Canad. Math. Bull. Vol. **46** (4), 2003 pp. 617–631

On Harmonic Theory in Flows

Hong Kyung Pak

Abstract. Recently [8], a harmonic theory was developed for a compact contact manifold from the viewpoint of the transversal geometry of contact flow. A contact flow is a typical example of geodesible flow. As a natural generalization of the contact flow, the present paper develops a harmonic theory for various flows on compact manifolds. We introduce the notions of H-harmonic and H^*-harmonic spaces associated to a Hörmander flow. We also introduce the notions of basic harmonic spaces associated to a weak basic flow. One of our main results is to show that in the special case of isometric flow these harmonic spaces are isomorphic to the cohomology spaces of certain complexes. Moreover, we find an obstruction for a geodesible flow to be isometric.

1 Introduction

Let $\mathcal{F}$ be a geodesible flow on a manifold M of dimension $m = 1 + q$ generated by a nonsingular vector field T. Then there exists a Riemannian metric g on M with respect to which $\mathcal{F}$ is geodesic, that is, the dual 1-form ω to T satisfies

$$\iota_T\omega = 1, \quad \mathcal{L}_T\omega = 0, \tag{1.1}$$

where ι_T (resp. $\mathcal{L}_T$) denotes the interior product (resp. the Lie derivative) with respect to T. The contact form ω on a contact manifold (M, ω) satisfies

$$\omega \wedge (d\omega)^{q/2} \neq 0. \tag{1.2}$$

The contact flow $\mathcal{F}_\omega$ generated by ω is a typical example of geodesible flows. Another important example of geodesible flows is an isometric flow which is defined by a nonsingular Killing vector field.

There have been extensively studied harmonic theory on a compact Sasakian manifold by many mathematicians since Sasaki introduced contact metric structures [13]. For instance, Tachibana [14] and Ogawa [7] considered special harmonic spaces, so-called C-harmonic and C^*-harmonic spaces, which seem to be closely associated with the contact structure. In the compact Sasakian case, they showed that such C-harmonic and C^*-harmonic spaces have nice relationships with the harmonic spaces on the manifold and moreover, satisfy several nice properties like decomposition theorem.

On the other hand, Rumin [10], [11] constructed a new De Rham complex on a compact contact manifold of dimension $2n + 1$ whose cohomology is isomorphic to

Received by the editors February 22, 2002; revised August 2, 2002, January 10, 2003, March 20, 2003.
This research is supported by Korean Research Foundation Korea,KRF-2000-042-D00007.
AMS subject classification: Primary: 53C20; secondary: 57R30.
Keywords: contact structure, geodesible flow, isometric flow, basic cohomology.

the De Rham cohomology on the manifold. Furthermore, he obtained a certain vanishing theorem of the k-th De Rham cohomology on a compact pseudo-Hermitian manifold where $k < n$.

Recently [8], a harmonic theory on various harmonic forms, such as basic harmonic, C-harmonic and C^*-harmonic forms and so on, was studied in a situation of a compact contact manifold. The results obtained in [8] extend those established in the K-contact or Sasakian case.

In the present paper, we develop a harmonic theory for various flows on a compact manifold. As a natural generalization of the case of contact flows, we introduce the notions of H-harmonic and H^*-harmonic spaces associated to a Hörmander flow. We also introduce the notions of basic harmonic spaces associated to a weak basic flow. One of our main results is to show that in the special case of isometric flow these harmonic spaces are isomorphic to the cohomology spaces of certain complexes. Moreover, we find an obstruction for a geodesible flow to be isometric.

The author would like to thank the referee for his useful suggestions.

2 Fundamental Materials for a Flow

Let $\mathcal{F}$ be a flow on a Riemannian manifold (M, g) of dimension $m = 1 + q$ generated by a unit vector field T. Let ω be the dual 1-form to T, which will be called a flow form. A pair $(\mathcal{F}, \omega)$ is denoted by a flow $\mathcal{F}$ whose flow form is ω. There is an orthogonal decomposition of the tangent bundle TM

$$TM = \mathcal{D} \oplus E,$$

where $\mathcal{D} := \ker \omega$ and E is the tangent bundle to $\mathcal{F}$. This gives rise to the associated bigrading of the graded algebra Ω^* of all differential forms on M

$$\Omega^u = \bigoplus_{k+\ell=u} \Omega^{k,\ell}, \tag{2.1}$$

where $\Omega^{k,\ell} := \Gamma(\bigwedge^k \mathcal{D}^* \otimes \bigwedge^\ell E^*)$. Correspondingly, the exterior differential operator d is also decomposed into

$$d = d^{0,1} + d^{1,0} + d^{2,-1} =: d_E + d_B + d^{2,-1},$$

where each $d^{i,j}$ is the bihomogeneous differential of bidegree (i, j).

And let $\delta := \delta^{0,1} + \delta^{1,0} + \delta^{2,-1} =: \delta_E + \delta_B + \delta^{2,-1}$ be the decomposition of the codifferential operator δ corresponding to the decomposition of d with respect to g. The following relations hold on $\phi \in \Omega^{s,r}$;

$$\delta_E \phi = (-1)^{m(s+r+1)+1} * d_E * \phi,$$

$$\delta_B \phi = (-1)^{m(s+r+1)+1} * d_B * \phi,$$

$$\delta^{2,-1} \phi = (-1)^{m(s+r+1)+1} * d^{2,-1} * \phi,$$

where $*$ is the Hodge star operator defined by the Riemannian metric induced from g. Note that $*\phi \in \Omega^{q-k,1}$ for $\phi \in \Omega^{k,0}$.

