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THE EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF NEUTRAL DIFFERENTIAL DIFFERENCE EQUATIONS

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Abstract

In this paper, we study the existence of periodic solutions of the NDDE (neutral differential difference equation):

$$(x(t)+cx(t-\tau))' = -f(x(t), x(t-\tau)) \qquad (*)$$

where $\tau > 0$ and c is a real number. We obtain a sufficient condition under which $(*)$ has at least k nonconstant oscillatory periodic solutions.

1. Introduction

In 1967, R. Brayton [1–2] considered the problem of lossless transmission lines used to connect switching circuits and obtained the following NDDE (neutral differential difference equation):

$$\dot{u}(t) - k\dot{u}(t-2/s) = f(u(t), u(t-2/s)) \qquad \text{(A)}$$

where $s = \sqrt{LC}$. In this paper, we study the existence of a periodic solution of (A). For the case $k = 0$, several papers [5–6] have given sufficient conditions for the existence of a periodic solution of (A). However, for the cases $k \neq 0$, there are few papers dealing with the existence of a periodic solution of (A). Now, we consider a class of NDDEs which is more general than (A):

$$(x(t)+cx(t-\tau))' = -f(x(t), x(t-\tau)) \qquad (1)$$

where $\tau > 0$, c is a real number, and $f(x, y)$ is a continuous function. Since the solutions of (1) may not be differentiable, (1) is more general than (A).

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Throughout this paper we assume that there exists a continuous function $g(x, y)$ such that

$$f(x, y) = g(x, y) - cg(y, x). \tag{2}$$

Usually a function $g(x, y)$ which satisfies (2) is easily obtained. For example if $c \neq \pm 1$, then, from (2), we have

$$f(y, x) = g(y, x) - cg(x, y). \tag{3}$$

By (2), (3) we obtain

$$g(x, y) = \frac{1}{1-c^2} f(x, y) + \frac{c}{1-c^2} f(y, x).$$

2. The main result

Consider the ordinary differential system:

$$\frac{dx}{dy} = -g(x, y), \qquad \frac{dy}{dx} = g(y, x). \tag{4}$$

We suppose that

(I) $g(x, y)$ is continuous on R^2;

(II) $\binom{-g(x,y)}{g(y,x)}$ satisfies the local Lipschitz condition on R^2.

It is easy to see that, under the conditions (I) and (II), (4) has a unique solution which satisfies the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$ and through any point (x_0, y_0) (4) has a unique orbit [3].

Lemma. *Suppose that*

(a) $g(x, -y) = -g(x, y)$, $g(-x, y) = g(x, y)$, $yg(x, y) > 0$ $(y \neq 0)$;

(b) *there exists some* $b > 0$ *such that*

$$g(y, x)/g(x, y) \leq A(x)B(y) \quad (x \geq 0,\ y \geq b > 0),$$

where $A(x)$ *is continuous on* $[0, +\infty)$, $B(y)$ *is continuous on* $[b, +\infty)$, $B(y) > 0$ $(y \geq b)$ *and* $\int^{+\infty}(1/B(y))\,dy = +\infty$;

(c)

$$\lim_{x^2+y^2\to 0^+} \frac{xg(y, x) + yg(x, y)}{x^2 + y^2} = p,$$

$$\lim_{x^2+y^2\to +\infty} \frac{xg(y, x) + yg(x, y)}{x^2 + y^2} = q;$$

(d) *there is some* $T > 0$ *such that* $p < 2\pi/T < q \leq +\infty$ *or* $q < 2\pi/T < p \leq +\infty$. *Then* (4) *has a periodic solution with period of* T.

