THE SPECTRAL SEQUENCE OF A COVERING

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1. Introduction.

Let $\mathscr{U}$ be a covering of a topological space X and $\mathscr{F}$ a sheaf of abelian groups over X. By a well known result of Leray, (**3**) II theorems 5.2.4. and 5.4.1., if $\mathscr{U}$ is open, or closed and locally finite, there exists a spectral sequence $\{E_r\}$ satisfying isomorphisms $E_2^{p,q} \cong H^p\{\mathscr{U}, \mathscr{H}^q(\mathscr{F})\}$ and $E_\infty^{p,q} \cong \mathscr{G}^p H^{p+q}(X, \mathscr{F})$ for some filtration of the graded group $H^*(X, \mathscr{F})$. $\mathscr{H}^q(\mathscr{F})$ denotes the system of coefficients over $\mathscr{U}: s \to H^q(|s|, \mathscr{F})$.

In this paper we shall derive another Leray sequence, given in Theorem 1 when $\mathscr{U}$ is locally finite, open or closed, which satisfies isomorphisms $E_2^{p,q} \cong H^p\{\mathscr{U}, \tilde{\mathscr{H}}^q(\mathscr{F})\}$ and $E_\infty^{p,q} \cong \mathscr{G}^p \check{H}^{p+q}(X, \mathscr{F})$ with a suitable filtration of the Čech cohomology $\check{H}^*(X, \mathscr{F})$. $\tilde{\mathscr{H}}^q(\mathscr{F})$ is the system: $s \to \tilde{H}^q(|s|, \mathscr{F})$, this being the " restricted " cohomology of $|s|$ as a subspace of X introduced in Definition 1 of § 2.

The method used is equivalent to taking the double complex $C^{*,*}\{\mathscr{U}, \mathscr{V}; \mathscr{F}\}$ defined by a pair of coverings $\mathscr{U}$ and $\mathscr{V}$, (**4**) p. 220, forming its spectral sequences, and taking their direct limit as $\mathscr{V}$ runs over " all " open coverings of X. One of these spectral sequences will degenerate provided $\mathscr{U}$ admits an open refinement; the other will then be the Leray sequence given in Theorem 1.

In § 4 we express the restricted cohomology $\tilde{H}^*(M, \mathscr{F})$ of a subspace $M \subset X$ as the Čech cohomology of the closure $\overline{M}$ with coefficients in an associated sheaf $\tilde{\mathscr{F}}$ which is the direct image of $\mathscr{F}$ under the inclusion map $M \to \overline{M}$. In Theorem 2 we obtain a spectral sequence relating the restricted and true cohomologies of M, which leads to a sufficient condition for them to be isomorphic.

Finally in Theorem 3 we obtain a map of spectral sequences from the sequence of Theorem 1 to the usual Leray sequence for an open covering, and characterise this map in the E_2 terms.

2. Basic Definitions and Operations

See (**3**) I 1.6., 2.1., 2.2., 2.6., 3.3., 4.4., 4.5., 4.8., II 5.1., 5.8., (**2**) V 5., VIII, and (**1**) XV 5.12.

We denote by $\prod_I A_i$ the direct product of a family of abelian groups $\{A_i\}$ $i \in I$; by $\lim_{\Lambda} A_\lambda$ the direct limit of a direct system of abelian groups $\{A_\lambda\}$ over a

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directed set Λ; and by $H^n A^*$ the nth cohomology group of a cochain com $A^* = (A^n)_{n \in Z}$.

If $\prod_I A_i^* = (\prod_I A_i^n)_{n \in Z}$ is the direct product of complexes A_i^* then $H^n \prod_I A_i^*$ $\prod_I H^n A_i^*$, so that direct products commute with the formation of cohomol groups. If A_λ^* is a direct system of complexes then $H^n \lim_\Lambda A_\lambda^* \cong \lim_\Lambda H^n A_\lambda^*$ that direct limits commute with the formation of cohomology groups.

If $\mathscr{U} = \{\mathscr{U}_i\}$ $i \in I$ is a covering of a topological space X we may call ordered sequence $s = (i_0 i_1 \ldots i_p)$ of $(p+1)$ elements of I an ordered p-simple $\mathscr{U}$. We denote by $|s|$ the (possibly empty) set $U_{i_0} \cap \ldots \cap U_{i_p}$ and by $S_p(\mathscr{U})$ set of ordered p-simplexes of $\mathscr{U}$. If $\mathscr{F}$ is a sheaf of abelian groups over X any system of local coefficients over $\mathscr{U}$, then the complex $C^*(\mathscr{U}, \mathscr{F})$ of coch of $\mathscr{U}$ with coefficients in $\mathscr{F}$ is defined with $C^p(\mathscr{U}, \mathscr{F}) = \prod_s \mathscr{F}(|s|)$, $s \in S_p(\mathscr{U})$.

