

$$\phi_{n,m} \colon A_n \xrightarrow{\pi} \widetilde{B} \xrightarrow{\bigoplus_s \phi_s} A_m,$$

where $\pi = \bigoplus_s \pi_s, \pi_s(f) = f|_{Y_i^{j,s}}$, and $\phi_s \colon M_{[n,i]}(C(Y_i^{j,s})) \to A_m^j$ is the homomorphism induced by $\phi^{i,j}_{n,m}$.

Property 2 We have

$$\mathrm{SP}(\phi_s)_y \cap B_{\frac{\eta}{2}}(x_0, Y_i^{j,s}) \neq \varnothing, \quad \forall x_0 \in Y_i^{j,s}, \forall y \in X_m^j.$$

In fact, if $x_0 \in Y_i^{j,s}$, then, by construction, we have $x_0 \in B_{\frac{\eta}{4}}(x_k) \subseteq Y_i^{j,s}$ for some k. Hence' $\mathrm{SP}(\phi'^{i,j}_{n,m})_y \cap B_{\frac{\eta}{4}}(x_k) \neq \varnothing$. Notice that

$$\mathrm{SP}(\phi_s)_y = \mathrm{SP}(\phi'^{i,j}_{n,m})_y \cap Y_i^{j,s}, \forall y \in X_m^j,$$

and $B_{\frac{\eta}{4}}(x_k) \subseteq Y_i^{j,s}$, and we have

$$\mathrm{SP}(\phi_s)_y \cap B_{\frac{\eta}{2}}(x_0, Y_i^{j,s}) = \\ (\mathrm{SP}(\phi'^{i,j}_{n,m})_y \cap Y_i^{j,s}) \cap B_{\frac{\eta}{2}}(x_0, Y_i^{j,s}) \supset \mathrm{SP}(\phi'^{i,j}_{n,m})_y \cap B_{\frac{\eta}{4}}(x_k) \cap Y_i^{j,s} \neq \varnothing.$$

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The following is the main theorem of this section.

Theorem 4.2 *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ be AI algebra with the ideal property, where $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_n^i))$, $X_n^i \equiv [0,1]$. For any fixed A_n, and any $\eta > 0$, there exist $\delta > 0$, a positive integer $m_0 > n$, subintervals $Y_i^1, Y_i^2, \ldots, Y_i^\bullet \subset X_n^i$, $i = 1, 2, \ldots, k_n$, and a homomorphism*

$$\phi\colon \widetilde{B} = \bigoplus_{i=1}^{k_n} \bigoplus_{s} M_{[n,i]}(C(Y_i^s)) \to A_m,$$

$(m > m_0)$ such that

(i) *$\phi_{n,m}$ factors as $\phi_{n,m}\colon A_n \xrightarrow{\pi} \widetilde{B} \xrightarrow{\phi} A_m$, where $\pi(f) = (f|_{Y_i^1}, f|_{Y_i^2}, \ldots, f|_{Y_i^\bullet}) \in \widetilde{B}$, for $f \in A_n^i$;*

(ii) *the homomorphism ϕ satisfies the dichotomy condition,* i.e., *for all Y_i^s, the partial map $\phi_s = \phi_i^{j,s} = M_{[n,i]}(C(Y_i^s)) \to A_m^j$ is either zero or has the property* $\mathrm{sdp}(\eta, \delta)$*. And for any $m' > m$, each $\phi_{m,m'} \circ \phi$ also satisfies the dichotomy condition.*

Proof For any fixed A_n^i and any η, we can find corresponding $m_0 > 0$, and subsets

$$Y_i^{1,1}, Y_i^{1,2}, \ldots, Y_i^{1,\bullet}, Y_i^{2,1}, \ldots, Y_i^{j,s}, \ldots, Y_i^{l_m,\bullet} \subset X_n^i,$$

renamed as $Y_i^1, Y_i^2, \ldots, Y_i^\bullet$ that satisfy conclusion (i) (by Property 1). And for all $x_0 \in Y_i^s$, by Property 2, we have

$$B_\eta(x_0, Y_i^s) \cap \mathrm{SP}(\phi_s)_y \neq \varnothing.$$

Choose $\delta = \min_{j,s}\{\frac{1}{\mathrm{rank}(\phi_s(1_{M_{[n,i]}(C(Y_i^{j,s}))}))}\}$, then for any $x \in Y_i^{j,s}$, we have

$$\#\,\mathrm{SP}(\phi_i^s)_y \cap B_\eta(x) \geq 1 \geq \delta \#\,\mathrm{SP}(\phi_i^s)_y.$$

Now we only need to prove that for any $m' > m$, each nonzero partial map of $\phi_{m,m'} \circ \phi$ also has the property $\mathrm{sdp}(\eta, \delta)$.

In fact, we only need to prove the following proposition. If the homomorphism

$$\phi\colon A := \bigoplus_{i=1}^{m} M_{n_i}(C(X^i)) \to B := \bigoplus_{j=1}^{L} M_{n_j}(C(Y^j))$$

satisfies the dichotomy condition, then for any homomorphism

$$\psi\colon B = \bigoplus_{j=1}^{L} M_{n_j}(C(Y^j)) \to C := \bigoplus_{k=1}^{N} M_{n_k}(C(Z^k)), \quad \psi \circ \phi$$

also satisfies the dichotomy condition, where $X^i = Y^j = Z^k = [0,1]$, for any i, j, k.

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Notice that for each pair (i,k), there is a partial map

$$(\psi\circ\phi)^{i,k}=\bigoplus_{j=1}^{L}\psi^{j,k}\circ\phi^{i,j}\colon M_{n_i}(C(X^i))\to M_{n_k}(C(Z^k)).$$

For any $z\in Z^k$,

$$\mathrm{SP}(\psi\circ\phi)^{i,k}_z=\bigcup_{j=1}^{L}\bigcup_{y\in\mathrm{SP}\,\psi^{j,k}_z}\mathrm{SP}(\phi^{i,j})_y.$$

Since ϕ satisfies the dichotomy condition, then for any $B_\eta(x)$ and j, we have

$$\#(\mathrm{SP}(\phi^{i,j})_y\cap B_\eta(x))\geq\delta\frac{\mathrm{rank}\,\phi^{i,j}(\mathbf{1}_{M_{n_i}(C(X^i))})}{\mathrm{rank}(\mathbf{1}_{M_{n_i}(C(X^i))})}.$$

(Notice that if $\phi^{i,j}=0$, then both sides of the equation are equal to zero, so it still holds.) For convenience, we let $\mathbf{1}_{M_{n_i}(C(X^i))}$ be $\mathbf{1}$. And for any projection $p\in M_{n_j}(C(Y^j))$,

$$\#(\mathrm{SP}(\psi^{j,k}_z))=\frac{\mathrm{rank}\,\psi^{j,k}(p)}{\mathrm{rank}(p)}.$$