Let $e(\omega)$ be the formal adjoint of ι_T with respect to the metric g defined by

$$e(\omega)\phi := \omega \wedge \phi.$$

Since $\dim \mathcal{F} = 1$, we have:

Lemma 2.1 *Let $(\mathcal{F}, \omega)$ be a flow on a Riemannian manifold (M, g). Then for any k the operator $e(\omega)\colon \Omega^{k,0} \longrightarrow \Omega^{k,1}$ is an isomorphism with inverse $\iota_T\colon \Omega^{k,1} \longrightarrow \Omega^{k,0}$.*

Proof Let $\phi \in \Omega^{k,0}$ satisfy $e(\omega)\phi = 0$. Then $\iota_T\phi = 0$, so that

$$\phi = \iota_T e(\omega)\phi + e(\omega)\iota_T\phi = 0.$$

Thus $e(\omega)$ is injective.

Now for $\psi \in \Omega^{k,1}$ take $\iota_T\psi \in \Omega^{k,0}$. Since $e(\omega)\iota_T\psi = \psi$, we conclude that $e(\omega)$ is surjective. ■

The metric g is decomposed into $g = g_{\mathcal{D}} + g_E$. We say that a flow $(\mathcal{F}, \omega)$ is geodesic with respect to g if $\mathcal{L}_X g_E = 0$ for all $X \in \Gamma(\mathcal{D})$. Or equivalently, ω satisfies (1.1). The following observation is immediate.

Lemma 2.2 *Let $(M, g, \mathcal{F}, \omega)$ be as in Lemma 2.1. Then $\mathcal{F}$ is geodesic if and only if $d\omega = d^{2,-1}\omega \in \Omega^{2,0}$.*

It is useful to introduce operators L, Λ on Ω^k

$$L\phi := d\omega \wedge \phi, \quad \Lambda\phi := (-1)^{m(k+1)+1} * L * \phi, \tag{2.2}$$

and the spaces J and J_*

$$J^k := \ker \Lambda \cap \Omega^{k,0}, \quad J_*^{k+1} := \ker L \cap \Omega^{k,1}. \tag{2.3}$$

These spaces were discussed in [11] for the contact flow case. Note that L is the formal adjoint of Λ with respect to g and that $*\phi \in J_*^{m-k}$ for $\phi \in J^k$.

Lemma 2.3 *Let $(M, g, \mathcal{F}, \omega)$ be as in Lemma 2.1. Then L (resp. Λ) commutes with operators $e(\omega)$ and d (resp. ι_T and δ). Moreover,*

$$L = e(\omega)d + de(\omega), \quad \Lambda = \delta\iota_T + \iota_T\delta.$$

In particular, if $\mathcal{F}$ is geodesic then L (resp. Λ) commutes with ι_T (resp. $e(\omega)$).

https://doi.org/10.4153/CMB-2003-057-9 Published online by Cambridge University Press

Proof It can be easily seen that L commutes with $e(\omega)$, d and $L = e(\omega)d + de(\omega)$. In particular, when $\mathcal{F}$ is geodesic we have from (1.1)

$$L\iota_T\phi = d\omega \wedge \iota_T\phi = \iota_T L\phi$$

for $\phi \in \Omega^k$. ∎

Furthermore, we find useful operator identities. These are only due to the fact that $(\mathcal{F}, \omega)$ is a flow.

Lemma 2.4 *Let $(M, g, \mathcal{F}, \omega)$ be as in Lemma 2.1. Then we have*

$$d^{2,-1} = \iota_T L \quad \text{on } \Omega^{k,1},$$
$$\delta^{2,-1} = e(\omega)\Lambda \quad \text{on } \Omega^{k,0}.$$

3 Harmonic Spaces for a Hörmander Flow

Let $(\mathcal{F}, \omega)$ be a flow on a Riemannian manifold (M, g) of dimension $m = 1 + q$. It should be noted that the condition $d^{2,-1}\omega = 0$ on M if and only if the distribution $\mathcal{D}$ is integrable. On the contrary, $d^{2,-1}\omega \neq 0$ on M means that $\mathcal{D}$ satisfies a Hörmander condition, that is, $\Gamma(\mathcal{D})$ generates $\Gamma(TM)$ as a Lie algebra. In this sense:

Definition A flow $(\mathcal{F}, \omega)$ is a Hörmander flow on a manifold M if the distribution $\mathcal{D} := \ker \omega$ satisfies a Hörmander condition.

A contact flow is an example of a Hörmander flow. In what follows we consider a Hörmander flow $(\mathcal{F}, \omega)$. Then $d\omega \neq 0$ on M. Such a 2-form $d\omega$ is called the fundamental form associated to a Hörmander flow $(\mathcal{F}, \omega)$.

From now on M is supposed to be compact. Let $\langle \cdot\,, \cdot \rangle$ be the global inner product on M

$$\langle \phi, \psi \rangle := \int_M \phi \wedge *\psi.$$

Introduce on Ω^* Laplacians Δ and Δ_B defined as

$$\Delta := d\delta + \delta d, \quad \Delta_B := d_B\delta_B + \delta_B d_B.$$

Δ_B is called the basic Laplacian. A form $\phi \in \Omega^*$ is said to be harmonic if $\Delta\phi = 0$. Let $\mathcal{H}^*$ be the space of all harmonic forms on (M, g). Observe that bigrading (2.1) gives rise to

$$\Delta = \Delta^{0,0} + \Delta^{-1,1} + \Delta^{1,-1}, \tag{3.1}$$

where $\Delta^{0,0} = d_E\delta_E + \delta_E d_E + d_B\delta_B + \delta_B d_B + d^{2,-1}\delta^{2,-1} + \delta^{2,-1}d^{2,-1}$. It is obvious that on $\Omega^{k,\ell}$

$$\ker \Delta^{0,0} = \ker \Delta. \tag{3.2}$$

https://doi.org/10.4153/CMB-2003-057-9 Published online by Cambridge University Press

Set $\mathcal{H}^{k,\ell} := \ker \Delta^{0,0}$ on $\Omega^{k,\ell}$. In view of (3.2) we have an orthogonal decomposition

$$\mathcal{H}^k = \mathcal{H}^{k,0} \oplus \mathcal{H}^{k-1,1}.$$

Tachibana [14] established notions of C-harmonic and C^*-harmonic forms on a compact Sasakian manifold which seem to be closely associated with the contact structure. Since a contact flow is a geodesible Hörmander flow, it is natural to extend these notions to the case of a Hörmander flow by considering the operators given in (2.2).