Let $R(X)$ be the set of all open coverings of X of the form $\mathscr{U} = \{U_x\}$ inde by $x \in X$, such that $x \in U_x$ all x. Define an ordering relation $\geqslant$ in $R(X)$ putting $\{U_x\} \geqslant \{V_x\}$ iff $U_x \supset V_x$ each $x \in X$. More generally, if M is any sul of X, put $\{U_x\} \geqslant_M \{V_x\}$ iff $U_x \cap M \supset V_x \cap M$ each $x \in X$. $R(X)$ is a directed with respect to each of the relations $\geqslant$ and $\geqslant_M$. Let $R_M(X)$ be the se coverings $\{V_x\} \in R(X)$ such that $V_x \subset X - M$ if $x \in X - M$. If M is closed i then $R_M(X)$ is cofinal in $R(X)$, so that $M \cap R_M(X)$ is cofinal in $M \cap R(X)$. $M \cap R_M(X)$ with the ordering induced by $\geqslant_M$ may be identified with $R(M)$ thus $R(M)$ is cofinal in $M \cap R(X)$. ....

$C^*(\mathscr{U}, \mathscr{F})$ is a direct system of complexes over $\mathscr{U} \in R(X)$ with the relation and $C^*(M \cap \mathscr{U}, \mathscr{F})$ is a direct system over $R(X)$ with each of the relation and $\geqslant_M$, the maps of the system being the same for $\mathscr{U} \geqslant \mathscr{V}$ as for $\mathscr{U} \geqslant_M$ Moreover $\lim_{\geqslant} C^*(M \cap \mathscr{U}, \mathscr{F}) \cong \lim_{\geqslant_M} C^*(M \cap \mathscr{U}, \mathscr{F})$.

Definition 1. If M is any subset of X and $\mathscr{F}$ a sheaf over M, then we pu

$$\tilde{C}^*(M, \mathscr{F}) = \lim_{\mathscr{U} \in R(X),\ \geqslant} C^*(M \cap \mathscr{U}, \mathscr{F})$$

and call it the complex of *restricted cochains of* M (*as a subset of* X) with coeffici in $\mathscr{F}$. When $M = X$ we have in particular the Čech complex $\check{C}^*(M, \mathscr{F}$ $\lim_{\mathscr{U} \in R(M)} C^*(\mathscr{U}, \mathscr{F})$.

We call the cohomology groups of these complexes the *restricted cohomol* $\tilde{H}^*(M, \mathscr{F})$ (*of* M *as a subset of* X) and the Čech cohomology $\check{H}^*(M,$ respectively. The restricted cohomology is that obtained by using only coch relative to those coverings of M which can be obtained by intersection fr coverings of the whole space X.

If $\mathscr{U}$ is any covering of X and $\mathscr{F}$ a sheaf over X, let $K^{*,*} = C^*\{\mathscr{U}, \tilde{\mathscr{C}}^*($ be the bigraded group defined by:

$$C^p\{\mathscr{U}, \tilde{\mathscr{C}}^q(\mathscr{F})\} = \prod_s \tilde{C}^q(|s|, \mathscr{F}), \ s \in S_p(\mathscr{U}) \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

i.e. the group of p-cochains of $\mathscr{U}$ with coefficients in the system: $s \to \tilde{C}^q(|s|, \mathscr{F})$. The differentiations in the complexes $C^*\{\mathscr{U}, \tilde{\mathscr{C}}^q(\mathscr{F})\}$ and $C^p\{\mathscr{U}, \tilde{\mathscr{C}}^*(\mathscr{F})\}$ define endomorphisms d' and d'' in $K^{*,*}$ of degrees (1, 0) and (0, 1) with $d'd'' = d''d'$. If d_1 and d_2 are the endomorphisms in $K^{*,*}$ with $d_1 = d'$ and $d_2 = (-1)^p d''$ on homogeneous elements of degree (p, q), then $d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$, so that $K^{*,*}$ is a double complex with differentiations d_1 and d_2.

