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ON THE INTEGER RING OF THE COMPOSITUM OF ALGEBRAIC NUMBER FIELDS

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§1. Statement of the results

Let k be an algebraic number field of finite degree. For a finite extension L/k we denote by $\mathfrak{D}_{L/k}$ the different of L/k, and by $\mathfrak{O}_L$ the integer ring of L. Let K_1 and K_2 be finite extensions of k. It is known that we have $\mathfrak{O}_{K_1K_2} = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ if K_1 and K_2 are linearly disjoint over k and $\mathfrak{D}_{K_1K_2/k} = \mathfrak{D}_{K_1/k}\mathfrak{D}_{K_2/k}$ holds (see Shimura [2], 1.2).

In this paper we compute the conductor of $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ with respect to $\mathfrak{O}_{K_1K_2}$ and the module index of $\mathfrak{O}_{K_1K_2}$ and $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ in terms of relevant differents and "Elements". We note that the conductor of $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ with respect to $O_{K_1K_2}$ is the largest ideal of $\mathfrak{O}_{K_1K_2}$ which is contained in $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$. For a Dedekind domain R whose quotient field is L and R-lattices M, N of the same finite dimensional vector space over L, we denote by $[M: N]_R$ the module index of M and N. We note that the index $[M: N]$ is the absolute norm of $[M: N]_R$ if L is a number field and R is its integer ring. For general properties of module indices we refer to Frölich [1]. For a finite extension L/K of algebraic number fields of finite degree and an embedding σ of L over K, we denote by $\mathfrak{e}_\sigma$ the element with respect to σ. We recall that $\mathfrak{e}_\sigma$ is the ideal generated by $x - x^\sigma$, $x \in \mathfrak{O}_L$.

We state our results.

THEOREM. *Let k be an algebraic number field of finite degree, and K_1, K_2 its finite extensions. Then we have*

(1) *the conductor $\mathfrak{f}$ of $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ with respect to $\mathfrak{O}_{K_1K_2}$ is $\prod_{\sigma\neq 1} \mathfrak{e}_{\sigma|K_2}\mathfrak{D}^{-1}_{K_1K_2/K_1}$, where σ runs through all the non-trivial embeddings of K_1K_2 over K_1,*

(2) $[\mathfrak{O}_{K_1K_2}: \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]^2_{\mathfrak{O}_L} = \mathrm{N}_{K_1K_2/L}(\mathfrak{f})$ *holds, where $L = k, K_1, K_2$.*

COROLLARY. *Let notations be as in Theorem. Then we have $\mathfrak{O}_{K_1K_2} = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ if and only if $\mathfrak{D}_{K_1K_2/K_1} = \prod_{\sigma\neq 1} \mathfrak{e}_{\sigma|K_2}$ holds.*

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We note that $\prod_{\sigma\neq 1}\mathfrak{e}_{\sigma|K_2}=\mathfrak{D}_{K_2/k}$ holds if K_1 and K_2 are linearly disjoint over k.

We shall give another description of the conductor and some examples in § 3.

§ 2. Proofs

2.1. *Proof of Theorem* (1). Firstly we claim that there exists an element $z\in\mathfrak{O}_{K_2}$ for any prime ideal $\mathfrak{p}$ of K_2 such that

(i) $k(z)=K_2$,

(ii) the $\mathfrak{P}$-component of the conductor of $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ is that of $\mathfrak{O}_{K_1}[z]$ for all prime ideals $\mathfrak{P}$ of K_1K_2 above $\mathfrak{p}$.

We take $z\in\mathfrak{O}_{K_2}$ which satisfies

(iii) $\operatorname{ord}_{\mathfrak{p}}f_z'(z)=\operatorname{ord}_{\mathfrak{p}}\mathfrak{D}_{K_2/k}$, and $\deg f_z=[K_2:k]$.

Here f_z is the minimal polynomial of z over k. We show z satisfies (ii). We recall that the conductor of $\mathfrak{O}_k[z]$ with respect to $\mathfrak{O}_{K_2}$ is $f_z'(z)\mathfrak{D}_{K_2/k}^{-1}$, where f_z' is the derivative of f_z. We have

$$\begin{aligned}\mathfrak{O}_{K_1}[z]&=\mathfrak{O}_{K_1}\mathfrak{O}_k[z]\supset\mathfrak{O}_{K_1}f_z'(z)\mathfrak{D}_{K_2/k}^{-1}\\&=\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}f_z'(z)\mathfrak{D}_{K_2/k}^{-1}\supset\mathfrak{f}f_z'(z)\mathfrak{D}_{K_2/k}^{-1}.\end{aligned}$$

Therefore the conductor of $\mathfrak{O}_{K_1}[z]$ contains $\mathfrak{f}f_z'(z)\mathfrak{D}_{K_2/k}^{-1}$. Since $\operatorname{ord}_{\mathfrak{p}}f_z'(z)\mathfrak{D}_{K_2/k}^{-1}=0$, we get the claim.