Definition Let $(\mathcal{F},\omega)$ be a Hörmander flow on a compact Riemannian manifold (M,g). A k-form $\phi \in \Omega^k$ is a H-harmonic (resp. H^*-harmonic) form if it satisfies

$$\iota_T\phi = 0,\ d\phi = 0,\ \delta\phi = e(\omega)\Lambda\phi \quad (\text{resp. } e(\omega)\phi = 0,\ d\phi = \iota_T L\phi,\ \delta\phi = 0).$$

Let $\mathcal{H}^k_H$ (resp. $\mathcal{H}^k_{H^*}$) be the space of all H-harmonic (resp. H^*-harmonic) k-forms for $(\mathcal{F},\omega)$.

In case of the contact flow, the spaces $\mathcal{H}_H$ and $\mathcal{H}_{H^*}$ coincide with $\mathcal{CH}$ and $\mathcal{C}^*\mathcal{H}$ respectively [8]. Then we obtain:

Theorem 3.1 *Let $(\mathcal{F},\omega)$ be a Hörmander flow on a compact Riemannian manifold (M,g) of dimension $m = 1+q$. Then for any k*

$$\mathcal{H}^{k,0} = \mathcal{H}^k_H \cap J^k, \quad \mathcal{H}^{k,1} = \mathcal{H}^{k+1}_{H^*} \cap J^{k+1}_*.$$

Proof Let $\phi \in \mathcal{H}^{k,0}$. By definition, ϕ satisfies

$$d_E\phi = d_B\phi = \delta_B\phi = \delta^{2,-1}\phi = 0.$$

From Lemma 2.4, this means that $\phi \in \mathcal{H}^k_H$ and $\delta^{2,-1}\phi = 0$. On the other hand, observe that the operator Λ restricted to $\Omega^{k,0}$ sends to $\Omega^{k-2,0}$. This, combined with Lemma 2.1, says that $\ker \delta^{2,-1} = \ker \Lambda$ on $\Omega^{k,0}$. Therefore, we deduce $\mathcal{H}^{k,0} = \mathcal{H}^k_H \cap J^k$.

Similarly, for $\psi \in \mathcal{H}^{k,1}$ we find

$$d_B\psi = d^{2,-1}\psi = \delta_E\psi = \delta_B\psi = 0.$$

This means that $\psi \in \mathcal{H}^{k+1}_{H^*}$ and $d^{2,-1}\psi = 0$. In this case the operator L restricted to $\Omega^{k,1}$ sends to $\Omega^{k+2,1}$, so that $\mathcal{H}^{k,1} = \mathcal{H}^{k+1}_{H^*} \cap J^{k+1}_*$. ∎

From the definition the following duality holds:

Corollary 3.2 *Under the same situation as in Theorem 3.1, we have for any k*

$$\mathcal{H}^k_H = \mathcal{H}^{m-k}_{H^*}.$$

Remarks (a) We may define looser notions than the above definition (*cf.* [7]). A k-form $\phi \in \Omega^k$ is called a $\tilde{H}$-harmonic (resp. $\tilde{H}^*$-harmonic) form for a Hörmander flow $(\mathcal{F}, \omega)$ if

$$d\phi = 0,\ \delta\phi = e(\omega)\Lambda\phi \quad (\text{resp. } d\phi = \iota_T L\phi,\ \delta\phi = 0).$$

Let $\mathcal{H}^k_{\tilde{H}}$ (resp. $\mathcal{H}^k_{\tilde{H}^*}$) be the space of all $\tilde{H}$-harmonic (resp. $\tilde{H}^*$-harmonic) k-forms. Considering bigrading gives rise to an orthogonal decomposition

$$\mathcal{H}^k_{\tilde{H}} = \mathcal{H}^{k,0}_{\tilde{H}} \oplus \mathcal{H}^{k-1,1}_{\tilde{H}},$$

where $\mathcal{H}^{k,0}_{\tilde{H}} := \mathcal{H}^k_{\tilde{H}} \cap \Omega^{k,0}$ and $\mathcal{H}^{k,1}_{\tilde{H}} := \mathcal{H}^{k+1}_{\tilde{H}} \cap \Omega^{k,1}$. It is immediate from the definition that $\mathcal{H}^{k-1,1}_{\tilde{H}} = \mathcal{H}^{k-1,1}$, so that

$$\mathcal{H}^k_{\tilde{H}} = \mathcal{H}^k_H \oplus \mathcal{H}^{k-1,1}. \tag{3.3}$$

Similarly if we set $\mathcal{H}^{k,0}_{\tilde{H}^*} := \mathcal{H}^k_{\tilde{H}^*} \cap \Omega^{k,0}$ and $\mathcal{H}^{k,1}_{\tilde{H}^*} := \mathcal{H}^{k+1}_{\tilde{H}^*} \cap \Omega^{k,1}$, then we have a corresponding orthogonal decomposition

$$\mathcal{H}^k_{\tilde{H}^*} = \mathcal{H}^{k,0}_{\tilde{H}^*} \oplus \mathcal{H}^{k-1,1}_{\tilde{H}^*} = \mathcal{H}^{k,0} \oplus \mathcal{H}^k_{H^*}. \tag{3.4}$$