If $K^{*,*}$ is any double complex and $\{'E_r\}$ its spectral sequence with respect to its first filtration, we have isomorphisms $'E_2^{p,q} \cong 'H^{p''}H^q K^{*',*''}$ the primes indicating which complexes the cohomology operators act on. If $L^* \xrightarrow{j} K^{*,*}$ is the inclusion map of the d_2 0-cocyles of $K^{*,*}$ then:

$$'E_2^{n,0} \cong 'H^{n''}H^0 K^{*',*''} \cong H^n L^*.$$

With this isomorphism the map $'E_2^{n,0} \to H^n(K)$ defined by the spectral sequence is the same as the induced map of total cohomology:

$$H^n L^* \xrightarrow{j} H^n(K^{*,*}). \qquad \text{.................................(3)}$$

In particular if the sequence degenerates, i.e. if $'E_2^{p,q} \cong 'H^{p''}H^q K^{*',*''} = 0$ all $q > 0$, then (3) will be bijective all n. We have a similar result for the inclusion map of the d_1 0-cocycles.

3. A Spectral Sequence defined by a Covering

Lemma 1. *If $\{M_i\}$ $i \in I$ is a locally finite family of subsets of a topological space X, and if $\{\mathscr{V}^i\}$ $i \in I$ is a family of coverings belonging to $R(X)$; then there exists $\mathscr{U}^0 \in R(X)$ such that $\mathscr{V}^i \gg_{M_i} \mathscr{U}^0$ each $i \in I$.*

Proof. Let $\mathscr{V}^i = \{V_x^i\}$ each i. Let $x \in X$; choose an open neighbourhood W_x of x such that W_x intersects only a finite number of members of $\{M_i\}$: $M_{i_0}, \ldots, M_{i_N}$ say. Put $U_x^0 = W_x \cap V_x^{i_0} \cap \ldots \cap V_x^{i_N}$. Then $U_x^0 \cap M_i \subset W_x \cap M_i = \varnothing$ if $i \notin (i_0, \ldots, i_N)$ and $U_x^0 \subset V_x^i$ if $i \in (i_0, \ldots, i_N)$. Choose U_x^0 similarly for each $x \in X$; then $\mathscr{U}^0 = \{U_x^0\}$ satisfies the required conditions.

Lemma 2. *If $\mathscr{U}$ is a locally finite covering of X then for all $p, q \geqq 0$:*

$$\lim_{\mathscr{V}} \prod_s C^p(|s| \cap \mathscr{V}, \mathscr{F}) \cong \prod_s \lim_{\mathscr{V}} C^p(|s| \cap \mathscr{V}, \mathscr{F})$$

over $\mathscr{V} \in R(X)$, $\gg$ and $s \in S_q(\mathscr{U})$.

Proof. Let θ: $\lim_{\gg} \prod_s C^p(|s| \cap \mathscr{V}, \mathscr{F}) \to \prod_s \lim_{\gg_{|s|}} C^p(|s| \cap \mathscr{V}, \mathscr{F})$ be the homomorphism defined by $\theta[\lim_{\gg} \prod_s c^p(s, \mathscr{V}^0)] = \prod_s \lim_{\gg_{|s|}} c^p(s, \mathscr{V}^0)$ for any set of elements $c^p(s, \mathscr{V}^0) \in C^p(|s| \cap \mathscr{V}^0, \mathscr{F})$. By the limit of an element we mean its projection in the limit group.

Then $\lim_{\gg} \prod_s c^p(s, \mathscr{V}^0) \in \text{kernel } \theta \Rightarrow \lim_{\gg_{|s|}} c^p(s, \mathscr{V}^0) = 0$ all $s \in S_q(\mathscr{U}) \Rightarrow$ for all s, $\exists\, \mathscr{V}^s \in R(X)$ such that $\mathscr{V}^0 \gg_{|s|} \mathscr{V}^s$ and $\pi_{\mathscr{V}^s}^{\mathscr{V}^0}[c^p(s, \mathscr{V}^0)] = 0$, where $\{\pi_{\mathscr{V}''}^{\mathscr{V}'}\}$ are the maps of the direct systems involved. But $\mathscr{U}$ is locally finite, so

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that the collection of subsets $\{|s|\}$ for $s \in S_q(\mathscr{U})$ is locally finite, and therefore by Lemma 1 there exists $\mathscr{U}^0 \in R(X)$ with $\mathscr{V}^s \geqslant_{|s|} \mathscr{U}^0$ all $s \in S_q(\mathscr{U})$; we can also take $\mathscr{V}^0 \geqslant \mathscr{U}^0$. It follows that $\pi^{\mathscr{V}^0}_{\mathscr{U}^0}[c^p(s, \mathscr{V}^0)] = \pi^{\mathscr{V}^s}_{\mathscr{U}^0}\pi^{\mathscr{V}^0}_{\mathscr{V}^s}[c^p(s, \mathscr{V}^0)] = 0$, so that $\lim\limits_{\geqslant} \prod\limits_s c^p(s, \mathscr{V}^0) = 0$ and θ is a monomorphism.

