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Transcendental Solutions of a Class of Minimal Functional Equations

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Abstract. We prove a result concerning power series $f(z) \in \mathbb{C}[[z]]$ satisfying a functional equation of the form

$$f(z^d) = \sum_{k=1}^{n} \frac{A_k(z)}{B_k(z)} f(z)^k,$$

where $A_k(z), B_k(z) \in \mathbb{C}[z]$. In particular, we show that if $f(z)$ satisfies a minimal functional equation of the above form with $n \geqslant 2$, then $f(z)$ is necessarily transcendental. Towards a more complete classification, the case $n = 1$ is also considered.

1 Introduction

We are concerned with the algebraic character of power series $f(z) \in \mathbb{C}[[z]]$ that satisfy a functional equation of the form

$$f(z^d) = \sum_{k=0}^{n} \frac{A_k(z)}{B_k(z)} f(z)^k, \tag{1.1}$$

where $A_k(z), B_k(z) \in \mathbb{C}[z]$. Functional equations of this type were studied by Mahler [8–11], and as such are sometimes called Mahler-type functional equations. Mahler proved that under certain conditions, *if $f(z) \in \mathbb{C}[[z]]$ is transcendental over $\mathbb{C}(z)$, then for $\alpha \in \overline{\mathbb{Q}}$ within the radius of convergence of $f(z)$, we have that $f(\alpha)$ is transcendental over $\mathbb{Q}$*.

Nishioka [12] subsequently proved the following.

Theorem 1.1 *A power series $f(z) \in \mathbb{C}[[z]]$ satisfying (1.1) is either rational or transcendental over $\mathbb{C}(z)$.*

Towards a classification of this rational-transcendental dichotomy, we proved the following result [3].

Theorem 1.2 *If $f(z)$ is a power series in $\mathbb{C}[[z]]$ satisfying*

$$f(z^d) = f(z) + \frac{A(z)}{B(z)},$$

where $d \geqslant 2$, $A(z), B(z) \in \mathbb{C}[z]$ with $A(z) \neq 0$ and $\deg A(z), \deg B(z) < d$, then $f(z)$ is transcendental over $\mathbb{C}(z)$.

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We applied Theorem 1.2 in [3] to yield quick transcendence results regarding the values of the series

$$\sum_{n\geqslant 0}\frac{z^{k^n}}{1-z^{k^n}} \quad \text{and} \quad \sum_{n\geqslant 0}\frac{z^{k^n}}{1+z^{k^n}},$$

when $k \geqslant 2$. These series were studied previously by Golomb [7] and Schwarz [13].

In this paper, we focus on a different, more general, class of functions satisfying (1.1); specifically, we wish to classify power series $f(z) \in \mathbb{C}[[z]]$ satisfying a functional equation of the form

$$f(z^d) = \sum_{k=1}^{n}\frac{A_k(z)}{B_k(z)}f(z)^k,$$

where $A_k(z), B_k(z) \in \mathbb{C}[z]$ and $d \geqslant 2$.

Examples of such functions are readily available. If one takes $d = 2$, $n = 1$, $A_1(z) = 1$, and $B_1(z) = 1 - z$, then the function $f(z) \in \mathbb{C}[[z]]$ satisfying

$$f(z^2) = \left(\frac{1}{1-z}\right)f(z)$$

is the generating function of a version of the Thue–Morse sequence. That is, $f(z) = \sum_{n\geqslant 0} t_n z^n$ where $t_n = 1 - 2a_n$ and $(a_n)_{n\geqslant 0} = 0110100110010110\ldots$ is the Thue–Morse sequence defined by

$$a_0 = 0, \quad a_{2n} = a_n, \quad a_{2n+1} = 1 - a_n \quad (n \geqslant 1).$$

The transcendence of the generating function of the Thue–Morse sequence was given by Dekking [4].

Another example of such a function is the generating function of the *Stern sequence* (sometimes called Stern's diatomic sequence). The Stern sequence $(s(n))_{n\geqslant 0}$ is given by $s(0) = 0$, $s(1) = 1$, and when $n \geqslant 1$, by

$$s(2n) = s(n) \quad \text{and} \quad s(2n+1) = s(n) + s(n+1).$$

Properties of this sequence have been studied by many authors; for references see [5]. If $A(z)$ is the generating function of the Stern sequence, then

$$A(z^2) = \left(\frac{z}{z^2+z+1}\right)A(z).$$

In [2], we proved the transcendence of the function $A(z)$, as well as the transcendence of the generating functions of some special subsequences of $(s(n))_{n\geqslant 0}$ which were conjectured by Dilcher and Stolarsky [6] (similar results were given independently by Adamczewski [1]).