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PROOF. Since $yg(x, y) > 0$ $(y \neq 0)$, $(0, 0)$ is the unique singular point of (4). By (I) and (II), through any point (x_0, y_0), (4) has a unique orbit [6]. Assume that, through the point (x_0, y_0), the orbit of (4) is L, where $x_0 \geq 0$, $y_0 \geq 0$, $(x_0, y_0) \neq (0, 0)$. If $x_0 = 0$, then it is easy to see that L intersects the positive y-axis. If $x_0 > 0$, then we claim that L intersects the positive y-axis. Otherwise, by $dy/dx = -g(y, x)/g(x, y) < 0$ $(x > 0,\ y > 0)$, L has an asymptotic line $x = a \geq 0$. Let L be $y = y(x)$ $(a < x \leq x_0)$. Then, by $\lim_{x \to a^+} y(x) = +\infty$, there is $x_1 : a < x_1 \leq x_0$ such that $y(x_1) \geq b$. Noting that $y(x)$ is decreasing, we have $y(x) \geq y(x_1) \geq b$ $(a < x \leq x_1)$. Hence, by the condition (b), we obtain

$$\frac{dy(x)}{dx} = -\frac{g(y(x), x)}{g(x, y(x))} \geq -A(x)B(y(x))$$

and

$$\int_x^{x_1} \frac{dy(s)}{B(y(s))} \geq -\int_x^{x_1} A(s)\, ds$$

or

$$-\int_{y(x_1)}^{y(x)} \frac{dy}{B(y)} \geq -\int_x^{x_1} A(s)\, ds \qquad (a < x \leq x_1).$$

As $x \to a^+$, the above inequality and the condition $\int^{+\infty} (1/B(y))\, dy = +\infty$ produce the desired contradiction and establish the claim that L intersects the positive y-axis. By the condition (b) and (4), for $y \geq 0$ and $x \geq b > 0$, we have

$$\frac{dx}{dy} = -\frac{g(x, y)}{g(y, x)} \geq -A(y)B(x).$$

Similarly, we can prove that L intersects the positive x-axis. Then, any orbit which passes through the point (x_0, y_0) intersects the positive x-axis and the positive y-axis, where $x_0 \geq 0$, $y_0 \geq 0$, $(x_0, y_0) \neq (0, 0)$. By (4) and the condition (a), we have $dy/dx = -g(y, x)/g(x, y)$, $dx/dy = -g(x, y)/g(y, x)$ and $g(x, -y) = -g(x, y)$, $g(-x, y) = g(x, y)$. Hence, the orbit of (4) is symmetric for the x-axis, y-axis, origin and the lines $y = \pm x$. Then, noting that L intersects the positive x-axis and the positive y-axis, we know that every orbit of (4) is a simple closed curve which is symmetric for the x-axis, y-axis, origin and the lines $y = \pm x$. Let $(x_c(t), y_c(t))$ be the solution of (4) which satisfies $x_c(0) = c$, $y_c(0) = c$ $(c > 0)$. Since the orbit of (4) is closed, the solution $(x_c(t), y_c(t))$ is bounded. Because $g(x, y)$ satisfies the conditions (I) and (II), the solution $(x_c(t), y_c(t))$ exists on $(-\infty, +\infty)$ [3].

Suppose that through the point (c, c) the orbit of (4) is L_c. Since L_c is closed, the solution $(x_c(t), y_c(t))$ is a periodic solution of (4). Let the period of $(x_c(t), y_c(t))$ be $w(c)$. Because the solutions continuously depend on the initial conditions, it is easy to show that $w(c)$ is a continuous function. Noting that L_c is symmetric for the x-axis, y-axis and

$$\frac{dy}{dx} = -\frac{g(y, x)}{g(x, y)} < 0 \qquad (x > 0,\ y > 0),$$

we have

$$|x| \geq c \quad \text{or} \quad |y| \geq c, \quad \forall (x, y) \in L_c.$$

Then

$$x^2 + y^2 \geq c^2, \qquad \forall (x, y) \in L_c. \tag{5}$$

Let

$$m(c) = \inf\{x_c^2(t) + y_c^2(t),\ 0 \leq t \leq w(c)\},$$
$$M(c) = \sup\{x_c^2(t) + y_c^2),\ 0 \leq t \leq w(c)\}.$$