From (3.3) and (3.4) observe that for any k

$$\mathcal{H}^k_{\tilde{H}} \cap \mathcal{H}^k_{\tilde{H}^*} = \mathcal{H}^k, \quad \mathcal{H}^k_{\tilde{H}} \cup \mathcal{H}^k_{\tilde{H}^*} = \mathcal{H}^k_H \oplus \mathcal{H}^k_{H^*}. \tag{3.5}$$

(b) The Hörmander condition imposed in Theorem 3.1 ensures $L \neq 0$ on M. It may be helpful to consider another extreme case where the distribution $\mathcal{D} = \ker\omega$ for $(\mathcal{F}, \omega)$ is integrable. In this case, we easily find that for any k

$$J^k = \Omega^{k,0}, \quad J^k_* = \Omega^{k-1,1}, \tag{3.6}$$

so that

$$\mathcal{H}^k_H = \mathcal{H}^{k,0}, \quad \mathcal{H}^k_{H^*} = \mathcal{H}^{k-1,1}$$

and

$$\mathcal{H}^k_{\tilde{H}} = \mathcal{H}^k = \mathcal{H}^k_{\tilde{H}^*}.$$

It follows from this observation that if we drop the Hörmander condition in Theorem 3.1 then $\mathcal{H}^{k,0}$ (and $\mathcal{H}^{k,1}$) varies according to L (and Λ) vanishes or not.

On the other hand, it should be noted that (3.6) does not hold in general. For example, if $\mathcal{F}_\omega$ is a contact flow of codimension $2n$ then for any $k = 1, \dots, n$

$$J^{2k} \neq \Omega^{2k,0}.$$

Indeed, we have the following formula [8]

$$(\Lambda L^p - L^p\Lambda)\phi = 2p[(2n + 2 - 2p - 2l)L^{p-1}\phi + 2e(\omega)\iota_T L^{p-1}\phi]$$

https://doi.org/10.4153/CMB-2003-057-9 Published online by Cambridge University Press

for any l-form $\phi \in \Omega^l$, where p is any non-negatve integer and $L^{-1} \equiv 0$. Then we see that

$$\Lambda(d\omega)^k = 2k(2n+2-2k)(d\omega)^{k-1}.$$

Therefore, $\Lambda(d\omega)^k \neq 0$ on M because $k \leq n$.

Since $J^k = \Omega^{k,0}$ for $k = 0, 1$, we deduce:

Corollary 3.3 *Let $(M, g, \mathcal{F}, \omega)$ be as in Theorem 3.1 and M be connected. Then*

$$\mathcal{H}^0_H = \mathcal{H}^0 = \mathbf{R}, \quad \mathcal{H}^1_H = \mathcal{H}^{1,0} \subset \mathcal{H}^1.$$

4 Harmonic Spaces for a Weak Basic Flow

Let $(\mathcal{F}, \omega)$ be a flow on a compact Riemannian manifold (M, g) of dimension $m = 1 + q$. We define a subspace $\Omega_B^{*,\ell}$ $(\ell = 0, 1)$ of $\Omega^{k,\ell}$ by

$$\Omega_B^{k,0} := \{\phi \in \Omega^{k,0} \mid d_E\phi = 0\}$$
$$\Omega_B^{k,1} := \{\phi \in \Omega^{k,1} \mid \delta_E\phi = 0\}.$$

An element in $\Omega_B^{*,\ell}$ is called a basic form for $\mathcal{F}$. We note that $(\Omega_B^{*,0}, d_B)$ is nothing but the ordinary basic complex $(\Omega_B^*(\mathcal{F}), d_B)$ with respect to the flow $\mathcal{F}$ (see [9], [16]). Observe that $\mathcal{L}_T\phi = 0$ for $\phi \in \Omega_B^{*,0}$.

Now we choose an orientation on $(M, g, \mathcal{F})$ as follows. In our situation, ω is the characteristic form for $\mathcal{F}$. Its transversal volume form is defined by $\nu := *\omega$. The volume form on M is, by convention, given by $\mu = \nu \wedge \omega$.

Let Δ_B be the basic Laplacian given in Section 3. Note that $\Delta_B(\Omega^{k,\ell}) \subset \Omega^{k,\ell}$. We start with the following observation.

Theorem 4.1 *Let $(\mathcal{F}, \omega)$ be a flow on a Riemannian manifold (M, g) of dimension $m = 1 + q$ with transversal volume form $\nu \in \Omega_B^{q,0}$. If $\mathcal{F}$ is isoparametric, i.e., the mean curvature 1-form κ for $\mathcal{F}$ satisfies $\kappa \in \Omega_B^{1,0}$, then $\Delta_B \colon \Omega_B^{*,\ell} \longrightarrow \Omega_B^{*,\ell}$ is well-behaved for $\ell = 0, 1$.*

Proof Let $*_{\mathcal{D}}$ be the star operator on $\Omega^{*,0}$ with respect to the horizontal metric $g_{\mathcal{D}}$. Since $\nu \in \Omega_B^{q,0}$, the restriction of $*_{\mathcal{D}}$ to $\Omega_B^{*,0}$ induces an isomorphism

$$*_{\mathcal{D}} \colon \Omega_B^{k,0} \longrightarrow \Omega_B^{q-k,0}.$$

It is obvious that for $\phi \in \Omega_B^{k,0}$

$$*\phi = *_{\mathcal{D}}\phi \wedge \omega, \quad *_{\mathcal{D}}\phi = (-1)^{q-k} * (\phi \wedge \omega). \tag{4.1}$$

We introduce an auxiliary codifferential operator δ_T on $\Omega^{k,0}$ defined by

$$\delta_T := (-1)^{q(k+1)+1} *_{\mathcal{D}} d_B *_{\mathcal{D}}.$$

https://doi.org/10.4153/CMB-2003-057-9 Published online by Cambridge University Press