Using a generalization of the methods in [2–4], we prove that under the condition that a functional equation like (1.1) is minimal with respect to n, if $f(z)$ is the power series expansion of a rational function, then $n = 1$.

2 Zero Constant-Term Functional Equations

Definition 2.1 Suppose that for $d \geqslant 2$ the power series $f(z) \in \mathbb{C}[[z]]$ satisfies

$$f(z^d) = \sum_{k=1}^{n} \frac{A_k(z)}{B_k(z)} f(z)^k, \tag{2.1}$$

where $A_k(z), B_k(z) \in \mathbb{C}[z]$. We call the functional equation (2.1) for $f(z)$ *minimal* provided n is the smallest positive integer so that $f(z)$ satisfies (2.1).

Note that the functional equation (2.1) has no $k = 0$ term; this is the reason behind the title of this section. Our main result is the following.

Theorem 2.2 *Let $d \geqslant 2$ and suppose that $f(z) \in \mathbb{C}[[z]]$ is the power series expansion of a rational function satisfying the minimal functional equation*

$$f(z^d) = \sum_{k=1}^{n} \frac{A_k(z)}{B_k(z)} f(z)^k,$$

where $A_k(z), B_k(z) \in \mathbb{C}[z]$ and $\gcd(A_k(z), B_k(z)) = 1$. Then $n = 1$.

Proof Suppose that $f(z) \in \mathbb{C}[[z]]$ is a rational function satisfying the minimal functional equation

$$f(z^d) = \sum_{k=1}^{n} \frac{A_k(z)}{B_k(z)} f(z)^k,$$

where $A_k(z), B_k(z) \in \mathbb{C}[z]$ and $d \geqslant 2$. Since $f(z)$ is rational, there exist polynomials $q_0(z), q_1(z) \in \mathbb{C}[z]$ such that $f(z)$ satisfies $q_1(z)f(z) + q_0(z) = 0$. Then for any rational functions $D(z), C(z) \in \mathbb{C}(z)$, we have that both

$$0 = C(z)(q_1(z)f(z) + q_0(z))^n = C(z)\sum_{k=0}^{n} \binom{n}{k} q_0(z)^{n-k} q_1(z)^k f(z)^k, \tag{2.2}$$

$$\begin{aligned} 0 &= (D(z) - 1)(q_1(z^d)f(z^d) + q_0(z^d)) \\ &= D(z)q_1(z^d)\sum_{k=1}^{n} \frac{A_k(z)}{B_k(z)} f(z)^k + D(z)q_0(z) - (q_1(z^d)f(z^d) + q_0(z^d)). \end{aligned} \tag{2.3}$$

Subtracting (2.2) from (2.3) and rearranging slightly we have

$$\begin{aligned} f(z^d) = {} & \frac{1}{q_1(z^d)}\Big[D(z)q_1(z^d)\frac{A_n(z)}{B_n(z)} - C(z)q_1(z)^n\Big] f(z)^n \\ & + \frac{1}{q_1(z^d)}\sum_{k=1}^{n-1}\Big[D(z)q_1(z^d)\frac{A_k(z)}{B_k(z)} - C(z)\binom{n}{k} q_0(z)^{n-k}q_1(z)^k\Big] f(z)^k \\ & + \frac{1}{q_1(z^d)}[D(z)q_0(z^d) - C(z)q_0(z)^n]. \end{aligned}$$

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Set $D(z) = q_1(z)^n$ and $C(z) = q_1(z^d)\frac{A_n(z)}{B_n(z)}$. Then

$$D(z)q_1(z^d)\frac{A_n(z)}{B_n(z)} - C(z)q_1(z)^n = q_1(z)^n q_1(z^d)\frac{A_n(z)}{B_n(z)} - q_1(z^d)\frac{A_n(z)}{B_n(z)}q_1(z)^n = 0,$$

so that we can write

$$f(z^d) = \sum_{k=0}^{n-1} H_k(z) f(z)^k, \tag{2.4}$$

where for $1 \leqslant k \leqslant n-1$ we have

$$H_k(z) = q_1(z)^n \frac{A_k(z)}{B_k(z)} - \frac{A_n(z)}{B_n(z)}\binom{n}{k} q_0(z)^{n-k} q_1(z)^k,$$

$$H_0(z) = \frac{q_1(z)^n q_0(z^d)}{q_1(z^d)} - \frac{A_n(z)}{B_n(z)} q_0(z)^n.$$