Also if $\prod\limits_s \lim\limits_{\geqslant_{|s|}} c^p(s, \mathscr{V}^s) \in \prod\limits_s \lim\limits_{\geqslant_{|s|}} C^p(|s| \cap \mathscr{V}, \mathscr{F})$ then by Lemma 1 there exists $\mathscr{U}^0 \in R(X)$ with $\mathscr{V}^s \geqslant_{|s|} \mathscr{U}^0$ each s. Thus

$$\prod_s \lim_{\geqslant_{|s|}} c^p(s, \mathscr{V}^s) = \prod_s \lim_{\geqslant_{|s|}} \{\pi^{\mathscr{V}^s}_{\mathscr{U}^0}[c^p(s, \mathscr{V}^s)]\}$$
$$= \theta[\lim_{\geqslant} \prod_s \{\pi^{\mathscr{V}^s}_{\mathscr{U}^0} c^p(s, \mathscr{V}^s)\}]$$

which shows that θ is also an epimorphism.

The isomorphism

$$\theta: \lim_{\geqslant} \prod_s C^p(|s| \cap \mathscr{V}, \mathscr{F}) \cong \prod_s \lim_{\geqslant_{|s|}} C^p(|s| \cap \mathscr{V}, \mathscr{F})$$

together with the isomorphisms

$$\lim_{\geqslant} C^p(|s| \cap \mathscr{V}, \mathscr{F}) \cong \lim_{\geqslant_{|s|}} C^p(|s| \cap \mathscr{V}, \mathscr{F})$$

prove the lemma.

We are now in a position to obtain a Leray sequence for a locally finite covering, as follows.

Theorem 1. *If $\mathscr{U}$ is a locally finite, open or closed, covering of a topological space X, admitting an open refinement, and if $\mathscr{F}$ is a sheaf of abelian groups over X; then the Čech cohomology group $\check{H}^*(X, \mathscr{F})$ has a filtration so that there exists a spectral sequence $\{E_r\}$ with isomorphisms*

$$E_2^{p,q} \cong H^p\{\mathscr{U}, \tilde{\mathscr{H}}^q(\mathscr{F})\}$$

and

$$E_\infty^{p,q} \cong \mathscr{G}^p \check{H}^{p+q}(X, \mathscr{F})$$

all $p, q \geqq 0$; where $\tilde{\mathscr{H}}^q(\mathscr{F})$ denotes the system of coefficients: $s \to \tilde{H}^q(|s|, \mathscr{F})$ (see Definition 1 of § 2).

Proof. Let $K^{*,*} = C^*\{\mathscr{U}, \tilde{\mathscr{C}}^*(\mathscr{F})\}$ be the double complex introduced in (2), and let $'E(K)$ and $''E(K)$ be its two spectral sequences; see (3) I 4.8. We shall show that $''E(K)$ degenerates and that $'E(K)$ fulfills the requirements of the theorem.

In the first place

$$H^q C^p\{\mathscr{U}, \tilde{\mathscr{C}}^*(\mathscr{F})\} = H^q \prod_s \tilde{C}^*(|s|, \mathscr{F}) \quad s \in S_p(\mathscr{U})$$
$$\cong \prod_s \tilde{H}^q(|s|, \mathscr{F}),$$

since direct products commute with the formation of cohomology groups; thus

$$H^q C^p\{\mathscr{U}, \tilde{\mathscr{C}}^*(\mathscr{F})\} \cong C^p\{\mathscr{U}, \tilde{\mathscr{H}}^q(\mathscr{F})\}$$

and hence

$$
\begin{aligned}
'E_2^{p,q}(K) &\cong 'H^{p}{}''H^{q}C^{*'}\{\mathscr{U}, \tilde{\mathscr{C}}^{*''}(\mathscr{F})\} \\
&\cong 'H^{p}C^{*'}\{\mathscr{U}, \tilde{\mathscr{H}}^{q}(\mathscr{F})\} \\
&= H^{p}\{\mathscr{U}, \tilde{\mathscr{H}}^{q}(\mathscr{F})\}.
\end{aligned}
$$