By the claim $\mathfrak{f}$ is the greatest common divisor of the conductors of $\mathfrak{O}_{K_1}[z]$, where z satisfies (i) and is contained in $\mathfrak{O}_{K_2}$. The conductor of $\mathfrak{O}_{K_1}[z]$ with respect to $\mathfrak{O}_{K_1K_2}$ is $g_z'(z)\mathfrak{D}_{K_1K_2/K_1}^{-1}$, where z is an element of $\mathfrak{O}_{K_2}$ with (i) and g_z is the minimal polynomial of z over K_1. We show that

$$\prod_{\sigma\neq 1}\mathfrak{e}_{\sigma|K_2}=(g_z'(z):z\in\mathfrak{O}_{K_2}\text{ with (i)}),$$

where σ runs through all the non-trivial embeddings of K_1K_2 over K_1 into a finite Galois extension L over k containing K_1K_2. Let $\mathfrak{p}$ be a prime ideal of K_2. Let z be an element of $\mathfrak{O}_{K_2}$ satisfying (iii). Then we have

$$\operatorname{ord}_{\mathfrak{P}}(z-z^\sigma)\geq\operatorname{ord}_{\mathfrak{P}}\mathfrak{e}_{\sigma|K_2}$$

for all the non-trivial embeddings σ of K_1K_2 over K_1 into L and all prime ideals $\mathfrak{P}$ of L above $\mathfrak{p}$, and

$$\sum_{\sigma\neq 1}\operatorname{ord}_{\mathfrak{P}}(z-z^\sigma)=\operatorname{ord}_{\mathfrak{P}}\mathfrak{D}_{K_2/k}=\sum_{\sigma\neq 1}\operatorname{ord}_{\mathfrak{P}}\mathfrak{e}_\sigma$$

for all prime ideals $\mathfrak{P}$ of L above $\mathfrak{p}$. Here the sums are taken over all the non-trivial embeddings σ of K_2 over k into L. Therefore we have

$$\mathrm{ord}_{\mathfrak{P}}(z - z^{\sigma}) = \mathrm{ord}_{\mathfrak{P}}\, \mathfrak{e}_{\sigma|K_2}$$

for all non-trivial embeddings σ of K_1K_2 over K_1 into L and all prime ideals $\mathfrak{P}$ of L above $\mathfrak{p}$. Using the decomposition $g_z'(z) = \prod_{\sigma \neq 1}(z - z^{\sigma})$, we have

$$\mathrm{ord}_{\mathfrak{P}}\, g_z'(z) = \mathrm{ord}_{\mathfrak{P}} \prod_{\sigma \neq 1} \mathfrak{e}_{\sigma|K_2}$$

for all prime ideals $\mathfrak{P}$ of L above $\mathfrak{p}$. Thus we proved the assertion. Hence we have (1).

2.2. *Proof of Theorem* (2). It suffices to prove the case $L = K_1$. For a $\mathfrak{O}_{K_1}$-lattice M of K_1K_2 we denote by M^* the dual module of M with respect to $\mathrm{Tr}_{K_1K_2/K_1}$ We have

$$[\mathfrak{O}_{K_1K_2} : \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]^2_{\mathfrak{O}_{K_1}} = [(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})^* : \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]_{\mathfrak{O}_{K_1}} [\mathfrak{O}^*_{K_1K_2} : \mathfrak{O}_{K_1K_2}]^{-1}_{\mathfrak{O}_{K_1}}$$

(see Fröhlich [1], Proposition 4, § 3). Since $\mathfrak{O}^*_{K_1K_2} = \mathfrak{D}^{-1}_{K_1K_2/K_1}$,

$$\begin{aligned}[\mathfrak{O}^*_{K_1K_2} : \mathfrak{O}_{K_1K_2}]_{\mathfrak{O}_{K_1}} &= [\mathfrak{D}^{-1}_{K_1K_2/K_1} : \mathfrak{O}_{K_1K_2}]_{\mathfrak{O}_{K_1}} \\ &= \mathrm{N}_{K_1K_2/K_1}(\mathfrak{D}_{K_1K_2/K_1})\end{aligned}$$

holds.