Since n was minimal and $f(z)$ satisfies (2.4), we have that $H_k(z) = 0$ for all $k = 0, 1, \ldots, n-1$. Thus for $1 \leqslant k \leqslant n-1$, we have

$$q_1(z)^{n-k} A_k(z) B_n(z) = A_n(z) B_k(z) \binom{n}{k} q_0(z)^{n-k},$$

and for $k = 0$, we have $q_1(z)^n q_0(z^d) B_n(z) = A_n(z) q_1(z^d) q_0(z)^n$. These two equations give both

$$\frac{A_k(z)}{B_k(z)} = \binom{n}{k} \frac{A_n(z)}{B_n(z)} \left(\frac{q_0(z)}{q_1(z)}\right)^{n-k}, \tag{2.5}$$

$$\frac{A_n(z)}{B_n(z)} = \left(\frac{q_1(z)}{q_0(z)}\right)^{n} \frac{q_0(z^d)}{q_1(z^d)}. \tag{2.6}$$

Substituting (2.6) into (2.5) gives for each k that

$$\frac{A_k(z)}{B_k(z)} = \binom{n}{k} \left(\frac{q_1(z)}{q_0(z)}\right)^{k} \frac{q_0(z^d)}{q_1(z^d)}. \tag{2.7}$$

But $q_1(z) f(z) + q_0(z) = 0$, so that

$$\frac{q_1(z)}{q_0(z)} = \frac{-1}{f(z)} \quad \text{and} \quad \frac{q_1(z^d)}{q_0(z^d)} = \frac{-1}{f(z^d)}.$$

Thus (2.7) becomes

$$f(z^d) = \frac{(-1)^k}{\binom{n}{k}} \frac{A_k(z)}{B_k(z)} f(z)^k$$

for each k satisfying $1 \leqslant k \leqslant n$. Since n is minimal, we have that $n = 1$. ■

In view of Nishioka's theorem (see the Introduction), we have the following immediate corollary of Theorem 2.2.

Corollary 2.3 *Let $d \geqslant 2$ and suppose that $n \geqslant 2$ and $f(z) \in \mathbb{C}[[z]]$ satisfies the minimal functional equation*

$$f(z^d) = \sum_{k=1}^{n} \frac{A_k(z)}{B_k(z)} f(z)^k,$$

where $A_k(z), B_k(z) \in \mathbb{C}[z]$ and $\gcd(A_k(z), B_k(z)) = 1$. Then $f(z)$ is transcendental over $\mathbb{C}(z)$.

3 The Linear ($n = 1$) Case

Towards a further classification of the rational-transcendental dichotomy of power series satisfying (2.1), we note that the results of Section 2 allow us to focus on the case $n = 1$ of (2.1). Recall that this is the case into which the generating functions of both the Thue–Morse sequence and the Stern sequence fall. To formalize, in this section we consider power series $f(z) \in \mathbb{C}[[z]]$ that satisfy

$$f(z^d) = \frac{A(z)}{B(z)} f(z), \tag{3.1}$$

where $d \geqslant 2$ and $A(z), B(z) \in \mathbb{C}[[z]]$. We do not assume Nishioka's theorem for the proofs in this section.

Theorem 3.1 *If $f(z)$ is a power series in $\mathbb{C}[[z]]$ satisfying*

$$f(z^d) = \frac{A(z)}{B(z)} f(z),$$

where $d \geqslant 2$, $A(z), B(z) \in \mathbb{C}[z]$, and $\gcd(A(z), B(z)) = 1$. If $\deg B(z) - \deg A(z)$ is not a multiple of $d - 1$, then $f(z)$ is transcendental over $\mathbb{C}(z)$.