Then $m(c) \geq 0$ $M(c) \geq 0$. By (5), we obtain

$$\lim_{c \to +\infty} m(c) = +\infty. \tag{6}$$

Since, under the conditions (I) and (II), the orbits of (4) are mutually disjoint [3], $M(c)$ is an increasing function and $\lim_{c \to 0^+} M(c)$ exists. Noting that $M(c) \geq 0$, we have $\lim_{c \to 0^+} M(c) \geq 0$. We claim that

$$\lim_{c \to 0^+} M(c) = 0. \tag{7}$$

Otherwise, we have $\lim_{c \to 0^+} M(c) = d > 0$. Consider the orbit L_A which passes through the point $A(\sqrt{d}/2, 0)$. Since L_A is a simple closed curve which is symmetric for the lines $y = \pm x$, L_A intersects the positive y-axis and the intersection point is $(0, \sqrt{d}/2)$. Noting that L_A is symmetric for the x-axis, the y-axis and $dy/dx > 0$ on $x > 0$, $y > 0$, we have

$$|x| \leq \sqrt{d}/2, \quad |y| \leq \sqrt{d}/2, \quad \forall (x, y) \in L_A,$$

and

$$x^2 + y^2 \leq (\sqrt{d}/2)^2 + (\sqrt{d}/2)^2 = d/2, \qquad \forall (x, y) \in L_A.$$

Let the intersection point of L_A and $y = x$ be (a, a). Then, we have $M(a) \leq d/2$ which contradicts the fact that $\lim_{c \to o^+} M(c) = d > 0$ and establishes the claim that (7) holds.

Let $H(t) = \arctan y_c(t)/x_c(t)$. Then

$$\begin{aligned} 2\pi &= \int_0^{2\pi} dH = \int_0^{w(c)} H'(t)\,dt = \int_0^{w(c)} \frac{y_c'(t)x_c(t) - x_c'(t)y_c'(t)}{x_c^2(t) + y_c^2(t)}\,dt \\ &= \int_0^{w(c)} \frac{x_c(t)g(y_c(t), x_c(t)) + y_c(t)g(x_c(t), y_c(t))}{x_c^2(t) + y_c^2(t)}\,dt. \end{aligned} \tag{8}$$

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Since (7) holds, $x_c^2(t) + y_c^2(t)$ uniformly tends to zero as $c \to 0^+$. Then, by (7), (8) and the conditions (c), (d), we have

$$2\pi = p \lim_{c \to 0^+} w(c). \tag{9}$$

Similarly, by (6), (8) and the conditions (c), (d), we have

$$2\pi = q \lim_{c \to +\infty} w(c). \tag{10}$$

Noting that $p < 2\pi/T < q$ or $q < 2\pi/T > p$ and (9), (10) hold, we obtain

$$\lim_{c \to 0^+} w(c) > T, \qquad \lim_{c \to +\infty} w(c) < T$$

or

$$\lim_{c \to 0^+} w(c) < T, \qquad \lim_{c \to +\infty} w(c) > T.$$

Hence, there exists $c^* \in (0, +\infty)$ such that $w(c^*) = T$ and the solution $(x^*(t), y^*(t))$ which satisfies the initial conditions $x^*(0) = c^*$, $y^*(0) = c^*$ is a nonconstant periodic solution with period of T. The proof of the lemma is now complete.