From now on we compute the module index $[(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})^* : \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]_{\mathfrak{O}_{K_1}}$.

LEMMA 1. *Let S be a finite set of prime ideals of K_1K_2. Let $\mathfrak{p}$ be a prime ideal of K_2. We denote by n a natural number. Then there exists an element γ of $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ such that $\gamma \equiv 1 \pmod{\mathfrak{P}^n}$ holds for all $\mathfrak{P}$ of S above $\mathfrak{p}$ and $\gamma \equiv 0 \pmod{\mathfrak{P}^n}$ for all $\mathfrak{P}$ of S not above $\mathfrak{p}$.*

Proof. We put

$$S_1 = \{\mathfrak{O}_{K_1}\mathfrak{O}_{K_2} \cap \mathfrak{P} : \mathfrak{P} \in S, \mathfrak{P} \mid \mathfrak{p}\},$$

and

$$S_2 = \{\mathfrak{O}_{K_1}\mathfrak{O}_{K_2} \cap \mathfrak{P} : \mathfrak{P} \in S, \mathfrak{P} \nmid \mathfrak{p}\}.$$

Elements of $S_1 \cup S_2$ are maximal ideals of $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$. $S_1 \cap S_2 = \varnothing$ holds. So by Chinese remainder theorem there exists an element γ of $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ such that $\gamma \equiv 1 \pmod{\mathfrak{M}^n}$ for all $\mathfrak{M} \in S_1$ and $\gamma \equiv 0 \pmod{\mathfrak{M}^n}$ for all $\mathfrak{M} \in S_2$ hold, which proves Lemma 1.

For a $\mathfrak{O}_k$-module M and a prime ideal p of k, we denote by M_p $S_p^{-1}M$, where S_p is $\mathfrak{O}_k - p$.

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LEMMA 2. *Let p be a prime ideal of k. Then there exist $\alpha, \beta \in K_1K_2$ such that*

$$\text{(1)} \qquad \operatorname{ord}_{\mathfrak{P}} \alpha = \operatorname{ord}_{\mathfrak{P}} \beta = \operatorname{ord}_{\mathfrak{P}} \mathfrak{O}_{K_1K_2}(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})^*$$

hold for all prime ideals $\mathfrak{P}$ of K_1K_2 above p,

$$\text{(2)} \qquad \alpha(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})_p \subset (\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})_p^* \subset \beta(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})_p .$$

Proof. Firstly we prove the existence of α satisfying the condition. Let $\mathfrak{p}$ be a prime ideal of K_2 above p. Let z be an element of $\mathfrak{O}_{K_2}$ with (iii) in 2.1. Since the dual module of $\mathfrak{O}_k[z]$ with respect to $\operatorname{Tr}_{K_2/k}$ is $f_z'(z)^{-1}\mathfrak{O}_k[z]$, we have

$$f_z'(z)^{-1}\mathfrak{O}_k[z] \supset \mathfrak{D}_{K_2/k}^{-1} .$$

We take $d \in \mathfrak{D}_{K_2/k}$ which satisfies $\operatorname{ord}_{\mathfrak{q}} d = \operatorname{ord}_{\mathfrak{q}} \mathfrak{D}_{K_2/k}$ for all prime ideals $\mathfrak{q}$ of K_2 above p. Then we have

$$df_z'(z)^{-1}\mathfrak{O}_k[z]_p \supset \mathfrak{O}_{K_2,p} .$$

So we have

$$df_z'(z)^{-1}\mathfrak{O}_{K_1}[z]_p \supset (\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})_p .$$

By taking dual, we get

$$g_z'(z)^{-1}d^{-1}f_z'(z)\mathfrak{O}_{K_1}[z]_p \subset (\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})_p^* .$$

We put $\alpha_{\mathfrak{p}} = g_z'(z)^{-1}d^{-1}f_z'(z)$. We take $\gamma_{\mathfrak{p}}$ which satisfies the conditions in Lemma 1, where S is the set of all prime ideals of K_1K_2 above p and n is sufficiently large. We put

$$\alpha = \sum_{\mathfrak{p}|p} \alpha_{\mathfrak{p}}\gamma_{\mathfrak{p}} .$$

α satisfies the condition of Lemma 2. In fact, for a prime ideal $\mathfrak{P}$ of K_1K_2 and a prime ideal $\mathfrak{p}$ of K_2 with $\mathfrak{P}|\mathfrak{p}|p$, we have

$$\operatorname{ord}_{\mathfrak{P}} \alpha = \operatorname{ord}_{\mathfrak{P}} \alpha_{\mathfrak{p}}\gamma_{\mathfrak{p}} = \operatorname{ord}_{\mathfrak{P}} \alpha_{\mathfrak{p}} = \operatorname{ord}_{\mathfrak{P}} \mathfrak{O}_{K_1K_2}(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})^* .$$

And clearly α satisfies (2), Lemma 2.