The primes indicate which complexes the cohomology operators act on. Moreover

$$
\begin{aligned}
C^{q}\{\mathscr{U}, \tilde{\mathscr{C}}^{p}(\mathscr{F})\} &= \prod_{s} \check{C}^{p}(|\, s\,|, \mathscr{F}) \quad s \in S_q(\mathscr{U}) \\
&= \prod_{s} \lim_{\mathscr{V} \in R(X)} C^{p}(|\, s\,| \cap \mathscr{V}, \mathscr{F}) \\
&\cong \lim_{\mathscr{V}} \prod_{s} C^{p}(|\, s\,| \cap \mathscr{V}, \mathscr{F})
\end{aligned}
$$

by Lemma 2, since $\mathscr{U}$ is locally finite; therefore

$$
\begin{aligned}
C^{q}\{\mathscr{U}, \tilde{\mathscr{C}}^{p}(\mathscr{F})\} &\cong \lim_{\mathscr{V}} \prod_{s}\prod_{t} \mathscr{F}(|\, s\,| \cap |\, t\,|) \quad t \in S_p(\mathscr{V}) \\
&\cong \lim_{\mathscr{V}} \prod_{t} \{\prod_{s} \mathscr{F}(|\, s\,| \cap |\, t\,|)\} \\
&= \lim_{\mathscr{V}} \prod_{t} C^{q}(|\, t\,| \cap \mathscr{U}, \mathscr{F});
\end{aligned}
$$

so that

$$
\begin{aligned}
H^{q}C^{*}\{\mathscr{U}, \tilde{\mathscr{C}}^{p}(\mathscr{F})\} &\cong H^{q} \lim_{\mathscr{V}} \prod_{t} C^{*}(|\, t\,| \cap \mathscr{U}, \mathscr{F}) \\
&\cong \lim_{\mathscr{V}} \prod_{t} H^{q}(|\, t\,| \cap \mathscr{U}, \mathscr{F}) \\
&= 0 \quad (\text{all } q > 0),
\end{aligned}
$$

since $\mathscr{U}$ admits an open refinement and hence a refinement by an element $\mathscr{V} \in R(X)$. For such an element $|\, t\,| \cap \mathscr{U}$ is a trivial covering of $|\, t\,|$ each $t \in S_p(\mathscr{V})$ and therefore $H^{q}(|\, t\,| \cap \mathscr{U}, \mathscr{F}) = 0$ for all $q > 0$.

Also

$$
\begin{aligned}
H^{o}C^{*}\{\mathscr{U}, \tilde{\mathscr{C}}^{p}(\mathscr{F})\} &\cong \lim_{\mathscr{V}} \prod_{t} H^{o}(|\, t\,| \cap \mathscr{U}, \mathscr{F}) \\
&\cong \lim_{\mathscr{V}} \prod_{t} \mathscr{F}(|\, t\,|) \quad \text{by (3) II 5.2.2.} \\
&= \lim_{\mathscr{V}} C^{p}(\mathscr{V}, \mathscr{F}) \\
&= \check{C}^{p}(X, \mathscr{F}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
''E_2^{p,q}(K) &\cong 'H^{p}{}''H^{q}C^{*''}\{\mathscr{U}, \tilde{\mathscr{C}}^{*'}(\mathscr{F})\} \\
&= 0 \quad (\text{all } q > 0),
\end{aligned}
$$

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and

$$''E_2^{n,0}(K) \cong 'H^{n}{}''H^{0}C^{*''}\{\mathscr{U}, \tilde{\mathscr{C}}^{*'}(\mathscr{F})\}$$
$$\cong 'H^{n}\check{C}^{*'}(X, \mathscr{F})$$
$$= \check{H}^{n}(X, \mathscr{F}).$$

This shows that the spectral sequence $''E(K)$ degenerates, giving isomorphisms (1) XV 5.12.:

$$\check{H}^{*}(X, \mathscr{F}) \cong {}''E_2^{*,0}(K) \cong H^{*}(K),$$

this being the total cohomology of the double complex $K^{*,*}$, and giving $\check{H}^{*}(X, \mathscr{F})$ two filtrations induced by those of K.

Finally

$$'E_\infty^{p,q}(K) \cong {}'\mathscr{G}^{p}H^{p+q}(K) \quad \text{see (3) I 4.2.2.}$$
$$\cong {}'\mathscr{G}^{p}\check{H}^{p+q}(K),$$

which completes the proof that $'E(K)$ is a spectral sequence satisfying the required conditions.

4. The Restricted Cohomology of a Subspace

The restricted cochains of a subspace $M \subset X$, with coefficients in a sheaf $\mathscr{F}$ over M, can be expressed as Čech cochains of the closure $\overline{M}$ with coefficients in an associated sheaf $\tilde{\mathscr{F}}$ as follows. Let $\tilde{\mathscr{F}}$ be the direct image (3) II 1.13. of $\mathscr{F}$ under the inclusion map $i: M \to \overline{M}$; this is the sheaf defined by $\tilde{\mathscr{F}}(\overline{M} \cap U) = \mathscr{F}(M \cap U)$ for open sets U of X. If $\mathscr{V}$ is an open covering of X then