For each pair i,j,k, $\phi^{i,j}(\mathbf{1})\neq 0$. If let $\phi^{i,j}(\mathbf{1})=p$, then

$$\begin{aligned}\#(\mathrm{SP}(\psi^{j,k}\circ\phi^{i,j})_z\cap B_\eta(x))&=\sum_{y\in\mathrm{SP}\,\psi^{j,k}_z}\#(\mathrm{SP}(\phi^{i,j})_y\cap B_\eta(x))\\&\geq\frac{\mathrm{rank}\,\psi^{j,k}(\phi^{i,j}(\mathbf{1}))}{\mathrm{rank}\,\phi^{i,j}(\mathbf{1})}\delta\frac{\mathrm{rank}\,\phi^{i,j}(\mathbf{1})}{\mathrm{rank}(\mathbf{1})}\\&=\delta\frac{\mathrm{rank}\,\psi^{j,k}(\phi^{i,j}(\mathbf{1}))}{\mathrm{rank}(\mathbf{1})}.\end{aligned}$$

Thus,

$$\begin{aligned}\#(\mathrm{SP}(\psi\circ\phi)^{i,k}_z\cap B_\eta(x))&=\sum_j\#(\mathrm{SP}(\psi^{j,k}\circ\phi^{i,j})_z\cap B_\eta(x))\\&\geq\delta\frac{\sum\mathrm{rank}\,\psi^{j,k}(\phi^{i,j}(\mathbf{1}))}{\mathrm{rank}(\mathbf{1})}=\delta\frac{\mathrm{rank}(\psi\circ\phi)^{i,k}(\mathbf{1})}{\mathrm{rank}(\mathbf{1})}.\end{aligned}$$

This completes the proof. ∎

5 Classification

The following theorem is the main result of this paper.

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Theorem 5.1 *For AI algebras with the ideal property*

$$A = \lim_{n\to\infty}(A_n, \phi_{n,m}) \quad \text{and} \quad B = \lim_{n\to\infty}(B_n, \psi_{n,m}),$$

where

$$A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_n^i)) \quad \text{and} \quad B_m = \bigoplus_{j=1}^{l_m} M_{\{m,j\}}(C(Y_m^j)),$$

with $X_n^i \equiv Y_m^j \equiv [0,1]$, *satisfying the following conditions:*

(i) *There exists a scaled ordered group isomorphism* $\alpha\colon K_0(A) \to K_0(B)$;

(ii) *For any* $e \in \mathcal{P}(A), f \in \mathcal{P}(B)$ *with* $\alpha[e] = [f]$, *there exists an isomorphism* $\xi^{e,f}\colon \operatorname{AffT}(eAe) \to \operatorname{AffT}(fBf)$ *such that for any* $e' < e, f' < f$ *with* $\alpha[e'] = [f'], \xi^{e,f}, \xi^{e',f'}$ *are compatible,* i.e., *the diagram*

$$\begin{array}{ccc} \operatorname{AffT}(eAe) & \xrightarrow{\ \xi^{e,f}\ } & \operatorname{AffT}(fBf) \\ \uparrow & & \uparrow \\ \operatorname{AffT}(e'Ae') & \xrightarrow{\ \xi^{e',f'}\ } & \operatorname{AffT}(f'Bf') \end{array}$$

is commutative.

Then there exists an isomorphism $\Gamma\colon A \to B$ *such that:*

(a) $K_0(\Gamma) = \alpha$;

(b) *if* $\Gamma_e\colon eAe \to \Gamma(e)B\Gamma(e)$ *is the restriction of* Γ *in eAe, then*

$$\operatorname{AffT}(\Gamma_e) = \xi^{e,f}, \forall [f] = [\Gamma(e)].$$

Remark 5.2 To complete the proof of the classification theorem, we need to do some preparation and give some lemmas.

Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$ be AI algebras with the ideal property satisfying the conditions of Theorem 5.1, where

$$A_n = \bigoplus_i A_n^i, \quad B_m = \bigoplus_j B_m^j,$$

$$A_n^i = P_n^i M_{[n,i]}(C(X_n^i))P_n^i, \quad B_m^j = Q_m^j M_{\{m,j\}}(C(Y_m^j))Q_m^j, \quad P_n^i, \quad Q_m^j$$

are projections of $M_{[n,i]}(C(X_n^i))$ and $M_{\{m,j\}}(C(Y_m^j))$ respectively.

Suppose that $\xi\colon \operatorname{AffT} A \to \operatorname{AffT} B$ and $\alpha\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ are both scaled ordered group isomorphisms. Furthermore, α and ξ are compatible. If A and B are both unital, then by Lemma 1.8 and Remark 1.9, there exists an intertwining at the K_0 stage

$$\begin{array}{ccccccccc} \mathrm{K}_0A_1 & \longrightarrow & \mathrm{K}_0A_2 & \longrightarrow & \mathrm{K}_0A_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0A \\ \downarrow{\scriptstyle\alpha_1} & \nearrow{\scriptstyle\beta_1} & \downarrow{\scriptstyle\alpha_2} & \nearrow{\scriptstyle\beta_2} & \downarrow{\scriptstyle\alpha_3} & \nearrow{\scriptstyle\beta_3} & & & \downarrow{\scriptstyle\alpha} \\ \mathrm{K}_0B_1 & \longrightarrow & \mathrm{K}_0B_2 & \longrightarrow & \mathrm{K}_0B_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0B \end{array},$$

where α_i, β_i are all scaled ordered group homomorphisms, and there exist homomorphisms $\widetilde{\Lambda}_i \colon A_i \to B_i$, $\widetilde{\mathcal{M}}_i \colon B_i \to A_{i+1}$ such that $\mathrm{K}_0(\widetilde{\Lambda}_i) = \alpha_i$, $\mathrm{K}_0(\widetilde{\mathcal{M}}_i) = \beta_i$.

Considering the proof of the main theorem, we need to construct a new inductive system to make the homomorphisms unital. To establish this, we only need to use the projections to cut down each summand of the original inductive sequence. The following is the progress:

Now for fixed A_n^i, define

$$[A_{n+k}]_i = \phi_{n,n+k}(\mathbf{1}_{A_n^i})A_{n+k}\phi_{n,n+k}(\mathbf{1}_{A_n^i}), \quad [A_n]_i = A_n^i,\ e_i = \phi_{n,\infty}(\mathbf{1}_{A_n^i}),$$
$$e_iAe_i = \phi_{n,\infty}(\mathbf{1}_{A_n^i})A\phi_{n,\infty}(\mathbf{1}_{A_n^i}), \quad k = 1, 2, \dots,$$

and

$$[B_n]_i = \widetilde{\Lambda}_i(\mathbf{1}_{A_n^i})B_n\widetilde{\Lambda}_i(\mathbf{1}_{A_n^i}), \quad [B_{n+k}]_i = \psi_{n,n+k}(\widetilde{\Lambda}_i(\mathbf{1}_{A_n^i}))B_{n+k}\psi_{n,n+k}(\widetilde{\Lambda}_i(\mathbf{1}_{A_n^i})),$$
$$f_i = \psi_{n,\infty}(\widetilde{\Lambda}_i(\mathbf{1}_{A_n^i})), \quad f_iBf_i = \psi_{n,\infty}(\widetilde{\Lambda}_i(\mathbf{1}_{A_n^i}))B\psi_{n,\infty}(\widetilde{\Lambda}_i(\mathbf{1}_{A_n^i})), \quad k = 1, 2, \dots.$$