Proof Towards a contradiction, suppose that $f(z)$ is algebraic and satisfies, say,

$$q_n(z) f(z)^n + q_{n-1}(z) f(z)^{n-1} + \cdots + q_0(z) = 0,$$

where $q_i(z) \in \mathbb{C}[z]$, $\gcd(q_n(z), q_{n-1}(z), \ldots, q_0(z)) = 1$, and n is chosen minimally. Using this algebraic property, we have

$$0 = \sum_{k=0}^{n} q_k(z^d) f(z^d)^k = \sum_{k=0}^{n} q_k(z^d) f(z)^k \left(\frac{A(z)}{B(z)} \right)^k,$$

and upon multiplying by $B(z)^n$, we obtain

$$0 = \sum_{k=0}^{n} q_k(z^d) B(z)^{n-k} A(z)^k f(z)^k.$$

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Thus

$$0 = A(z)^n q_n(z^d) \sum_{k=0}^{n} q_k(z) f(z)^k - q_n(z) \sum_{k=0}^{n} q_k(z^d) B(z)^{n-k} A(z)^k f(z)^k \tag{3.2}$$

$$= \sum_{k=0}^{n} \left[q_n(z^d) q_k(z) A(z)^n - q_n(z) q_k(z^d) B(z)^{n-k} A(z)^k \right] f(z)^k.$$

The coefficient of $f(z)^n$ in (3.2) is $q_n(z^d) q_n(z) A(z)^n - q_n(z) q_n(z^d) A(z)^n = 0$, so that

$$0 = \sum_{k=0}^{n-1} \left[q_n(z^d) q_k(z) A(z)^n - q_n(z) q_k(z^d) B(z)^{n-k} A(z)^k \right] f(z)^k.$$

The minimality of n gives

$$q_n(z^d) q_k(z) A(z)^n = q_n(z) q_k(z^d) B(z)^{n-k} A(z)^k \tag{3.3}$$

for $k = 0, 1, \ldots, n-1$.

The equality in (3.3) gives the degree relationship

$$(d-1)(\deg q_n(z) - \deg q_k(z)) = (n-k)(\deg B(z) - \deg A(z)), \tag{3.4}$$

for $k = 0, 1, \ldots, n-1$. In particular, setting $k = n-1$ gives

$$(d-1)(\deg q_n(z) - \deg q_{n-1}(z)) = \deg B(z) - \deg A(z).$$

Thus $d-1$ divides $\deg B(z) - \deg A(z)$. ■

Continuing this line of reasoning, for algebraic $f(z)$ satisfying the functional equation (3.1), set

$$w := \frac{\deg B(z) - \deg A(z)}{d-1}.$$

Then again using (3.4), we have $\deg q_k(z) = \deg q_n(z) - w(n-k)$, which gives the following result.

Proposition 3.2 *Let $f(z) \in \mathbb{C}[[z]]$ be a power series satisfying*

$$f(z^d) = f(z) \frac{A(z)}{B(z)},$$

where $d \geqslant 2$, and $A(z), B(z) \in \mathbb{C}[z]$ with $\gcd(A(z), B(z)) = 1$. If $f(z)$ is rational, satisfying $q_1(z) f(z) + q_0(z) = 0$, where $q_1(z), q_0(z) \in \mathbb{C}[z]$ and $\gcd(q_1(z), q_0(z)) = 1$, then

$$\deg q_0(z) = \deg q_1(z) - \frac{\deg B(z) - \deg A(z)}{d-1},$$

for $k = 0, 1, \ldots, n$.

For the following theorem, it is convenient to define the following notation. For $p(z) \in \mathbb{C}[z]$ denote by $\operatorname{ord}_a p(z)$ the multiplicity of the root $z = a$ of $p(z)$. Also write $\zeta_m := e^{2\pi i/m}$.

Theorem 3.3 *Let $f(z) \in \mathbb{C}[[z]]$ be a power series satisfying*

$$f(z^d) = f(z)\frac{A(z)}{B(z)},$$

where $d \geqslant 2$, and $A(z), B(z) \in \mathbb{C}[z]$ with $\gcd(A(z), B(z)) = 1$. If $f(z)$ is algebraic, then for any $j \in \mathbb{Z}$ we have $\operatorname{ord}_{\zeta_{d+1}^j} B(z) = \operatorname{ord}_{\zeta_{d+1}^{-j}} A(z)$.

Proof We start with the terminology and statement of (3.3), that is,

$$q_n(z^d)q_k(z)A(z)^{n-k} = q_n(z)q_k(z^d)B(z)^{n-k} \tag{3.5}$$

for $k = 0, 1, \ldots, n-1$.