THEOREM 1. *Suppose that there is a function* $g(x, y)$ *such that*

$$f(x, y) = g(x, y) - cg(y, x) \tag{11}$$

where $g(x, y)$ *satisfies the conditions* (I), (II). *If the conditions* (a), (b), (c) *of the lemma and*

$$p < \frac{(1+4n)\pi}{2\tau} < q \quad \textit{or} \quad < \frac{(1+4n)\pi}{2\tau} < p \qquad (n = m, n+1, \dots, m+k-1) \tag{d$'$}$$

hold, then (1) *has at least* k *nonconstant oscillatory solutions, where* m *is some nonnegative integer and* k *is some positive integer.*

PROOF. By the lemma, (4) has a periodic solution with period of $T_n = 4\tau/(1+4m)$. Since $n = m, m+1, \dots, m+k-1$, we obtain k nonconstant solutions $(x_n(t), y_n(t))$ of (4), where $(x_n(t), y_n(t))$ satisfies the initial conditions $x_n(0) = c_n$, $y_n(0) = c_n$ and the period of $(x_n(t), y_n(t))$ is $w(c_n) = 4\tau/(1+4n)$. Assume that, through the point (c_n, c_n), the orbit of (4) is L_n. By the proof of the lemma, we know that L_n is a simple closed curve which is symmetric for the x-axis, y-axis, origin and the lines $y = \pm x$. Since L_n is symmetric for the origin, it is easy to show that the point $(-x_n(t), -y_n(t)) \in L_n$ for any $t \in (-\infty, +\infty)$ and that $(-x_n(t), -y_n(t))$ is a solution of (4). Then the solution $(x_n(t), y_n(t))$ will meet the solution $(-x_n(t), -y_n(t))$ after a translation of time τ_1, i.e. there is some $\tau_1 \in (0, \frac{4\tau}{1+4n})$ such that

$$\begin{aligned} x_n(t) &= -x_n(t+\tau_1) = x_n(t+2\tau_1), \\ y_n(t) &= -y_n(t+\tau_1) = y_n(t+2\tau_1). \end{aligned} \tag{12}$$

Noting that $x_n(t)$ has period of $4\tau/(1+4n)$, by (12), we have

$$2\tau_1 = h_1 \cdot \frac{4\tau}{1+4n} \quad \text{or} \quad \tau_1 = h_1 \cdot \frac{2\tau}{1+4n}$$

where h_1 is some positive integer. By $\tau_1 \in (0, \frac{4\tau}{1+4n})$ and $\tau_1 = h_1 \cdot \frac{2\tau}{1+4n}$ (h_1 is positive integer), we have $h_1 = 1$ and $\tau_1 = 2\tau/(2+4n)$. Then, by (12), we have

$$\begin{aligned} x_n(t) &= -x_n\left(t + \frac{2\tau}{1+4n}\right) = -x_n\left(t - \frac{2\tau}{1+4n}\right), \\ y_n(t) &= -y_n\left(t + \frac{2\tau}{1+4n}\right) = -y_n\left(t - \frac{2\tau}{1+4n}\right). \end{aligned} \tag{13}$$

On the other hand, since the closed curve L_n is symmetric for the x-axis, y-axis and the lines $y = \pm x$, it is easy to show that the point $(-y_n(t), x_n(t)) \in L_n$ $(t \in (-\infty, +\infty))$ and $(-y_n(t), x_n(t))$ is a solution of (4). Then the solution $(x_n(t), y_n(t))$ will meet the solution $(-y_n(t), x_n(t))$ after a translation of time τ_2, i.e. there is some $\tau_2 \in (0, \frac{4\tau}{1+4n})$ such that

$$-y_n(t) = x_n(t+\tau_2), \qquad x_n(t) = y_n(t+\tau_2). \tag{14}$$

By (14), we have

$$x_n(t) = y_n(t+\tau_2) = -x_n(t+2\tau_2). \tag{15}$$

By (13) and (15), we have

$$x_n\left(t - \frac{2\tau}{1+4n}\right) = x_n(t+2\tau_2).$$

Then

$$2\tau_2 + \frac{2\tau}{1+4n} = h_2 \cdot \frac{4\tau}{1+4n} \quad \text{or} \quad \tau_2 = \frac{4\tau}{1+4n} \cdot \left(\frac{h_2}{2} - \frac{1}{4}\right) \tag{16}$$