Then we can get the new inductive limits

$$e_iAe_i = \lim_{k\to\infty}([A_{n+k}]_i, [\phi_{n+k,n+l}]_i),\ f_iBf_i = \lim_{k\to\infty}([B_{n+k}]_i, [\psi_{n+k.n+l}]_i),$$

where $\mathbf{1}_{A_n^i}$ denotes the unit of A_n^i, and $[\phi_{n+k,n+l}]_i, [\psi_{n+k.n+l}]_i$ denote the unital homomorphisms induced by $\phi_{n,n+k}$ and $\psi_{n+k,n+l}$ respectively. We also can get the following intertwining

$$\begin{array}{ccccccccc}
\mathrm{K}_0[A_n]_i & \longrightarrow & \mathrm{K}_0[A_{n+1}]_i & \longrightarrow & \mathrm{K}_0[A_{n+2}]_i & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0 e_iAe_i \\
\downarrow \alpha_1^i & \nearrow \beta_1^i & \downarrow \alpha_2^i & \nearrow \beta_2^i & \downarrow \alpha_3^i & \nearrow \beta_3^i & & & \downarrow \alpha^{e_i,f_i} \\
\mathrm{K}_0[B_n]_i & \longrightarrow & \mathrm{K}_0[B_{n+1}]_i & \longrightarrow & \mathrm{K}_0[B_{n+2}]_i & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0 f_iBf_i
\end{array},$$

where $\alpha_k^i, \beta_k^i, \alpha^{e_i,f_i} (k = 1, 2, \dots)$ are all scaled ordered, and $\alpha^{e_i,f_i}[e_i] = [f_i]$.

Similarly, for fixed B_m^j, we can also get other two new inductive limits $\widetilde{f}_jB\widetilde{f}_j$ and $\widetilde{e}_jA\widetilde{e}_j$, where

$$\widetilde{f}_j = \psi_{m,\infty}(\mathbf{1}_{B_m^j}), \quad \widetilde{e}_j = \phi_{m+1,\infty} \circ \widetilde{\mathcal{M}}_m(\mathbf{1}_{B_m^j}), \quad \text{and} \quad \alpha[\widetilde{e}_j] = [\widetilde{f}_j].$$

If we let

$$\{B_m\}_j = B_m^j, \quad \{B_{m+k}\}_j = \psi_{m,m+k}(\mathbf{1}_{B_m^j})B_{m+k}\psi_{m,m+k}(\mathbf{1}_{B_m^j}),$$

and $\{\psi_{m+k,m+l}\}_j \colon \{B_{m+k}\}_j \to \{B_{m+l}\}_j$ be the unital homomorphism induced by $\psi_{m+k,m+l}$ $(k = 0, 1, 2 \dots)$, and let

$$\{A_{m+1}\}_j = \widetilde{\mathcal{M}}_m(\mathbf{1}_{B_m^j})A_{m+1}\widetilde{\mathcal{M}}_m(\mathbf{1}_{B_m^j}),$$
$$\{A_{m+k}\}_j = \phi_{m+1,m+k}(\mathbf{1}_{\{A_{m+1}\}_j})A_{m+k}\phi_{m+1,m+k}(\mathbf{1}_{\{A_{m+1}\}_j}),$$

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$\{\phi_{m+k,m+l}\}_j\colon \{A_{m+k}\}_j \to \{A_{m+l}\}_j$ be the unital homomorphism induced by $\phi_{m+k,m+l}$, then we have

$$\widetilde{e}_j A \widetilde{e}_j = \lim_{k\to\infty}(\{A_{m+k}\}_j, \{\phi_{m+k,m+l}\}_j), \quad \widetilde{f}_j B \widetilde{f}_j = \lim_{k\to\infty}(\{B_{m+k}\}_j, \{\psi_{m+k,m+l}\}_j).$$

Later we will discuss the cut down algebra, $q_s B^j_m q_s$, where $\{q_s\}_{s=1}^{\infty}$ is a set of mutually orthogonal projections. Then, for any non-zero projection $q_s \in B^j_m$, considering $q_s B^j_m q_s$ instead of B^j_m, we also can obtain the following inductive limits:

$$\widetilde{e}_{s,j} A \widetilde{e}_{s,j} = \lim_{k\to\infty}(\{A_{m+k}\}_{s,j}, \{\phi_{m+k,m+l}\}_{s,j}), \ \widetilde{f}_{s,j} B \widetilde{f}_{s,j} = \lim_{k\to\infty}(\{B_{m+k}\}_{s,j}, \{\psi_{m+k,m+l}\}_{s,j}),$$

and $\widetilde{e}_{s,j} < \widetilde{e}_j$, $\widetilde{f}_{s,j} < \widetilde{f}_j$, $\alpha[\widetilde{e}_{s,j}] = [\widetilde{f}_{s,j}]$, where the symbols $\widetilde{e}_{s,j}$, $\widetilde{f}_{s,j}$, $\{A_{m+k}\}_{s,j}$, $\{B_{m+k}\}_{s,j}$, and $\{\psi_{m+k,m+l}\}_{s,j}$ can be defined in the same way as $\widetilde{e}_j$, $\widetilde{f}_j$, $\{A_{m+k}\}_j$, $\{B_{m+k}\}_j$, and $\{\psi_{m+k,m+l}\}_j$.

To avoid confusion, we need to point out the differences between the notations above. The symbols $[\,\cdot\,]_i$,$\{\,\cdot\,\}_j$ always denote the algebras cut down by the image of unit of A^i_n, B^j_m under related maps respectively.

Using the definitions and symbols mentioned above, we can obtain the following lemmas.

Lemma 5.3 *Let $\{q_s\}_{s=1}^{\bullet}$ be a set of finitely many nonzero projections in $B^j_{m_1}$, $q_s q_{s'} = q_{s'} q_s = 0, s \neq s'$, $m_1 > 0$, and let $F_s \subset \mathrm{Aff}\,\mathrm{T}(q_s B^j_{m_1} q_s)$ be a finite set. For any $\varepsilon > 0$, there exists $\delta > 0$ and finite set $G \subset \mathrm{Aff}\,\mathrm{T}\, B^j_{m_1}$, such that the following statement is true.*