δ_T can be extended to Ω^* by defining $\delta_T = \delta_T \otimes \mathrm{id}$. From the construction it is clear that

$$\delta_T(\Omega_B^{k,0}) \subset \Omega_B^{k-1,0}.$$

In addition, a direct computation similar as in [16] shows that on $\Omega_B^{k,0}$

$$\delta_B = \delta_T + \iota_N, \tag{4.2}$$

where N denotes the mean curvature vector field for $\mathcal{F}$ dual to κ. Indeed, by using a Rummler's formula

$$d_B\omega = -\kappa \wedge \omega, \tag{4.3}$$

we have from (4.1)

$$\begin{aligned}\delta_B\phi &= (-1)^{m(k+1)+1} * d_B(*_{\mathcal{D}}\phi \wedge \omega)\\ &= (-1)^{m(k+1)+1} * \left(d_B *_{\mathcal{D}} \phi \wedge \omega + (-1)^{q-k} *_{\mathcal{D}} \phi \wedge d_B\omega\right)\\ &= (-1)^{q(k+1)+1}\left(*_{\mathcal{D}} d_B *_{\mathcal{D}} \phi - *_{\mathcal{D}} e(\kappa) *_{\mathcal{D}} \phi\right),\end{aligned}$$

which proves (4.2). The hypothesis $\kappa \in \Omega_B^{1,0}$ implies

$$\delta_B(\Omega_B^{k,0}) \subset \Omega_B^{k-1,0}. \tag{4.4}$$

Therefore, we conclude that

$$\Delta_B \colon \Omega_B^{k,0} \longrightarrow \Omega_B^{k,0}$$

is well-behaved.

Next, from $\delta_B\delta_E + \delta_E\delta_B = 0$ it is easy to see that $\delta_B(\Omega_B^{k,1}) \subset \Omega_B^{k-1,1}$. Moreover, if we notice

$$* \colon \Omega_B^{k,1} \text{ (resp. } \Omega_B^{k,0}) \longrightarrow \Omega_B^{q-k,0} \quad \text{(resp. } \Omega_B^{q-k,1}), \tag{4.5}$$

then (4.4) implies $\delta_E d_B \psi = * d_E \delta_B * \psi = 0$ for $\psi \in \Omega_B^{k,1}$. It follows that $\Delta_B \colon \Omega_B^{k,1} \longrightarrow \Omega_B^{k,1}$ is also well-behaved. ∎

Remarks (a) The tenseness problem for foliations has been attacked by several mathematicians: Given a foliation $\mathcal{F}$ on a compact manifold M, is there a Riemannian metric g with respect to which the mean curvature form κ for $\mathcal{F}$ is basic? Recently, it was answered in the affirmative when $\mathcal{F}$ is Riemannian [2], [5]. That is, if $\mathcal{F}$ is a Riemannian foliation then there exists a bundle-like metric stisfying $\kappa \in \Omega_B^{1,0}$. Thus a Riemannian flow satisfies two hypotheses, $\kappa, \nu \in \Omega_B^{*,0}$, imposed on Theorem 4.1. However, this problem is still open when $\mathcal{F}$ is a flow admitting basic transversal volume form but is not Riemannian.

For the contact flow case Theorem 4.1 was proved in [8]. Observe that a contact flow $(\mathcal{F}, \omega)$ satisfies the hypotheses of Theorem 4.1.

(b) When $p := \dim E > 1$, we replace the space $\Omega_B^{*,1}$ by $\Omega_B^{*,p}$. Then by a similar way we can show that $\Delta_B \colon \Omega_B^{*,\ell} \longrightarrow \Omega_B^{*,\ell}$ is well-behaved for $\ell = 0, p$.

In the sense of Theorem 4.1 we may define following harmonic spaces for an isoparametric flow with basic transversal volume form. For simplicity, such a flow is called a weak basic flow.

Definition Let $(\mathcal{F}, \omega)$ be a weak basic flow on a compact Riemannian manifold (M, g). A form $\phi \in \Omega_B^{*,\ell}$ $(\ell = 0, 1)$ is said to be basic harmonic if $\Delta_B \phi = 0$. Let $\mathcal{H}_B^{*,\ell}$ $(\ell = 0, 1)$ be the space of all basic harmonic forms for $(\mathcal{F}, \omega)$.

Remark We observe that $\mathcal{H}_B^{*,1}$ is a new basic harmonic space for $\mathcal{F}$, while $\mathcal{H}_B^{*,0}$ coincides with the ordinary basic harmonic space $\mathcal{H}_B^*(\mathcal{F})$ with respect to the flow $\mathcal{F}$. It was proved in [8] that in case of the contact flow there is a basic Hodge isomorphism for any k

$$\mathcal{H}_B^{k,0} = H_B^k(\mathcal{F}), \tag{4.6}$$

where $H_B(\mathcal{F}) := \frac{\ker d_B}{\operatorname{im} d_B}$ is the ordinary basic cohomology space of the basic complex $\big(\Omega_B^*(\mathcal{F}), d_B\big)$ with respect to $\mathcal{F}$. It is well-known that (4.6) holds for the case of Riemannian foliations (see [3], [16]).

By a similar way as in the proof of Theorem 3.1, we have:

Theorem 4.2 *Let $(\mathcal{F}, \omega)$ be a weak basic flow on a compact Riemannian manifold (M, g) of dimension $m = 1 + q$. Then for any k*

$$\mathcal{H}^{k,0} = \mathcal{H}_B^{k,0} \cap J^k, \quad \mathcal{H}^{k-1,1} = \mathcal{H}_B^{k-1,1} \cap J_*^k.$$

Since $*\Delta_B * = \Delta_B$, it holds a duality property from (4.5).