Now for any $k = 0, 1, \ldots, n-1$ if $z - \zeta_{d+1}^j \mid q_k(z)$, then $z^d - \zeta_{d+1}^j \mid q_k(z^d)$. Since

$$(\zeta_{d+1}^{-j})^d - \zeta_{d+1}^j = \zeta_{d+1}^{-j(d+1-1)} - \zeta_{d+1}^j = \zeta_{d+1}^j - \zeta_{d+1}^j = 0,$$

we have $z - \zeta_{d+1}^{-j} \mid q_k(z^d)$. Conversely, if $z - \zeta_{d+1}^{-j} \mid q_k(z^d)$, then since

$$q_k(z^d) = \alpha \prod_{i=0}^{\deg q_k(z)} (z^d - y_i),$$

we have that there is a y_i such that $z - \zeta_{d+1}^{-j} \mid z^d - y_i$. Thus

$$y_i = (\zeta_{d+1}^{-j})^d = \zeta_{d+1}^{-j(d+1-1)} = \zeta_{d+1}^j.$$

Hence $z - \zeta_{d+1}^j \mid q_k(z)$. This gives $\operatorname{ord}_{\zeta_{d+1}^j} q_k(z) = \operatorname{ord}_{\zeta_{d+1}^{-j}} q_k(z^d)$. The relationship (3.5) gives for $k = 0, 1, \ldots, n-1$, the two identities

$$\operatorname{ord}_{\zeta_{d+1}^j} q_n(z^d) + \operatorname{ord}_{\zeta_{d+1}^j} q_k(z) + (n-k)\operatorname{ord}_{\zeta_{d+1}^j} A(z)$$
$$= \operatorname{ord}_{\zeta_{d+1}^j} q_n(z) + \operatorname{ord}_{\zeta_{d+1}^j} q_k(z^d) + (n-k)\operatorname{ord}_{\zeta_{d+1}^j} B(z),$$

and

$$\operatorname{ord}_{\zeta_{d+1}^{-j}} q_n(z^d) + \operatorname{ord}_{\zeta_{d+1}^{-j}} q_k(z) + (n-k)\operatorname{ord}_{\zeta_{d+1}^{-j}} A(z)$$
$$= \operatorname{ord}_{\zeta_{d+1}^{-j}} q_n(z) + \operatorname{ord}_{\zeta_{d+1}^{-j}} q_k(z^d) + (n-k)\operatorname{ord}_{\zeta_{d+1}^{-j}} B(z).$$

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Since $\operatorname{ord}_{\zeta_{d+1}^{j}} q_k(z) = \operatorname{ord}_{\zeta_{d+1}^{-j}} q_k(z^d)$, the substitution of the second identity into the first gives

$$\operatorname{ord}_{\zeta_{d+1}^{j}} q_n(z^d) + \operatorname{ord}_{\zeta_{d+1}^{j}} q_k(z) + (n-k)(\operatorname{ord}_{\zeta_{d+1}^{j}} A(z) + \operatorname{ord}_{\zeta_{d+1}^{-j}} A(z))$$
$$= \operatorname{ord}_{\zeta_{d+1}^{-j}} q_n(z) + \operatorname{ord}_{\zeta_{d+1}^{-j}} q_k(z^d) + (n-k)(\operatorname{ord}_{\zeta_{d+1}^{-j}} B(z) + \operatorname{ord}_{\zeta_{d+1}^{j}} B(z)),$$

which reduces to $\operatorname{ord}_{\zeta_{d+1}^{j}} A(z) + \operatorname{ord}_{\zeta_{d+1}^{-j}} A(z) = \operatorname{ord}_{\zeta_{d+1}^{-j}} B(z) + \operatorname{ord}_{\zeta_{d+1}^{j}} B(z)$.

Since $\gcd(A(z), B(z)) = 1$, if $\operatorname{ord}_{\zeta_{d+1}^{j}} B(z) \neq 0$, we have that $\operatorname{ord}_{\zeta_{d+1}^{j}} A(z) = 0$. Thus we have $\operatorname{ord}_{\zeta_{d+1}^{-j}} A(z) = \operatorname{ord}_{\zeta_{d+1}^{-j}} B(z) + \operatorname{ord}_{\zeta_{d+1}^{j}} B(z)$. Taking into account that $\gcd(A(z), B(z)) = 1$, since $\operatorname{ord}_a p(z)$ is a non–negative integer, it must be the case that $\operatorname{ord}_{\zeta_{d+1}^{-j}} B(z) = 0$, and so $\operatorname{ord}_{\zeta_{d+1}^{-j}} A(z) = \operatorname{ord}_{\zeta_{d+1}^{j}} B(z)$. The case $\operatorname{ord}_{\zeta_{d+1}^{j}} A(z) \neq 0$ follows similarly. ■