If a homomorphism $\mathcal{M}_j\colon B^j_{m_1} \to \{A_{n_2}\}_j$ satisfies that

$$\|\,\mathrm{Aff}\mathrm{T}\{\phi_{n_2,\infty}\}_j \circ \mathrm{Aff}\mathrm{T}\,\mathcal{M}_j(g) - (\xi^{\widetilde{e}_j,\widetilde{f}_j})^{-1} \circ \mathrm{Aff}\mathrm{T}\{\psi_{m_1,\infty}\}_j(g)\| < \delta, \ \ \forall g \in G,$$

then the unital homomorphism $\mathcal{M}_{s,j}\colon q_s B^j_{m_1} q_s \to \{A_{n_2}\}_{s,j}$ induced by $\mathcal{M}_j$ satisfies that

$$\|\,\mathrm{Aff}\mathrm{T}\{\phi_{n_2,\infty}\}_{s,j} \circ \mathrm{Aff}\mathrm{T}\,\mathcal{M}_{s,j}(f) - (\xi^{\widetilde{e}_{s,j},\widetilde{f}_{s,j}})^{-1} \circ \mathrm{Aff}\mathrm{T}\{\psi_{m_1,\infty}\}_{s,j}(f)\| < \varepsilon, \ \ \forall f \in F_s.$$

Proof Let $I_s\colon q_s B^j_{m_1} q_s \to B^j_{m_1}$ be the imbedding map, and $G \triangleq \bigcup_s \mathrm{Aff}\mathrm{T}\, I_s(F_s)$. By the conditions of this lemma, we can get $\mathrm{Aff}\mathrm{T}\, I_s(f) \in G, \forall f \in F_s$. Now let $\delta = \min_s \frac{\mathrm{rank}\, q_s}{\mathrm{size} B^j_{m_1}} \cdot \varepsilon$. Let the unital homomorphism $\mathcal{M}_j$ satisfy that

$$\Delta_s \triangleq \big\|\,\mathrm{Aff}\mathrm{T}\{\phi_{n_2,\infty}\}_j \circ \mathrm{Aff}\mathrm{T}\,\mathcal{M}_j(\mathrm{Aff}\mathrm{T}\, I_s(f)) \\ - (\xi^{\widetilde{e}_j,\widetilde{f}_j})^{-1} \circ \mathrm{Aff}\mathrm{T}\{\psi_{m_1,\infty}\}_j(\mathrm{Aff}\mathrm{T}\, I_s(f))\big\| < \delta, \quad \forall f \in F_s;$$

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and notice that if AffT is a covariant functor, then the following diagrams are all commutative:

$$(5.1)\qquad \begin{array}{ccc} \mathrm{AffT}\, B^j_{m_1} & \xrightarrow{\mathrm{AffT}\,\mathcal{M}_j} & \mathrm{AffT}\{A_{n_2}\}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}\{B_{m_1}\}_{s,j} & \xrightarrow{\mathrm{AffT}\,\mathcal{M}_{s,j}} & \mathrm{AffT}\{A_{n_2}\}_{s,j} \end{array}$$

$$(5.2)\qquad \begin{array}{ccc} \mathrm{AffT}\, B^j_{m_1} & \xrightarrow{\mathrm{AffT}\{\psi_{m_1,\infty}\}_j} & \mathrm{AffT}\, \widetilde{f}_j B \widetilde{f}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}\{B_{m_1}\}_{s,j} & \xrightarrow{\mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}} & \mathrm{AffT}\, \widetilde{f}_{s,j} B \widetilde{f}_{s,j} \end{array}$$

$$(5.3)\qquad \begin{array}{ccc} \mathrm{AffT}\{A_{n_2}\}_j & \xrightarrow{\mathrm{AffT}\{\phi_{n_2,\infty}\}_j} & \mathrm{AffT}\, \widetilde{e}_j A \widetilde{e}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}\{A_{n_2}\}_{s,j} & \xrightarrow{\mathrm{AffT}\{\phi_{n_2,\infty}\}_{s,j}} & \mathrm{AffT}(\widetilde{e}_{s,j} A \widetilde{e}_{s,j}). \end{array}$$

By the compatibility of AffT eAe and AffT $e'Ae'$ $(e' < e)$ (Theorem 5.1(ii)), the diagram

$$(5.4)\qquad \begin{array}{ccc} \mathrm{AffT}\, \widetilde{e}_j A \widetilde{e}_j & \xrightarrow{\xi^{\widetilde{e}_j,\widetilde{f}_j}} & \mathrm{AffT}\, \widetilde{f}_j B \widetilde{f}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}(\widetilde{e}_{s,j} A \widetilde{e}_{s,j}) & \xrightarrow{\xi^{\widetilde{e}_{s,j},\widetilde{f}_{s,j}}} & \mathrm{AffT}(\widetilde{f}_{s,j} B \widetilde{f}_{s,j}) \end{array}$$

is also commutative.

For simplicity, we still use I_s to denote the following imbedding maps:

$$I^1_s\colon \{A_{n_2}\}_{s,j} \to \{A_{n_2}\}_j,\ I^2_s\colon \widetilde{f}_{s,j} B \widetilde{f}_{s,j} \to \widetilde{f}_j B \widetilde{f}_j,\ I^3_s\colon \widetilde{e}_{s,j} A \widetilde{e}_{s,j} \to \widetilde{e}_j A \widetilde{e}_j.$$

Since both diagrams (5.1) and (5.2) are commutative, we have

$$\Delta_s = \|\,\mathrm{AffT}(\{\phi_{n_2,\infty}\}_j \circ I_s \circ \mathcal{M}_{s,j})(f) - (\xi^{\widetilde{e}_j,\widetilde{f}_j})^{-1} \circ \mathrm{AffT}(I_s \circ \{\psi_{m_1,\infty}\}_{s,j})(f)\| < \delta.$$

Since both diagrams (5.3) and (5.4) are also commutative, we have

$$\Delta_s = \Big\|\,\mathrm{AffT}\, I_s\big(\mathrm{AffT}\{\phi_{n_2,\infty}\}_{s,j} \circ \mathrm{AffT}\,\mathcal{M}_{s,j}(f) - (\xi^{\widetilde{e}_{s,j},\widetilde{f}_{s,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}(f)\big)\Big\| < \delta.$$

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By Remark 1.10, we have

$$\| \operatorname{AffT} I_s(f')\| = \frac{\operatorname{rank} q_s}{\operatorname{size} B^j_{m_1}} \|f'\|, \quad \forall f' \in \operatorname{AffT} \widetilde{e}_{s,j} A \widetilde{e}_{s,j}.$$

Since $\delta = \min_s(\frac{\operatorname{rank} q_s}{\operatorname{size} B^j_{m_1}}) \cdot \varepsilon$, then we have

$$\| \operatorname{AffT}\{\phi_{n_2,\infty}\}_{s,j} \circ \operatorname{AffT} \mathcal{M}_{s,j}(f) - (\xi^{\widetilde{e}_{s,j},\widetilde{f}_{s,j}})^{-1} \circ \operatorname{AffT}\{\psi_{m_1,\infty}\}_{s,j}(f)\| < \varepsilon,$$

for any $f \in F_s$. This completes the proof. ■

Lemma 5.4 *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$ be AI algebras with the ideal property and satisfying the conditions of Theorem* 5.1*, where $A_n = \bigoplus_i A^i_n$ and $B_m = \bigoplus_j B^j_m$. For fixed A_{n_1} $(n_1 > 0)$, let $F_i \subset \operatorname{AffT} A^i_{n_1}$ be a finite set, $i = 1, 2, \dots, k_{n_1}$, and $\varepsilon > 0$, then there exist homomorphisms $\Lambda^i_1 \colon A^i_{n_1} \to [B_{m_1}]_i$ with following properties:*

(i) $K_0\Lambda^i_1 = K_0[\psi_{n_1,m_1}]_i \circ \alpha^i_{n_1}$, *and*
(ii) $\| \operatorname{AffT}[\psi_{m_1,\infty}]_i \circ \operatorname{AffT} \Lambda^i_1(f) - \xi^{e_i,f_i} \circ \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| < \frac{\varepsilon}{4}, \ \forall f \in F_i$.