Corollary 4.3 *Let $(\mathcal{F}, \omega)$ be as in Theorem 4.2. Then for any k*

$$\mathcal{H}_B^{k,0} = \mathcal{H}_B^{q-k,1}.$$

In the low degree case we easily verify from the definition:

Corollary 4.4 *Let $(M, g, \mathcal{F}, \omega)$ be as in Theorem 4.2 and M be connected. Then*

$$\mathcal{H}_B^{0,0} = \mathcal{H}^0 = \mathbf{R}, \quad \mathcal{H}_B^{1,0} = \mathcal{H}^{1,0} \subset \mathcal{H}^1.$$

If, in particular, $\mathcal{F}$ is geodesic, we have further:

Corollary 4.5 *Let $(M, g, \mathcal{F}, \omega)$ be as in Theorem 4.2. If, moreover, $\mathcal{F}$ is geodesic then*

$$\dim \mathcal{H}_B^{q,0} \geq 1.$$

Proof It suffices to show $\dim \mathcal{H}_B^{0,1} \geq 1$. Indeed, the flow form $\omega \in \Omega^{0,1}$ satisfies

$$\delta\omega = 0,\ d_B\omega = 0$$

since $\mathcal{F}$ is geodesic. That is, $\omega \in \mathcal{H}_B^{0,1}$. ■

Remarks (a) Corollary 4.4 is a special case of the following result in the cohomology terminology: for any foliation $\mathcal{F}$ on a compact manifold the inclusion $H_B^1(\mathcal{F}) \longrightarrow H^1$ is injective (say, see [16, 9.9]). Corollary 4.5 in the cohomology terminology is also found in [16, 9.21]. Here H^k denotes the k-th De Rham cohomology space on M.

(b) Carrière [1] gave an example of Riemannian flows on compact 3-dimensional manifolds with $H_B^2(\mathcal{F}) = 0$. His example is not geodesic.

We abbreviate "geodesic weak basic flow" to "basic flow". From Carrière's example it is natural to consider the problem when a basic flow achieves $\dim \mathcal{H}_B^{q,0} = 1$. A characterization to this problem will be given in Section 5.

Now we investigate the relationship between the spaces $\mathcal{H}_H^k$, $\mathcal{H}_{H^*}^k$ and $\mathcal{H}_B^{k,\ell}$. In case of the contact flow, the following result is found in [8]. Note that a contact flow is a basic Hörmander flow. We have:

Theorem 4.6 *Let $(\mathcal{F},\omega)$ be a weak basic Hörmander flow on a compact Riemannian manifold (M,g) of dimension $m = 1 + q$. Then for any k*

$$\mathcal{H}_B^{k,0} = \mathcal{H}_H^k, \quad \mathcal{H}_B^{k,1} = \mathcal{H}_{H^*}^{k+1}.$$

Proof If a form $\phi \in \Omega^k$ is H-harmonic, then $\iota_T\phi = 0$ means that $\phi \in \Omega^{k,0}$, and so $d\phi = 0$ is equivalent to $d_E\phi = d_B\phi = 0$. Furthermore, we see from Lemma 2.4 that $\delta\phi = e(\omega)\Lambda\phi = \delta^{2,-1}\phi$, so that $\delta_B\phi = 0$. Therefore $\phi \in \mathcal{H}_B^{k,0}$, and vice versa.

If $\psi \in \Omega^{k+1}$ is H^*-harmonic, then $\psi \in \Omega^{k,1}$ because $e(\omega)\psi = 0$. It follows that $\delta\psi = 0$ is equivalent to $\delta_E\psi = \delta_B\psi = 0$. Since $d\psi = \iota_T L\psi = d^{2,-1}\psi$, we see that $d_B\psi = 0$. Thus $\psi \in \mathcal{H}_B^{k,1}$, and vice versa. ■

5 Harmonic Spaces for an Isometric Flow

In this section, we are interested in the following question: when is the operator $e(\omega)\colon \Omega_B^{k,0} \longrightarrow \Omega_B^{k,1}$ well-behaved? This is related to the following question: when does $(\Omega_B^{*,1}, d_B)$ become a differential complex?

Lemma 5.1 *Let $(\mathcal{F},\omega)$ be a Riemannian flow on a Riemannian manifold (M,g) with bundle-like metric. Then for any k we have an isomorphism*

$$e(\omega)\colon \Omega_B^{k,0} \longrightarrow \Omega_B^{k,1}$$

with inverse ι_T.

Proof For a Riemannian foliation, it is useful to take an basic adapted orthonormal frame for a distinguished chart $\big(U,(x,y)\big)$ of the foliation, say $\{\omega_1 = \omega, \omega_a\}$ with

$\omega_a \in \Omega_B^{1,0}$ $(a = 1, \dots, q)$ (see [4]). Then we can compute locally for $\phi \in \Omega_B^{k,0}$ as follows. Write $\phi = \phi_A(y)\omega_A$. Then

$$\begin{aligned}\delta_E\big(e(\omega)\phi\big) &= (-1)^{mk+1} * d_E * \big(\phi_A(y)\omega \wedge \omega_A\big)\\ &= (-1)^{mk+1} * d_E\big(\phi_A(y) *_{\mathcal{D}} \omega_A\big)\\ &= (-1)^{mk+1} * \big(d_E\phi_A(y) \wedge *_{\mathcal{D}}\omega_A + \phi_A(y) d_E *_{\mathcal{D}} \omega_A\big) = 0.\end{aligned}$$

This means that $e(\omega)\colon \Omega_B^{k,0} \longrightarrow \Omega_B^{k,1}$ is well-defined. Hence $e(\omega)$ is injective by Lemma 2.1.