Secondly we prove the existence of β. For a prime ideal $\mathfrak{p}$ of K_2 above p we take an element z of $\mathfrak{O}_{K_2}$ with (iii) in 2.1. We put $\beta_{\mathfrak{p}} = g_z'(z)$. By taking dual of $\mathfrak{O}_{K_1}[z] \subset \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$, we have

$$\beta_{\mathfrak{p}}(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})^* \subset \mathfrak{O}_{K_1}[z] .$$

Therefore we have

$$(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*\Big(\sum_{\mathfrak{p}|p} \beta_{\mathfrak{p}}\mathfrak{D}_{K_1}\mathfrak{D}_{K_2}\Big) \subset \mathfrak{D}_{K_1}\mathfrak{D}_{K_2} .$$

For a prime ideal $\mathfrak{p}$ of K_2 we take $\gamma_{\mathfrak{p}}$ satisfying the condition in Lemma 1, where S is the set of all prime ideals of K_1K_2 above p and n is sufficiently large. We put

$$\beta = 1\Big/\sum_{\mathfrak{p}|p} \beta_{\mathfrak{p}}\gamma_{\mathfrak{p}} .$$

β satisfies the conditions of Lemma 2. In fact, for a prime ideal $\mathfrak{P}$ of K_1K_2 and a prime ideal $\mathfrak{p}$ of K_2 with $\mathfrak{P}|\mathfrak{p}|p$, we have

$$\mathrm{ord}_{\mathfrak{P}}\, \beta = -\mathrm{ord}_{\mathfrak{P}}\, \beta_{\mathfrak{p}}\gamma_{\mathfrak{p}} = -\mathrm{ord}_{\mathfrak{P}}\, \beta_{\mathfrak{p}} = \mathrm{ord}_{\mathfrak{P}}\, \mathfrak{D}_{K_1K_2}(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* .$$

And clearly β satisfies (2), Lemma 2. Thus we proved Lemma 2.

Let p be a prime ideal of k and α, β elements of K_1K_2 satisfying (1), (2) in Lemma 2. Since

$$\begin{aligned}[]&[\mathfrak{D}_{K_1K_2}(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*_p : \alpha(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{o}_{k,p}} \\ &\quad = [\mathfrak{D}_{K_1K_2}(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*_p : \beta(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{o}_{k,p}} \\ &\quad = [\mathfrak{D}_{K_1K_2,p} : (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{o}_{k,p}} ,\end{aligned}$$

we have

$$\alpha(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p = (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*_p = \beta(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p .$$

So we have

$$\begin{aligned}&([(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* : \mathfrak{D}_{K_1}\mathfrak{D}_{K_2}]_{\mathfrak{o}_{K_1}})_p \\ &\quad = [(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*_p : (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{o}_{K_1,p}} \\ &\quad = [\alpha(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p : (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{o}_{K_1,p}} \\ &\quad = \mathrm{N}_{K_1K_2/K_1}(\alpha^{-1})\mathfrak{D}_{K_1,p} \\ &\quad = \mathrm{N}_{K_1K_2/K_1}\big(\textstyle\prod_{\sigma\neq 1} \mathfrak{e}_{\sigma|K_2}\big)\mathfrak{D}_{K_1,p} .\end{aligned}$$

Hence we get

$$([\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* : \mathfrak{D}_{K_1}\mathfrak{D}_{K_2}]_{\mathfrak{o}_{K_1}} = \mathrm{N}_{K_1K_2/K_1}\Big(\prod_{\sigma\neq 1} \mathfrak{e}_{\sigma|K_2}\Big) .$$