And let $\Lambda_1 \colon \bigoplus_i A^i_{n_1} \to \bigoplus_j B^j_{m_1}$ be defined by $\Lambda_1 = \bigoplus_i \Lambda^i_1$.

Proof For $A^i_{n_1}$ and the unital inductive limits

$$e_iAe_i = \lim_{k\to\infty}([A_{n_1+k}]_i, [\phi_{n_1+k,n_1+l}]_i), \qquad f_iBf_i = \lim_{k\to\infty}([B_{n_1+k}]_i, [\psi_{n_1+k,n_1+l}]_i),$$

applying the existence theorem, we can find unital homomorphisms $\overline{\Lambda}^i_1 \colon A^i_{n_1} \to [B_{K_i}]_i \triangleq \overline{\Lambda}^i_1(\mathbf{1}_{A^i_{n_1}})B_{K_i}\overline{\Lambda}^i_1(\mathbf{1}_{A^i_{n_1}})$ such that

$$\| \operatorname{AffT}[\psi_{K_i,\infty}]_i \circ \operatorname{AffT} \overline{\Lambda}^i_1(f) - \xi^{e_i,f_i} \circ \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| < \frac{\varepsilon}{4}, \ \forall f \in F,$$

and $\mathrm{K}_0(\overline{\Lambda}^i_1) = \mathrm{K}_0[\psi_{n_1,K_i}]_i \circ \alpha^i_{n_1}$. Let $m_1 = \max\{K_1, K_2, \dots, K_{k_{n_1}}\}$, $\Lambda^i_1 = [\psi_{K_i,m_1}]_i \circ \overline{\Lambda}^i_1$, then

$$\begin{aligned}
&\| \operatorname{AffT}[\psi_{m_1,\infty}]_i \circ \operatorname{AffT} \Lambda^i_1(f) - \xi^{e_i,f_i} \circ \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| \\
&\quad = \| \operatorname{AffT}[\psi_{m_1,\infty}]_i \circ \operatorname{AffT}([\psi_{K_i,m_1}]_i \circ \overline{\Lambda}^i_1)(f) - \xi^{e_i,f_i} \circ \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| \\
&\quad = \| \operatorname{AffT}[\psi_{K_i,\infty}]_i \circ \operatorname{AffT} \overline{\Lambda}^i_1(f) - \xi^{e_i,f_i} \circ \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| < \frac{\varepsilon}{4}.
\end{aligned}$$

And $\mathrm{K}_0\Lambda^i_1 = \mathrm{K}_0[\psi_{n_1,m_1}]_i \circ \alpha^i_{n_1}$. ■

Remark 5.5 Similarly with the proof of Lemma 5.4, we can prove the following statement. Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$ be AI algebras with the ideal property mentioned in Lemma 5.4, where $A_n = \bigoplus_i A^i_n$ and $B_m = \bigoplus_j B^j_m$. For any fixed B_{m_1}, let $G_j \subset \operatorname{AffT} B^j_{m_1}$ be a finite set, $j = 1, 2, \dots l_{m_1}$, and $\delta > 0$, then there exist homomorphisms $\mathcal{M}^j_1 \colon B^j_{m_1} \to \{A_{n'_2}\}_j$ with the following properties:

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(i) $\mathrm{K}_0\mathcal{M}_1^j = \mathrm{K}_0\{\psi_{m_1+1,n_2}\}_j \circ \beta_{m_1}^j$, and
(ii) $\|\operatorname{AffT}\{\phi_{n_2',\infty}\}_j \circ \operatorname{AffT}\mathcal{M}_1^j(g) - (\xi^{\widetilde{e}_j,\widetilde{f}_j})^{-1} \circ \operatorname{AffT}\{\psi_{m_1,\infty}\}_j(g)\| < \delta, \forall g \in G_j$.

Lemma 5.6 *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$ be AI algebras with the ideal property mentioned in Lemma 5.4. Let $F_i \subset \operatorname{AffT} A_{n_1}^i$ be a finite set, $\varepsilon > 0$, and let $\Lambda_1^i \colon A_{n_1}^i \to [B_{m_1}]_i$ $i = 1, 2, \dots k_{n_1}$ be the homomorphisms described in Lemma 5.4, then there exist finite sets $G_j \subset \operatorname{AffT} B_{m_1}^j$, $\delta > 0$, $j = 1, 2, \dots l_{m_1}$ such that the following statements hold.*

If the homomorphism $\mathcal{M}_1^j \colon B_{m_1}^j \to \{A_{n_2}\}_j$ satisfies the properties described in Remark 5.5, then there exists $n_2 > 0$ such that the homomorphism $\mathcal{M}_1 := [\phi_{n_2,n_2'}]_i \circ \bigoplus_j \mathcal{M}_1^j$ satisfies the following conditions:

(i) *$K_0[\mathcal{M}_1 \circ \Lambda_1]_i = K_0[\phi_{n_1,n_2}]_i$, and*
(ii) *$\|\operatorname{AffT}[\phi_{n_1,n_2}]_i(f) - \operatorname{AffT}[\mathcal{M}_1 \circ \Lambda_1]_i(f)\| < \varepsilon$, $\forall f \in F_i$, where*

$$[\mathcal{M}_1 \circ \Lambda_1]_i \colon A_{n_1}^i \to (\mathcal{M}_1 \circ \Lambda_1)(\mathbf{1}_{A_{n_1}^i})A_{n_2}(\mathcal{M}_1 \circ \Lambda_1)(\mathbf{1}_{A_{n_1}^i})$$

is unital.

Proof Let Λ_1^i and Λ_1 be the homomorphisms we mentioned in Lemma 5.4, and let $\Lambda_1^{i,j} \colon A_{n_1}^i \to \Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})B_{m_1}^j\Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})$ be the partial map of Λ_1^i.