$$C^{p}(M \cap \mathscr{V}, \mathscr{F}) \cong \prod_{s} \mathscr{F}(M \cap |s|) \quad s \in S_p(\mathscr{V})$$
$$= \prod_{s} \tilde{\mathscr{F}}(\overline{M} \cap |s|)$$
$$\cong C^{p}(\overline{M} \cap \mathscr{V}, \tilde{\mathscr{F}}),$$

giving an isomorphism of complexes $C^{*}(M \cap \mathscr{V}, \mathscr{F}) \cong C^{*}(\overline{M} \cap \mathscr{V}, \tilde{\mathscr{F}})$. So that

$$\tilde{C}^{*}(M, \mathscr{F}) = \lim_{\mathscr{V} \in R(X)} C^{*}(M \cap \mathscr{V}, \mathscr{F})$$
$$\cong \lim_{\mathscr{V} \in R(X)} C^{*}(\overline{M} \cap \mathscr{V}, \tilde{\mathscr{F}})$$
$$\cong \lim_{\mathscr{V} \in R(\overline{M})} C^{*}(\mathscr{V}, \tilde{\mathscr{F}})$$

by (1) since $\overline{M}$ is closed in X; i.e. $\tilde{C}^{*}(M, \mathscr{F}) \cong \check{C}^{*}(\overline{M}, \tilde{\mathscr{F}})$ and thus

$$\tilde{H}^{*}(M, \mathscr{F}) \cong \check{H}^{*}(\overline{M}, \tilde{\mathscr{F}}). \qquad \text{....................(4)}$$

This shows in particular that the restricted cohomology may differ from the true cohomology. For if X is a 2-sphere and $M = X - p$ where p is any point

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of X, and if Z is the simple sheaf of integers over M, then $\tilde{Z}$ is the simple sheaf of integers over $\bar{M} = X$; therefore $\tilde{H}^2(M, Z) \cong \check{H}^2(X, Z) \neq 0$, while

$$H^2(M, Z) = 0.$$

The relation between the restricted and true cohomologies of a subspace can in general be expressed in terms of a spectral sequence, according to the following theorem.

Theorem 2. *If $\mathscr{F}$ is a sheaf over $M \subset X$ then there exists a spectral sequence $\{E_r\}$ satisfying isomorphisms*

$$E_2^{n,0} \cong \tilde{H}^n(M, \mathscr{F}) \quad n \geqq 0$$

and

$$E_\infty^{p,q} \cong \mathscr{G}^p H^{p+q}(M, \mathscr{F}) \quad p, q \geqq 0$$

for some filtration of $H^*(M, \mathscr{F})$.

Proof. Consider the double complex $K^{*,*} = \tilde{C}^*(M, \mathscr{L}^*)$, where $\mathscr{L}^* = \mathscr{C}^*(M, \mathscr{F})$ is the canonical flabby resolution (3) II 4.3.

Since the operations of taking sections, direct products and direct limits are left exact at least, the exact sequence of sheaves $0 \to \mathscr{F} \xrightarrow{j_1} \mathscr{L}^0 \to \mathscr{L}^1$ gives an exact sequence of groups $0 \to \tilde{C}^p(M, \mathscr{F}) \xrightarrow{j_1} \tilde{C}^p(M, \mathscr{L}^0) \to \tilde{C}^p(M, \mathscr{L}^1)$ each $p \geqq 0$, which shows that

$$j_1 : \tilde{C}^*(M, \mathscr{F}) \to \tilde{C}^*(M, \mathscr{L}^*) \quad \text{.........................(5)}$$

embeds $\tilde{C}^*(M, \mathscr{F})$ as the subcomplex of d_2 0-cocycles of $K^{*,*}$; and therefore

$$H^0 K^{p,*} \cong \tilde{C}^p(M, \mathscr{F}). \quad \text{...............................(6)}$$

Similarly the exact sequence (3) II 5.2.1. $0 \to \mathscr{L}^q \xrightarrow{j_2} \mathscr{C}^0(M \cap \mathscr{V}, \mathscr{L}^q) \to \mathscr{C}^1(M \cap \mathscr{V}, \mathscr{L}^q)$ for each $\mathscr{V} \in R(X)$ and $q \geqq 0$, gives an embedding

$$j_2 : C^*(M, \mathscr{F}) \to \tilde{C}^*(M, \mathscr{L}^*) \quad \text{............................(7)}$$

of $C^*(M, \mathscr{F})$ as the subcomplex of d_1 0-cocycles of $K^{*,*}$.