Finally it is enough to show $\iota_T\psi \in \Omega_B^{k,0}$ for $\psi \in \Omega_B^{k,1}$. Since $\mathcal{F}$ is Riemannian, ψ can be written as $\psi = \psi_A(y)\omega \wedge \omega_A$. Hence a similar way gives rise to

$$\begin{aligned}d_E(\iota_T\psi) &= d_E\big(\psi_A(y)\omega_A\big)\\ &= d_E\psi_A(y) \wedge \omega_A + \psi_A(y) d_E\omega_A\\ &= 0,\end{aligned}$$

so that $\iota_T\psi \in \Omega_B^{k,0}$. ■

In order to develop a harmonic theory, we consider an isometric flow which is generated by a nonsingular Killing vector field. It is well-known [1], [16] that an isometric flow is equivalent to a geodesible Riemannian flow. Given an isometric flow $\mathcal{F}$ on M, there exists a Riemannian metric g on M with respect to T defining $\mathcal{F}$ is a unit Killing vector field. In this case, we have further relations.

Lemma 5.2 *Let $(\mathcal{F}, \omega)$ be an isometric flow on a Riemannian manifold (M, g) generated by a unit Killing vector field T. Then*

$$\mathcal{L}_T = -\delta e(\omega) - e(\omega)\delta.$$

In particular, if $\phi \in \Omega_B^{*,1}$ then $\mathcal{L}_T\phi = 0$.

Proof It suffices to notice that

$$\mathcal{L}_T = *\mathcal{L}_T*$$

because T is Killing. Then Lemma 2.3 completes the proof. ■

A contact flow $(\mathcal{F}, \omega)$ is said to be R-contact if $\mathcal{F}$ is Riemannian. Observe that a R-contact flow is an isometric flow. In the compact case we extend a result [8] obtained for the case of a R-contact flow.

Theorem 5.3 *Let $(\mathcal{F}, \omega)$ be an isometric flow on a compact Riemannian manifold (M, g) of dimension $m = 1 + q$. Then for any k the operator $e(\omega)\colon \mathcal{H}_B^{k,0} \longrightarrow \mathcal{H}_B^{k,1}$ is an isomorphism with inverse ι_T.*

Proof For $\phi \in \mathcal{H}_B^{k,0}$, we see from Lemma 5.1 $e(\omega)\phi \in \Omega_B^{k,1}$. Since $\mathcal{F}$ is geodesic, Lemma 2.2 implies

$$d\big(e(\omega)\phi\big) = d\omega \wedge \phi \in \Omega^{k+2,0}. \tag{5.1}$$

Thus $d_B\big(e(\omega)\phi\big) = 0$.

On the other hand, by Lemma 5.2 we have

$$\delta\big(e(\omega)\phi\big) = -\mathcal{L}_T\phi - e(\omega)\delta\phi = 0, \tag{5.2}$$

which implies $\delta_B\big(e(\omega)\phi\big) = 0$. Hence $e(\omega)\phi \in \mathcal{H}_B^{k,1}$.

Now we show that $\iota_T \colon \mathcal{H}_B^{k,1} \longrightarrow \mathcal{H}_B^{k,0}$ is also well-behaved. Indeed, for $\psi \in \mathcal{H}_B^{k,1}$ a similar way shows

$$d(\iota_T\psi) = \mathcal{L}_T\psi - \iota_T d\psi = 0,$$
$$\delta(\iota_T\psi) = *de(\omega) * \psi = *(d\omega \wedge *\psi) \in \Omega^{k-2,1}.$$

The last formula says that $\delta_B(\iota_T\psi) = 0$. Hence $\iota_T\psi \in \mathcal{H}_B^{k,0}$. ∎

Theorem 5.3, together with Theorem 4.2, yields a Poincare type duality on $\mathcal{H}_B$ as follows.

Corollary 5.4 *Under the same situation as in Theorem 5.3, we have an isomorphism* $\bar{*} := * \circ e(\omega) \colon \mathcal{H}_B^{k,0} \longrightarrow \mathcal{H}_B^{q-k,0}$ *for any k.*

In addition, the following result shows that $\mathcal{H}_B^{q,0}$ provides an obstruction for a geodesible flow to be isometric.

Corollary 5.5 *Let* $(\mathcal{F}, \omega)$ *be a basic flow on a compact connected Riemannian manifold* (M, g)*. Then* $\dim \mathcal{H}_B^{q,0} \geq 1$*. Furthermore, the following are equivalent.*

(a) *$\mathcal{F}$ is isometric with a bundle-like metric g,*
(b) $\dim \mathcal{H}_B^{q,0} = 1$.

Proof Notice that

$$\mathcal{H}_B^{q,1} = \mathcal{H}^m = \mathbf{R}.$$

Thus Theorem 5.3 yields

$$\mathcal{H}_B^{q,0} = \mathcal{H}_B^{q,1} = \mathbf{R}. \tag{5.3}$$

For the converse, refer to [6]. ∎

From Lemma 5.1 and Lemma 5.2 we can discuss the question when $(\Omega_B^{*,1}, d_B)$ becomes a differential complex.

Theorem 5.6 *Let* $(M, g, \mathcal{F}, \omega)$ *be as in Lemma 5.2. Then* $(\Omega_B^{*,1}, d_B)$ *is a differential complex.*

Proof First we claim that $d_B\colon \Omega_B^{k,1} \longrightarrow \Omega_B^{k+1,1}$ is well-behaved for any k. Take an basic adapted orthonormal frame $\{\omega_1 = \omega, \omega_a\}$ for a distinguished chart. By Lemma 5.2 we can write $\psi \in \Omega_B^{k,1}$ as $\psi = \psi_A(y)\omega \wedge \omega_A$. Then Lemma 2.2 implies

$$\begin{aligned}\delta_E(d_B\psi) &= (-1)^{m(k+1)+1} * d_E * \big(d_B\psi_A(y) \wedge \omega \wedge \omega_A - \psi_A(y)\omega \wedge d_B\omega_A\big)\\ &= (-1)^{m(k+1)+1} * d_E\Big((-1)^k *_{\mathcal{D}} \big(d_B\psi_A(y) \wedge \omega_A\big) + (-1)^k\psi_A(y) *_{\mathcal{D}} d_B\omega_A\Big)\\ &= 0.\end{aligned}$$

Thus $d_B\psi \in \Omega_B^{k+1,1}$.