Corollary 3.4 *Let $f(z) \in \mathbb{C}[[z]]$ be a power series satisfying*

$$f(z^d) = \frac{A(z)}{B(z)} f(z),$$

where $d \geqslant 2$, and $A(z), B(z) \in \mathbb{R}[z]$ with $\gcd(A(z), B(z)) = 1$. If there is a $j \in \mathbb{Z}$ such that $A(\zeta_{d+1}^{j}) = 0$ or $B(\zeta_{d+1}^{j}) = 0$, then $f(z)$ is transcendental over $\mathbb{C}(z)$.

Proof Note that if $p(z) \in \mathbb{R}[z]$, then $\operatorname{ord}_a p(z) = \operatorname{ord}_{\overline{a}} p(z)$, where $\overline{a}$ is the complex conjugate of a (for real a we have $a = \overline{a}$). Suppose that $f(z)$ is algebraic over $\mathbb{C}(z)$ and satisfies the above assumptions. Applying the previous theorem, we have

$$\operatorname{ord}_{\zeta_{d+1}^{j}} A(z) = \operatorname{ord}_{\zeta_{d+1}^{-j}} A(z) = \operatorname{ord}_{\zeta_{d+1}^{j}} B(z). \tag{3.6}$$

If one of $A(\zeta_{d+1}^{j}) = 0$ or $B(\zeta_{d+1}^{j}) = 0$, then (3.6) gives that $\gcd(A(z), B(z)) \neq 1$, which is a contradiction. Thus $f(z)$ is transcendental over $\mathbb{C}(z)$. ■

We note that Corollary 3.4 implies that the generating functions of both the Thue–Morse sequence and the Stern sequence are transcendental.

References

[1] B. Adamczewski, *Non-converging continued fractions related to the Stern diatomic sequence*. Acta Arith. **142**(2010), no. 1, 67–78. http://dx.doi.org/10.4064/aa142-1-6

[2] M. Coons, *The transcendence of series related to Stern's diatomic sequence*. Int. J. Number Theory **6**(2010), no.1, 211–217. http://dx.doi.org/10.1142/S1793042110002958

[3] ———, *Extension of some theorems of W. Schwarz*. Canad. Math. Bull. Published online March 10, 2011 http://dx.doi.org/10.4153/CMB-2011-037-9

[4] M. Dekking, *Transcendance du nombre de Thue-Morse*. C. R. Acad. Sci. Paris Sér. A-B **285**(1977), no. 4, A157–A160.

[5] K. Dilcher and K. B. Stolarsky, *A polynomial analogue to the Stern sequence*. Int. J. Number Theory **3**(2007), no. 1, 85–103. http://dx.doi.org/10.1142/S179304210700081X

[6] ———, *Stern polynomials and double-limit continued fractions*. Acta Arith. **140**(2009), no. 2, 119–134. http://dx.doi.org/10.4064/aa140-2-2

[7] S. W. Golomb, *On the sum of the reciprocals of the Fermat numbers and related irrationalities*. Canad. J. Math. **15**(1963), 475–478. http://dx.doi.org/10.4153/CJM-1963-051-0
[8] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*. Math. Ann. **101**(1929), no. 1, 342–366. http://dx.doi.org/10.1007/BF01454845
[9] ______, *Arithmetische Eigenschaften einer Klasse transzendental-transzendenter Funktionen*. Math. Zeit. **32**(1930), no. 1, 545–585. http://dx.doi.org/10.1007/BF01194652
[10] ______, *Über das Verschwinden von Potenzreihen mehrerer Veränderlicher in speziellen Punktfolgen*. Math. Ann. **103**(1930), no. 1, 573–587. http://dx.doi.org/10.1007/BF01455711
[11] ______, *Remarks on a paper by W. Schwarz*, J. Number Theory **1**(1969), 512–521. http://dx.doi.org/10.1016/0022-314X(69)90013-4
[12] K. Nishioka, *Algebraic function solutions of a certain class of functional equations*. Arch. Math. **44**(1985), no. 4, 330–335.
[13] W. Schwarz, *Remarks on the irrationality and transcendence of certain series*. Math. Scand **20**(1967), 269–274.

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