§ 3. Examples

3.1. Let k be a number field and K its finite Galois extension with

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Galois group G. Let K_1, K_2 be intermediate fields of K/k. We denote by H_1, H_2 the subgroups of G corresponding to K_1, K_2 respectively. We define $\sum_{K_1,K_2}$, a subset of G, by $H_1H_2 - H_1 - H_2$, where H_1H_2 is $\{h_1h_2 : h_1 \in H_1, h_2 \in H_2\}$. *Then the conductor* $\mathfrak{f}_{K_1,K_2}$ *of* $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ *with respect to* $\mathfrak{O}_{K_1K_2}$ *is* $\prod_{\sigma \in \Sigma_{K_1,K_2}} \mathfrak{e}_\sigma$, *where* $\mathfrak{e}_\sigma$ *is* $(x - x^\sigma : x \in \mathfrak{O}_K)$. This can be proved by fundamental properties of elements. From this fact we know that $\mathfrak{O}_{K_1K_2} = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ holds if and only if for any prime ideal $\mathfrak{P}$ of K and any $\sigma \in \sum_{K_1,K_2}$, σ is not contained in the inertia group of $\mathfrak{P}$.

3.2. We give some examples.

1. $G = \mathrm{Gal}(K/k) = \langle \sigma, \tau : \sigma^2 = \tau^2 = (\sigma\tau)^2 = 1 \rangle$.

Let K_1, K_2, K_3 be the fixed fields of $\langle\sigma\rangle, \langle\tau\rangle, \langle\sigma\tau\rangle$ respectively. Then we have

$$\sum\nolimits_{K_1,K_2} = \{\sigma\tau\}\,,$$
$$\mathfrak{f}_{K_1,K_2} = \mathfrak{e}_{\sigma\tau} = \mathfrak{D}_{K/K_3}\,.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ holds if and only if K/K_3 is unramified.

2. $G = \mathrm{Gal}(K/k) = \langle \sigma, \tau : \sigma^3 = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$.

Let K_i be the fixed field of $\langle \tau\sigma^{i-1}\rangle$ $(i = 1, 2, 3)$ and M the fixed field of $\langle\sigma\rangle$.

3.2.1 $\sum_{K_1,K_2} = \{\sigma\}$,

$$\mathfrak{f}_{K_1,K_2} = \mathfrak{e}_\sigma = \mathfrak{D}_{K/M}^{1/2}\,.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ holds if and only if K/M is unramified.

3.2.2 $\sum_{K_1,M} = \{\tau\sigma, \tau\sigma^2\}$,

$$\mathfrak{f}_{K_1,M} = \mathfrak{e}_{\tau\sigma}\mathfrak{e}_{\tau\sigma^2} = \mathfrak{D}_{K/K_2}\mathfrak{D}_{K/K_3}\,.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1}\mathfrak{O}_M$ holds if and only if M/k is unramified.

3. $G = \mathrm{Gal}(K/k) = A_4 \hookrightarrow \mathrm{Aut}(\{a, b, c, d\})$.

We put $x = (a\,b)(c\,d)$, $y = (a\,c)(b\,d)$, $z = (a\,d)(b\,c)$, $t = (a\,b\,c)$ and $H = \{1, x, y, z\}$. Let K_1, K_2, K_3, K_4 be the fixed fields of $\langle t\rangle, \langle tx\rangle, \langle ty\rangle, \langle tz\rangle$ respectively. Let L_1, L_2, L_3 be the fixed fields of $\langle x\rangle, \langle y\rangle, \langle z\rangle$ respectively. Let M be the fixed field of H.

3.3.1 $\sum_{K_1,M} = \{t^2x, t^2y, t^2z, tx, ty, tz\}$.

$$\mathfrak{f}_{K_1,M} = \mathfrak{D}_{K/K_2}\mathfrak{D}_{K/K_3}\mathfrak{D}_{K/K_4}\,.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1}\mathfrak{O}_M$ holds if and only if M/k is unramified.

3.3.2 $\sum_{K_1,K_2} = \{x, tz, t^2x, z\}$,

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$$\mathfrak{f}_{K_1,K_2} = \mathfrak{D}_{K/L_1}\mathfrak{D}_{K/L_3}\mathfrak{D}^{1/2}_{K/K_3}\mathfrak{D}^{1/2}_{K/K_4}\,.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ holds if and only if K/k is unramified.

3.3.3 $\sum_{K_1,L_2} = \{t^2y, ty\}$,

$$\mathfrak{f}_{K_1,L_2} = \mathfrak{D}^{1/2}_{K/K_3}\mathfrak{D}^{1/2}_{K/K_4}\,.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1}\mathfrak{O}_{L_2}$ holds if and only if M/k is unramified.

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