For

$$\widetilde{e}_jA\widetilde{e}_j = \lim_{k\to\infty}(\{A_{m_1+k}\}_j, \{\phi_{m_1+k,m_1+l}\}_j), \quad \widetilde{f}_jB\widetilde{f}_j = \lim_{k\to\infty}(\{B_{m_1+k}\}_j, \{\psi_{m_1+k,m_1+l}\}_j),$$

$\delta > 0$ and the finite subset $G_{i,j} := \operatorname{AffT} I_{i,j}(\operatorname{AffT}\Lambda_1^{i,j}(F))$, $G_j = \bigcup_i G_{i,j}$, by the statement of Remark 5.5, we can obtain a unital homomorphism $\mathcal{M}_1^j \colon B_{m_1}^j \to \{A_{n_2'}\}_j$, such that

$$\|\operatorname{AffT}\{\phi_{n_2',\infty}\}_j \circ \operatorname{AffT}\mathcal{M}_1^j(g) - (\xi^{\widetilde{e}_j,\widetilde{f}_j})^{-1} \circ \operatorname{AffT}\{\psi_{m_1,\infty}\}_j(g)\| < \delta, \forall g \in G_j,$$

where

$$\delta \triangleq \min_{i,j}\left\{\frac{\operatorname{rank}\Lambda_1^{i,j}\left(\mathbf{1}_{A_{n_1}^i}\right)}{\operatorname{size}B_{m_1}^j}\right\} \cdot \frac{\varepsilon}{4}, \quad (\text{and } \operatorname{rank}\Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i}) \neq 0)$$

as that of chosen in Lemma 5.3 for $\frac{\varepsilon}{4}$, and $I_{i,j}$ is the imbedding map from $\Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})B_{m_1}^j\Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})$ to $B_{m_1}^j$.

By Lemma 5.3, if

$$\mathcal{M}_1^{i,j} \colon \Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})B_{m_1}^j\Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i}) \to \mathcal{M}_1^{i,j} \circ \Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})A_{n_2'}\mathcal{M}_1^{i,j} \circ \Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})$$

is the unital homomorphism induced by $\mathcal{M}_1^j$, where projections $\{\Lambda_1^{i,j}(\mathbf{1}_{A_{n_1}^i})\}_{i=1}^{\bullet} = \{q_s\}_{s=1}^{\bullet}$ (see q_s in Remark 5.2 or Lemma 5.3, here let i=s). Then by Lemma 5.3, we have

$$\|\operatorname{AffT}\{\phi_{n_2',\infty}\}_{i,j} \circ \operatorname{AffT}\mathcal{M}_1^{i,j}(g) - (\xi^{\widetilde{e}_{i,j},\widetilde{f}_{i,j}})^{-1} \circ \operatorname{AffT}\{\psi_{m_1,\infty}\}_{i,j}(g)\| < \frac{\varepsilon}{4},$$

$$\forall g \in \operatorname{AffT}\Lambda_1^{i,j}(F).$$

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Let $\overline{I}_{ij}$ be the imbedding map from $\Lambda_1^{i,j}(\mathbf{1}_{A^i_{n_1}})B^j_{m_1}\Lambda_1^{i,j}(\mathbf{1}_{A^i_{n_1}})$ to $\Lambda_1^i(\mathbf{1}_{A^i_{n_1}})B_{m_1}\Lambda_1^i(\mathbf{1}_{A^i_{n_1}}) = \bigoplus_j \Lambda_1^{i,j}(\mathbf{1}_{A^i_{n_1}})B^j_{m_1}\Lambda_1^{i,j}(\mathbf{1}_{A^i_{n_1}})$. Then

$$\operatorname{AffT}\overline{I}_{i,j}(f) = \underbrace{0 \oplus 0 \oplus \cdots 0 \oplus f}_{j} \oplus 0 \cdots \oplus 0.$$

Let $\mathcal{M}'^i$ be the restriction of $M_1 \triangleq \bigoplus_j \mathcal{M}_1^j$ on $\Lambda_1^i(\mathbf{1}_{A^i_{n_1}})B_{m_1}\Lambda_1^i(\mathbf{1}_{A^i_{n_1}})$. Then $\mathcal{M}'^i = \bigoplus_j \mathcal{M}_1^{i,j}$.

Completely similar to the proof of Lemma 5.3, we have

$$\|\operatorname{AffT}\overline{I}_{i,j}\big(\operatorname{AffT}\{\phi_{n_2',\infty}\}_{i,j} \circ \operatorname{AffT}\mathcal{M}_1^{i,j}(g) - (\xi^{\widetilde{e}_{i,j},\widetilde{f}_{i,j}})^{-1} \circ \operatorname{AffT}\{\psi_{m_1,\infty}\}_{i,j}(g)\big)\| < \frac{\varepsilon}{4},$$

for any $g \in \operatorname{AffT}(\Lambda_1^{i,j}(F))$. And for any $f \in F_i$, we have

$$\begin{aligned}
&\|\operatorname{AffT}([\phi_{n_2',\infty}]_i \circ \mathcal{M}'^i \circ \Lambda_1^i)(f) - (\xi^{e_i,f_i})^{-1} \circ \operatorname{AffT}([\psi_{m_1,\infty}]_i \circ \Lambda_1^i)(f)\| \\
&\quad = \|\operatorname{AffT}([\phi_{n_2',\infty}]_i \circ \mathcal{M}'^i)(\bigoplus_j \operatorname{AffT}\Lambda_1^{i,j}(f)) \\
&\qquad - (\xi^{e_i,f_i})^{-1} \circ \operatorname{AffT}[\psi_{m_1,\infty}]_i(\bigoplus_j \operatorname{AffT}\Lambda_1^{i,j}(f))\| \\
&\quad \le \max_j \|\operatorname{AffT}[\phi_{n_2',\infty}]_i \circ \operatorname{AffT}\mathcal{M}'^i(\operatorname{AffT}\overline{I}_{i,j}(\operatorname{AffT}\Lambda_1^{i,j}(f))) \\
&\qquad - (\xi^{e_i,f_i})^{-1} \circ \operatorname{AffT}[\psi_{m_1,\infty}]_i(\operatorname{AffT}\overline{I}_{i,j}(\operatorname{AffT}\Lambda_1^{i,j}(f)))\| \\
&\quad \le \max_j \|\operatorname{AffT}\{\phi_{n_2',\infty}\}_{i,j} \circ \operatorname{AffT}\mathcal{M}_1^{i,j}(\operatorname{AffT}\Lambda_1^{i,j}(f)) \\
&\qquad - (\xi^{\widetilde{e}_{i,j},\widetilde{f}_{i,j}})^{-1} \circ \operatorname{AffT}\{\psi_{m_1,\infty}\}_{i,j}(\operatorname{AffT}\Lambda_1^{i,j}(f))\| \le \frac{\varepsilon}{4}.
\end{aligned}$$

Then

$$\|\xi^{e_i,f_i}(\operatorname{AffT}([\phi_{n_2',\infty}]_i \circ \mathcal{M}'^i \circ \Lambda_1^i))(f) - \operatorname{AffT}([\psi_{m_1,\infty}]_i \circ \Lambda_1^i)(f))\| < \frac{\varepsilon}{4},$$

and for each i,

$$\|\operatorname{AffT}[\psi_{m_1,\infty}]_i \circ \operatorname{AffT}\Lambda_1^i(f) - \xi^{e_i,f_i} \circ \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| < \frac{\varepsilon}{4},$$

so we have

$$\|\operatorname{AffT}[\phi_{n_2',\infty}]_i \circ \operatorname{AffT}(\mathcal{M}'^i \circ \Lambda_1^i)(f) - \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| < \frac{\varepsilon}{2}.$$