where h_2 is some positive integer. By $\tau_2 \in (0, \frac{4\tau}{1+4n})$ and (16), it is easy to see $h_2 = 1$ or 2. Then $\tau_2 = \frac{\tau}{1+4n}$ or $\tau_2 = \frac{3\tau}{1+4n}$. We choose t_0 such that the point $(x_n(t_0), y_n(t_0))$ belongs to the first quadrant. Then $x_n(t_0) > 0$, $y_n(t_0) > 0$. Hence the point $(-y_n(t_0), x_n(t_0))$ should belong to the second quadrant and the point $(-x_n(t_0), -y_n(t_0))$ should belong to the third quadrant.

By (13), and (14), we have

$$\left(x_n\left(t_0+\frac{2\tau}{1+4n}\right), y_n\left(t_0+\frac{2\tau}{1+4n}\right)\right) = (x_n(t_0), -y_n(t_0)),$$
$$(x_n(t_0+\tau_2), y_n(t_0+\tau_2)) = (-y_n(t_0), x_n(t_0)).$$

Then $(x_n(t_0+\tau_2), y_n(t_0+\tau_2))$ belongs to the second quadrant and $(x_n(t_0+\frac{2\tau}{1+4n}), y_n(t_0+\frac{2\tau}{1+4n}))$ belongs to the third quadrant. Hence $\tau_2 \neq \frac{3\tau}{1+4n}$ $(h_2 \neq 2)$ and $\tau_2 = \frac{\tau}{1+4n}$ $(h_2 = 1)$.

By (14) and $\tau_2 = \frac{\tau}{1+4n}$, we obtain

$$x_n(t) = y_n(t+\tau_2) = -x_n(t+2\tau_2) = -y_n(t+3\tau_2) = x_n(t+4\tau)$$
$$= y_n(t+5\tau_2) = \cdots = y_n(t+(1+4n)\tau_2) = y_n(t+\tau).$$

Then

$$x_n(t-\tau) = y_n(t). \tag{17}$$

By (4) and (17), we have

$$\frac{dx_n(t)}{dt} = -g(x_n(t), y_n(t)) = -g(x_n(t), x_n(t-\tau)), \tag{18}$$

$$\frac{dx_n(t-\tau)}{dt} = \frac{dy_n(t)}{dt} = g(y_n(t), x_n(t)) = g(x_n(t-\tau), x_n(t)). \tag{19}$$

By (18), (19) and (11), we have

$$\frac{d(x_n(t)+cx_n(t-\tau))}{dt} = -g(x_n(t), x_n(t-\tau)) + cg(x_n(t-\tau), x_n(t))$$
$$= -f(x_n(t), x_n(t-\tau)).$$

Hence $x_n(t)$ is a periodic solution of (1). By the proof of the lemma, $x_n(t)$ is nonconstant oscillatory and has period of $\frac{4\tau}{1+4n}$. Since $n = m, m+1, \ldots, m+k-1$ we obtain k nonconstant oscillatory periodic solutions. The proof of Theorem 1 is now complete.

REMARK 1. By Theorem 1, if $p < +\infty$, $q = +\infty$ or $p = +\infty$, $q < +\infty$, then (1) has an infinite number of periodic solutions.