But

$$\begin{aligned} H^q K^{*,p} &= H^q \tilde{C}^*(M, \mathscr{L}^p) \\ &= H^q \lim_{\mathscr{V} \in R(X)} C^*(M \cap \mathscr{V}, \mathscr{L}^p) \\ &\cong \lim_{\mathscr{V} \in R(X)} H^q(M \cap \mathscr{V}, \mathscr{L}^p) \\ &= 0 \quad (\text{all } q > 0) \end{aligned}$$

by (3) II 5.2.3. since $\mathscr{L}^p$ is flabby all $p \geqq 0$; and therefore applying (3) we see that (7) induces an isomorphism all $n \geqq 0$

$$j_2 : H^n(M, \mathscr{F}) \cong H^n(K^{*,*}). \quad \text{.........................(8)}$$

If $\{E_r\}$ is the spectral sequence of $K^{*,*}$ with respect to its first filtration, then $E_2^{n,0} \cong {}'H^{n''} H^0 K^{*',*''} \cong \tilde{H}^n(M, \mathscr{F})$ by (6); and $E_\infty^{p,q} \cong \mathscr{G}^p H^{p+q}(K) \cong \mathscr{G}^p H^{p+q}(M, \mathscr{F})$

where $H^*(M, \mathscr{F})$ is filtered by the isomorphism (8) and the first filtration of $K^{*,*}$. $\{E_r\}$ is therefore a spectral sequence satisfying the required conditions.

Corollary. *There exists a homomorphism* $\tilde{H}^*(M, \mathscr{F}) \to H^*(M, \mathscr{F})$. *This is bijective if* $\overline{M}$ *is paracompact and if* $\lim_{U \in \psi(x)} H^n(M \cap U, \mathscr{F}) = 0$ *for all* $n > 0$, *each* $x \in \overline{M}$; *where* $\Psi(x)$ *is the directed set of open neighbourhoods of* x *in* $\overline{M}$, *ordered by inclusion.*

Proof. The spectral sequence of the previous theorem defines a homomorphism, (3) I 4.5. $E_2^{n,0} \to H^n(K)$ i.e. $\tilde{H}^n(M, \mathscr{F}) \to H^n(M, \mathscr{F})$ each $n \geqq 0$. This is bijective if the sequence degenerates, i.e. if $E_2^{p,q} = 0$ all $q > 0$.

Now

$$H^q\tilde{C}^p(M, \mathscr{L}^*) \cong H^q\check{C}^p(\overline{M}, \tilde{\mathscr{L}}^*) \quad \text{by (4)}$$
$$= H^q \lim_{\mathscr{V} \in R(\overline{M})} \prod_{s \in S_p(\mathscr{V})} \tilde{\mathscr{L}}^*(|s|)$$
$$\cong \lim_{\mathscr{V}} \prod_{s} H^q\mathscr{L}^*(M \cap |s|)$$
$$\cong \lim_{\mathscr{V}} \prod_{s} H^q(M \cap |s|, \mathscr{F})$$

by (3) II Lemma 4.9.1. since each $M \cap |s|$ is open in M and $\mathscr{L}^*$ is the canonical resolution of $\mathscr{F}$ over M.

Thus $H^q\tilde{C}^p(M, \mathscr{L}^*) \cong \check{C}^p(\overline{M}, \mathscr{H}^q)$ where $\mathscr{H}^q$ denotes the presheaf over $\overline{M}$: $\mathscr{H}^q(U) = H^q(M \cap U, \mathscr{F})$; and therefore

$$E_2^{p,q} \cong {}'H^{p''}H^q\tilde{C}^{*'}(M, \mathscr{L}^{*''})$$
$$\cong \check{H}^p(\overline{M}, \mathscr{H}^q)$$
$$= 0 \quad (\text{all } q > 0)$$

by (3) II 5.10.2. since the sheaf generated (3) II 1.2. by $\mathscr{H}^q$ over $\overline{M}$ has stalk over x: $\mathscr{H}^q(x) = \lim_{U \in \psi(x)} H^q(M \cap U, \mathscr{F})$ which is given to be zero all $x \in \overline{M}$, $q > 0$.

5. Relation to the Leray Sequence

In (3) theorems II 5.4.1. and II 5.2.4. the Leray spectral sequence of an open, or a closed locally finite, covering $\mathscr{U}$ is given satisfying isomorphisms $E_2^{p,q} \cong H^p\{\mathscr{U}, \mathscr{H}^q(\mathscr{F})\}$ and $E_2^{p,q} \cong \mathscr{G}^pH^{p+q}(X, \mathscr{F})$ where $\mathscr{H}^q(\mathscr{F})$ denotes the system of coefficients $s \to H^q(|s|, \mathscr{F})$.

If $\mathscr{U}$ is closed and X paracompact the Čech and restricted and true cohomology groups of $|s|$ are all isomorphic for simplexes s of $\mathscr{U}$. The sequence of Theorem 1 will then be isomorphic to the Leray sequence. In the case of an open covering we have the following result.