Since $\mathcal{M}'^i \circ \Lambda_1^i = M_1 \circ \Lambda_1^i : A^i_{n_1} \to M_1 \circ \Lambda_1^i(\mathbf{1}_{A^i_{n_1}})A_{n_2}M_1 \circ \Lambda_1^i(\mathbf{1}_{A^i_{n_1}})$, then

$$\operatorname{AffT}(\mathcal{M}'^i \circ \Lambda_1^i)(f) = \operatorname{AffT}(M_1 \circ \Lambda_1^i)(f).$$

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That is

$$\| \operatorname{AffT}[\phi_{n_2',\infty}]_i \circ \operatorname{AffT}(M_1 \circ \Lambda_1^i)(f) - \operatorname{AffT}[\phi_{n_1,\infty}]_i(f)\| < \frac{\varepsilon}{2}.$$

By the definition of inductive limit, there exists $n_2 > 0$ such that

$$\| \operatorname{AffT}[\phi_{n_2',n_2}]_i \circ \operatorname{AffT}(M_1 \circ \Lambda_1^i)(f) - \operatorname{AffT}[\phi_{n_1,n_2}]_i(f)\| < \varepsilon.$$

So we only need to let $\mathcal{M}_1 = [\phi_{n_2',n_2}]_i \circ M_1$.

Then we have

$$\| \operatorname{AffT}[\phi_{n_1,n_2}]_i(f) - \operatorname{AffT}([\mathcal{M}_1 \circ \Lambda_1]_i)(f)\| < \varepsilon.$$

By Lemma 5.4 and the statement of Remark 5.5, we naturally have $\mathrm{K}_0([\mathcal{M}_1 \circ \Lambda_1]_i) = \mathrm{K}_0[\phi_{n_1,n_2}]_i$, and the proof is completed. ■

Proof of the main theorem Let there be given AI algebras with the ideal property, $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ and $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$, and an scaled ordered group isomorphism $\alpha\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$. There exist scaled ordered group maps

$$\alpha_i\colon \mathrm{K}_0A_i \to \mathrm{K}_0B_i, \quad \beta_i\colon \mathrm{K}_0B_i \to \mathrm{K}_0A_{i+1}$$

making following the diagram commutative:

$$\begin{array}{ccccccccc}
\mathrm{K}_0A_1 & \longrightarrow & \mathrm{K}_0A_2 & \longrightarrow & \mathrm{K}_0A_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0A \\
\downarrow \alpha_1 & \nearrow \beta_1 & \downarrow \alpha_2 & \nearrow \beta_2 & \downarrow \alpha_3 & \nearrow \beta_3 & & & \downarrow \alpha \\
\mathrm{K}_0B_1 & \longrightarrow & \mathrm{K}_0B_2 & \longrightarrow & \mathrm{K}_0B_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0B.
\end{array}$$

To prove the classification theorem, we need to construct an approximate intertwining of the two sequences of C*-algebras.

In this process, we will pass to subsequences several times. Let $\varepsilon_1, \varepsilon_2, \dots$ be positive numbers with $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. We choose the subsequences of $\{A_n\}_{n=1}^{\infty}$, $\{B_m\}_{m=1}^{\infty}$:

$$A_{n_1} \longrightarrow A_{n_2} \longrightarrow \cdots \longrightarrow A$$

$$B_{n_1} \longrightarrow B_{n_2} \longrightarrow \cdots \longrightarrow B$$

and maps $\Lambda_i\colon A_{n_i} \to B_{m_i}$, $\mathcal{M}_i\colon B_{m_i} \to A_{n_{i+1}}$, satisfying certain conditions so that the diagram

$$\begin{array}{ccccccccc}
A_{n_1} & \longrightarrow & A_{n_2} & \longrightarrow & A_{n_3} & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow \Lambda_1 & \nearrow \mathcal{M}_1 & \downarrow \Lambda_2 & \nearrow \mathcal{M}_2 & \downarrow \Lambda_3 & \nearrow \mathcal{M}_3 & & & \\
B_{m_1} & \longrightarrow & B_{m_2} & \longrightarrow & B_{m_3} & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

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is an approximate intertwining, *i.e.*, homomorphisms $\Lambda_i, \mathcal{M}_i$, and the finite generating subsets $F_{n_i} \subset A_{n_i}$, $G_{m_i} \subset B_{m_i}$ satisfy that

$$\|\Lambda_i \circ \mathcal{M}_{i-1}(f) - \psi_{m_{i-1},m_i}(f)\| < \varepsilon_i, \quad \forall f \in G_{m_{i-1}},$$
$$\|\mathcal{M}_i \circ \Lambda_i(f) - \phi_{m_i,m_{i+1}}(f)\| < \varepsilon_i, \quad \forall f \in F_{n_i},$$

and $F_{n_i} \supseteq \mathcal{M}_{n_{i-1}}(G_{n_{i-1}}) \bigcup \phi_{n_{i-1},n_i}(F_{n_{i-1}})$, $G_{m_i} \supseteq \Lambda_{n_i}(F_{n_i}) \bigcup \psi_{m_{i-1},m_i}(G_{m_{i-1}})$. Then, by [12, Theorem 2.1], it follows that A, B are isomorphic.

Now let $F_i \subset A_i$, $G_i \subset B_i$ be finite sets such that

$$F_1 \subset F_2 \subset \cdots \subset \overline{\bigcup_i^{\infty} F_i} = A, \quad G_1 \subset G_2 \subset \cdots \subset \overline{\bigcup_i^{\infty} G_i} = B.$$

Choose $k_1 = 1$. For $\varepsilon_1 > 0$ and $F_1 \subset A_1$, we can find $\eta, \delta > 0$ (to be defined later) in the uniqueness theorem and the finite set $H(\eta, \delta, X)$, $X = [0, 1]$.

For the given η, δ (see η, δ in Theorem 4.2), by the dichotomy theorem, there exists n_1 such that $\phi_{1,n_1} \colon A_1 \to A_{n_1}$ factors as

$$\phi_{1,n_1} \colon A_1 \xrightarrow{\pi} \widetilde{B} = \bigoplus_i \bigoplus_j M_{[1,i]}(C(Y_i^s)) \xrightarrow{\phi = \oplus_s \phi_s} A_{n_1} = \bigoplus_{i'} A_{n_1}^{i'},$$

where ϕ_s has the property sdp(η, δ), and each partial map of $\phi_{n,m} \circ \phi$ also has the property sdp(η, δ) $(\forall m > n_1)$. Notice that

$$\phi_s = \phi_i^{i',s} \colon M_{[1,i]}(C(Y_i^{i',s})) \to A_{n_1}^{i'}.$$

Now let $A_{n_1} = \bigoplus_{i'} A_{n_1}^{i'}$. For each fixed $A_{n_1}^{i'}$, by Remark 5.2, we can find AI algebras with the ideal property,

$$e_{i'} A e_{i'}, \quad f_{i'} B f_{i'} (e_{i'} = \phi_{n_1,\infty}(\mathbf{1}_{A_{n_1}^{i'}}), \quad f_{i'} = \psi_{n,\infty}(\widetilde{\Lambda_{i'}}(\mathbf{1}_{A_{n_1}^{i'}})),$$

and an isomorphism $\xi^{e_{i'},f_{i'}}$ between them. Naturally, $e_{i'} A e_{i'}$, $f_{i'} B f_{i'}$ still satisfy the conditions of the existence theorem.