In the case $c = 0$, $f(x, y) = F(y)$, we can choose $g(x, y) = F(y)$. Then, by Theorem 1, we have the following corollary:

COROLLARY 1. *Suppose that*

(a) *$F(y)$ is a continuous odd function, $yF(y) > 0$ $(y \neq 0)$ and $\int^{+\infty} F(y)\,dy = +\infty$;*

(b) *$\lim_{y\to 0} F(y)/y = p$, $\lim_{y\to+\infty} F(y)/y = q$ and*

$$p < \frac{(1+4n)\pi}{2\tau} < q \qquad or \qquad q < \frac{(1+4n)\pi}{2\tau} < p,$$

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where $n = m, \ m+1, \ldots, m+k-1$ *and* m *is some nonnegative integer,* k *is some positive integer. Then the equation*

$$x'(t) = -F(x(t-\tau)) \qquad (\tau > 0) \tag{20}$$

has at least k *nonconstant oscillatory periodic solutions. Specifically, if* $p < +\infty$, $q = +\infty$ *or* $p = +\infty$, $q < +\infty$, *then* (20) *has an infinite number of periodic solutions.*

REMARK 2. Corollary 1 generalises the result of Kaplan and Yorke [5].

3. Some examples

EXAMPLE 1. Consider

$$(x(t) + x(t-\tau))' = -a(x^s(t-\tau) - x^s(t)) \tag{21}$$

where $a > 0$, $\tau > 0$, $s > 1$ and s is a ratio of two positive odd numbers. Then $c = 1$, $f(x, y) = a(y^s - x^s)$. We choose $g(x, y) = ay^s$. It is easy to show that $g(x, y)$ satisfies the conditions of Theorem 1 and $p = 0$, $q = +\infty$. By Theorem 1 and Remark 1, (21) has an infinite number of periodic solutions.

EXAMPLE 2. Consider

$$(x(t) - x(t-\tau))' = -(x(t) + x(t-\tau))\exp(-x^2(t) - x^2(t-\tau)), \tag{22}$$

where $\tau > \frac{(1+4(k-1))\pi}{2}$ and k is some positive integer. Then $c = -1$ and $f(x, y) = (x+y)\exp(-x^2 - y^2)$. We choose $g(x, y) = y\exp(-x^2 - y^2)$. Hence, we have

$$\frac{g(y, x)}{g(x, y)} = \frac{x\exp(-x^2 - y^2)}{y\exp(-x^2 - y^2)} = x \cdot \frac{1}{y} \qquad (x \geq 0, \ y \geq b > 0)$$

and

$$\frac{xg(y, x) + yg(x, y)}{x^2 + y^2} = \exp(-x^2 - y^2).$$

It is easy to show that $g(x, y)$ satisfies the conditions of Theorem 1 and

$$q = 0 < \frac{(1+4n)\pi}{2\tau} < 1 = p, \qquad n = 0, 1, \ldots, k-1.$$

Then, by Theorem 1, (22) has at least k periodic solutions.

EXAMPLE 3. Consider

$$(x(t) + cx(t-\tau))' = -a(1 + x^2(t) + x^2(t-\tau))(x(t-\tau) - cx(t)) \tag{23}$$

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where $a>0$, $\tau>0$ and c is a constant. Then, $f(x,y)=a(1+x^2+y^2)(y-cx)$. We choose $g(x,y)=a(1+x^2+y^2)y$. Then we have

$$\frac{g(y,x)}{g(x,y)}=\frac{a(1+y^2+x^2)x}{a(1+x^2+y^2)y}=x\cdot\frac{1}{y}\qquad(x\geq 0,\ y\geq b>0)$$

and

$$\frac{xg(y,x)+yg(x,y)}{x^2+y^2}=a(1+x^2+y^2).$$

It is easy to show that $g(x,y)$ satisfies the conditions of Theorem 1 and $p=a$, $q=+\infty$. By Theorem 1 and Remark 1, (23) has an infinite number of periodic solutions. Indeed, it is easy to show that

$$x_n(t)=\left(\frac{(1+4n)\pi}{2a\tau}-1\right)^{1/2}\cdot\sin\frac{(1+4n)\pi t}{2\tau},\qquad n=m,m+1,\dots,$$

are the periodic solutions of (23), where $m=[\frac{2\tau a-\pi}{4\pi}]+1$. This is the same conclusion as we obtain by Theorem 1 and Remark 1.

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