Theorem 3. *If* $\mathscr{U}$ *is a locally finite open covering there exists a map of spectral sequences from the sequence of Theorem* 1 *to the Leray sequence of* (3) II 5.4.1. *This is induced in the* E_2 *terms by the map of local coefficients over* $\mathscr{U}$: $\tilde{\mathscr{H}}^*(\mathscr{F}) \to \mathscr{H}^*(\mathscr{F})$ *defined by the homomorphism of the corollary to Theorem* 2.

Proof. Consider the double complexes $K_1^{*,*} = C^*(\mathscr{U}, \tilde{\mathscr{C}}^*(\mathscr{F}))$ and $K_2^{*,*} = C^*\{\mathscr{U}, \mathscr{C}^*(X, \mathscr{F})\}$; and the triple complex $K^{*,*,*} = C^*\{\mathscr{U}, \tilde{\mathscr{C}}^*[\mathscr{C}^*(X, \mathscr{F})]\}$, the latter having differentiations d_1, d_2 and d_3. The sequences of Theorem 1 and of (3) II 5.4.1. are the spectral sequences of like K_1 and K_2 respectively with respect to their first filtrations.

The embeddings (5) and (7):

$$\tilde{C}^*(|s|, \mathscr{F}) \xrightarrow{j_1} \tilde{C}^*[|s|, \mathscr{C}^*(X, \mathscr{F})] \xleftarrow{j_2} C^*(|s|, \mathscr{F}) \quad \text{.............. (9)}$$

for each $s \in S_p(\mathscr{U})$, give embeddings:

$$C^*\{\mathscr{U}, \tilde{\mathscr{C}}^*(\mathscr{F})\} \xrightarrow{j_1} C^*\{\mathscr{U}, \tilde{\mathscr{C}}^*[\mathscr{C}^*(X, \mathscr{F})]\} \xleftarrow{j_2} C^*\{\mathscr{U}, \mathscr{C}^*(X, \mathscr{F})\}$$

i.e. $$K_1^{*,*} \xrightarrow{j_1} K^{*,*,*} \xleftarrow{j_2} K_2^{*,*} \quad \text{................................ (10)}$$

We have used the fact (3) II Lemma 4.9.1. that $\mathscr{C}^*(X, \mathscr{F})\big|_{|s|} \cong \mathscr{C}^*(|s|, \mathscr{F})$ since $\mathscr{U}$ is open.

In the induced map of total cohomologies of (9):

$$\tilde{H}^q(|s|, \mathscr{F}) \xrightarrow{j_1} H^q\{\tilde{C}^*[|s|, \mathscr{C}^*(X, \mathscr{F})]\} \xleftarrow{j_2} H^q(|s|, \mathscr{F}), \quad \text{........... (11)}$$

we have from (8) that j_2 is bijective, and by (3)

$$j_2^{-1} \cdot j_1 : \tilde{H}^q(|s|, \mathscr{F}) \to H^q(|s|, \mathscr{F}) \quad \text{...................... (12)}$$

is the homomorphism of the corollary to Theorem 2.

Let $\mathscr{K}^{*,*}$ be the double complex defined by $\mathscr{K}^{p,q} = \sum_{q'+r'=q} K^{p,q',r'}$ with differentiations d_1 and d_2+d_3. Then (10) defines maps of double complexes

$$K_1^{*,*} \xrightarrow{j_1} K^{*,*} \xleftarrow{j_2} K_2^{*,*}$$

and hence, by (1) XV 6., maps of spectral sequences

$$'E(K_1) \xrightarrow{j_1} {}'E(\mathscr{K}) \xleftarrow{j_2} {}'E(K_2), \quad \text{......................... (13)}$$

taking the first filtration of each double complex.

The $E_2^{p,q}$ terms in (13) are

$$'H^{p''}H^q K_1^{*',*''} \xrightarrow{j_1} {}'H^{p''}H^q \mathscr{K}^{*',*''} \xleftarrow{j_2} {}'H^{p''}H^q K_2^{*',*''}, \quad \text{........... (14)}$$

which are just the maps of the pth cohomology of $\mathscr{U}$ induced by the maps of local coefficients (11) over simplexes s of $\mathscr{U}$. Therefore j_2 in (14) is bijective and hence, by (1) XV 3.2., j_2 in (13) is an isomorphism of spectral sequences.

Thus $j_2^{-1} \cdot j_1 : {}'E(K_1) \to {}'E(K_2)$ is a map of spectral sequences induced in the E_2 terms by the maps of local coefficients (12) for simplexes s of $\mathscr{U}$; which completes the proof of the theorem.

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