On the other hand, it follows from Lemma 2.2, Lemma 2.4 and Lemma 5.1 that

$$d^{2,-1}\psi = \iota_T(d\omega \wedge \psi) \in \Omega_B^{k+2,0}. \tag{5.4}$$

Now on Ω^* we get $d_B^2 + d_Ed^{2,-1} + d^{2,-1}d_E = 0$ because $d^2 = 0$. Thus (5.4) implies

$$d_B^2\psi = d_Ed^{2,-1}\psi = 0,$$

which completes the proof. ∎

Theorem 5.6 allows us to define a new cohomology space associated to the differential complex $(\Omega_B^{*,1}, d_B)$

$$H_B^{*,1} := H(\Omega_B^{*,1}, d_B). \tag{5.5}$$

Recall that the cohomology space $H_B^{*,0}$ of the differential complex $(\Omega_B^{*,0}, d_B)$ is nothing but the ordinary basic cohomology space $H_B^*(\mathcal{F})$. Theorem 5.6, combined with Lemma 5.1, says that:

Corollary 5.7 *Let $(M, g, \mathcal{F}, \omega)$ be as in Lemma 5.2. Then we have a complex isomorphism*

$$e(\omega)\colon (\Omega_B^{*,0}, d_B) \longrightarrow (\Omega_B^{*,1}, d_B)$$

with inverse ι_T.

Furthermore, we can show that $e(\omega)$ induces a cohomology isomorphism.

Theorem 5.8 *Let $(M, g, \mathcal{F}, \omega)$ be as in Theorem 5.3. Then*

$$e(\omega)\colon H_B^{*,0} \longrightarrow H_B^{*,1}$$

is an isomorphism with inverse ι_T.

Proof Take $\phi \in \Omega_B^{k,0}$ with $d_B\phi = 0$. Then by Lemma 2.2 and Lemma 5.1 we see that $e(\omega)\phi \in \Omega_B^{k,1}$ and $d_B\big(e(\omega)\phi\big) = 0$. Thus $e(\omega)[\phi] := [e(\omega)\phi]$ is well-defined.

Suppose that $[e(\omega)\phi] = 0$. Then there exists $\psi \in \Omega_B^{k-1,1}$ such that $e(\omega)\phi = d_B\psi$. A direct computation by using Lemma 2.4 and Lemma 5.2 gives rise to

$$\phi = \iota_T e(\omega)\phi = \iota_T d\psi = -d\iota_T\psi.$$

It follows that $[\phi] = 0$, that is, $e(\omega)$ is injective.

Now given $[\psi] \in H_B^{k,1}$, take $\psi \in \Omega_B^{k,1}$ with $d_B\psi = 0$. Then $\iota_T\psi \in \Omega_B^{k,0}$ and

$$d_B\iota_T\psi = \mathfrak{L}_T\psi - \iota_T d\psi = 0,$$

so that $[\iota_T\psi] \in H_B^{k,0}$. ■

Corollary 5.9 *Under the same situation as in Theorem 5.8, we have a basic Hodge isomorphism*

$$\mathcal{H}_B^{k,1} = H_B^{k,1}$$

for any k.

Corollary 5.9 can transfer previous results in terms of harmonic spaces into those in the cohomology terminology under the situation of an isometric flow. For example, Corollary 5.4 in the cohomology terminology is found in [16]. Corollary 5.5 in the cohomology terminology is found in [16], [6]. In particular, (5.3) was obtained in [12] by a different method, namely, by constructing the Gysin sequence for an isometric flow.

References

[1] Y. Carriére, *Flots riemannienes.* Astérisque **116**(1984), 31–52.
[2] D. Dominguez, *Finiteness and tenseness for Riemannian foliations.* Preprint.
[3] A. El Kacimi-Alaoui and G. Hector, *Décomposition de Hodge basique pour un feuilletage riemannien.* Ann. Inst. Fourier **36**(1986), 207–227.
[4] H. Kitahara, *Differential geometry of Riemannian foliations.* Kyungpook Nat. Univ., 1986.
[5] P. March, M. Min-Oo and R. A. Ruh, *Mean curvature of Riemannian foliations.* Canad. Math. Bull. **39**(1996), 95–105.
[6] P. Molino and V. Sergiescu, *Deux remarques sur les flots riemanniens.* Manuscripta Math. **51**(1985), 145–161.
[7] Y. Ogawa, *On C-harmonic forms in a compact Sasakian space.* Tohoku Math. J. **19**(1967), 267–296.
[8] H. K. Pak and T. Takahashi, *Harmonic forms in a compact contact manifold.* Tohoku Math. Publ. **20**(2001), 125–138.
[9] B. Reinhart, *Harmonic integrals on foliated manifolds.* Amer. J. Math. **91**(1959), 529–536.
[10] M. Rumin, *Un complexe de formes différentielles sur les variétés de contact.* C. R. Acad. Sci. Paris **310**(1990), 401–404.
[11] ———, *Formes différentielles sur les variétés de contact.* J. Differential Geom. **39**(1994), 281–330.
[12] M. Saralegui, *The Euler class for flows of isometries.* Pitman Res. Notes in Math. **131**(1985), 220–227.
[13] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure.* Tohoku Math. J. **12**(1960), 459–476.
[14] S. Tachibana, *On a decomposition of C-harmonic forms in a compact Sasakian space.* Tohoku Math. J. **19**(1967), 198–212.

[15] S. Tanno, *Harmonic forms and Betti numbers of certain contact Riemannian manifolds.* J. Math. Soc. Japan **19**(1967), 308–316
[16] Ph. Tondeur, *Foliations on Riemannian manifolds.* Universitext, Springer, New York, 1988.

Department of Mathematics
Kyungsan University
Kyungsan City 712-240
Korea
email: hkpak@kyungsan.ac.kr