So for $F_{i'}^s \triangleq \mathrm{AffT}(\phi_s \circ \pi_s)(H(\eta, \delta, X))$, $F_{i'} = \bigoplus_s F_{i'}^s$, and $\delta > 0$, applying Lemmas 5.4 and 5.6 and Remark 5.5, we can obtain homomorphisms

$$\Lambda_1^{i'} \colon A_{n_1}^{i'} \to B_{m_1} = \bigoplus_j B_{m_1}^j, \quad \mathcal{M}_1 \colon B_{m_1} \to A_{n_2}$$

such that

$$\| \mathrm{AffT}[\phi_{n_1,n_2}]_{i'}(f) - \mathrm{AffT}[\mathcal{M}_1 \circ \Lambda_1]_{i'}(f)\| < \delta, \ \ \forall f \in F_{i'},$$

where $\Lambda_1 \triangleq \bigoplus_{i'} \Lambda_1^{i'}$ is just the homomorphism Λ_1 of Lemma 5.6, and

$$[\mathcal{M}_1 \circ \Lambda_1]_{i'} \colon A_{n_1}^{i'} \to \mathcal{M}_1 \circ \Lambda_1(\mathbf{1}_{A_{n_1}^{i'}}) A_{n_2} \mathcal{M}_1 \circ \Lambda_1(\mathbf{1}_{A_{n_1}^{i'}})$$

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is unital.

By simple calculation, for any $f \in \pi_s(H(\eta,\delta,X))$, we have

$$\|\operatorname{AffT}([\phi_{n_1,n_2}]_s \circ \phi_s)(f) - \operatorname{AffT}[\mathcal{M}_1 \circ \Lambda_1]_s \circ \operatorname{AffT}\phi_s(f)\| < \delta =$$
$$\|\operatorname{AffT}([\phi_{n_1,n_2}]_{i'} \circ \phi_s)(f) - \operatorname{AffT}[\mathcal{M}_1 \circ \Lambda_1]_{i'} \circ \operatorname{AffT}\phi_s(f)\| < \delta,$$

where

$$[\phi_{n_1,n_2}]_s \colon \phi_s(\mathbf{1})A_{n_1}^{i'}\phi_s(\mathbf{1}) \to [\phi_{n_1.n_2}]_{i'}(\phi_s(\mathbf{1}))A_{n_2}[\phi_{n_1.n_2}]_{i'}(\phi_s(\mathbf{1}))$$

and

$$[\mathcal{M}_1 \circ \Lambda_1]_s \colon \phi_s(\mathbf{1})A_{n_1}^{i'}\phi_s(\mathbf{1}) \to [\mathcal{M}_1 \circ \Lambda_1]_{i'}(\phi_s(\mathbf{1}))A_{n_2}[\mathcal{M}_1 \circ \Lambda_1]_{i'}(\phi_s(\mathbf{1}))$$

are both unital. So "$[\,\cdot\,]_s$" is induced by the projection $\phi_s(\mathbf{1})$ similar to the notation defined in Remark 5.2. (Here we use the fact $\mathrm{K}_0[\phi_{n_1,n_2}]_s = \mathrm{K}_0[\mathcal{M}_1 \circ \Lambda_1]_s$.)

By Theorem 4.2, we know that ϕ_s has the property $\operatorname{sdp}(\eta,\delta)$, and the partial maps of $[\phi_{n_1,n_2}]_i \circ \phi_s$ also have the property $\operatorname{sdp}(\eta,\delta)$. Thus, we only need to choose appropriate η and δ and apply the uniqueness theorem (Theorem 3.5) to find unitary $U_s \in A_{n_2}$ such that

$$\|[\phi_{n_1,n_2}]_{i'} \circ \phi_s(f) - U_s([\mathcal{M}_1 \circ \Lambda_1]_{i'} \circ \phi_s)(f))U_s^*\| < \varepsilon_1, \quad \forall f \in \pi_s(F).$$

Notice that $\phi_s = \phi_i^{i',s} \colon M_{[n,i]}(C(Y_i^{i',s})) \to A_{n_1}^{i'}$, $\phi_{1,n_1} = \bigoplus_s(\phi_s \circ \pi_s) = \phi \circ \pi$. Setting $\Lambda_1 = (\bigoplus_{i'} \Lambda_{i'}) \circ \phi_{1,n_1}$, $(\bigoplus_s U_s)\mathcal{M}_1(\bigoplus_s U_s)^* = \mathcal{M}_1$, then for each $f \in F_1$, we have

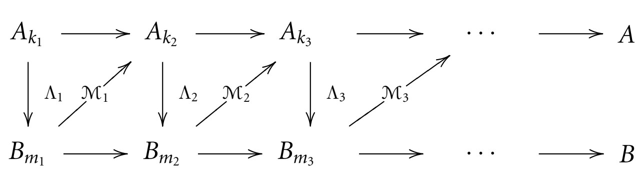
$$\|\phi_{1,n_2}(f) - \mathcal{M}_1 \circ \Lambda_1(f)\| \le$$
$$\max_s \|[\phi_{n_1,n_2}]_{i'} \circ \phi_s \circ \pi_s(f) - U_s(\mathcal{M}_1 \circ \Lambda_{i'} \circ \phi_s)(\pi_s(f))U_s^*\| < \varepsilon_1.$$

Similarly, we can construct $\Lambda_i, \mathcal{M}_i$ such that

$$\|\Lambda_{i+1} \circ \mathcal{M}_i(f) - \psi_{m_i,m_{i+1}}(f)\| < \varepsilon_i, \forall f \in \widetilde{G}_{m_i},$$
$$\|\mathcal{M}_i \circ \Lambda_i(f) - \phi_{n_i,n_{i+1}}(f)\| < \varepsilon_i, \forall f \in \widetilde{F}_{n_i},$$

where $\widetilde{G}_{m_i} = G_{m_i} \cup \Lambda_i(\widetilde{F}_i) \cup \psi_{m_{i-1},m_i}(\widetilde{G}_{m_{i-1}})$, $\widetilde{F}_{n_i} = F_{n_i} \cup \mathcal{M}_i(G_{m_i}) \cup \phi_{n_{i-1},n_i}(\widetilde{F}_{n_{i-1}})$.

Then



is an approximate intertwining. Hence A and B are isomorphic, and conclusions (i) and (ii) also hold by the proof above. ∎

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