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REMARKS TO THE UNIQUENESS PROBLEM OF MEROMORPHIC MAPS INTO $P^N(C)$, II

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§ 1. Introduction

In [7], R. Nevanlinna gave the following uniqueness theorem of meromorphic functions as an improvement of a result of G. Pólya ([8]).

THEOREM A. *Let f, g be non-constant meromorphic functions on C. If there are five mutually distinct values $a_1, \cdots, a_5$ such that $f^{-1}(a_i) = g^{-1}(a_i)$ $(1 \leqq i \leqq 5)$, then $f \equiv g$.*

The author attempted to generalize this to the case of meromorphic maps of C^n into $P^N(C)$ and obtained some results in the previous papers [4], [5] and [6]. One of them is the following;

THEOREM B. *Let f and g be meromorphic maps of C^n into $P^N(C)$ one of which is algebraically non-degenerate. If there are $2N+3$ hyperplanes H_i $(1 \leqq i \leqq 2N+3)$ in general position such that $\nu(f, H_i) = \nu(g, H_i)$ for pull-backs $\nu(f, H_i)$, $\nu(g, H_i)$ of the divisors (H_i) by f and g respectively, then $f \equiv g$.*

Relating to this, the following theorem will be proved.

THEOREM I. *Let f, g be algebraically non-degenerate meromorphic maps of C^n into $P^N(C)$. If there are hyperplanes H_i in general position such that*

$$\nu(f, H_i) = \nu(g, H_i) = 0\ ,$$

namely, $f(C^n) \cap H_i = g(C^n) \cap H_i = \phi$ for $i = 1, 2, \cdots, N+1$ and

$$\min(\nu(f, H_j), N) = \min(\nu(g, H_j), N)$$

for $j = N+2, \cdots, 2N+3$, then $f \equiv g$.

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This will be given as a consequence of the following generalization of a classical result of R. Nevanlinna ([7], Satz 7, p. 388).

THEOREM II. *Let f, g be algebraically non-degenerate meromorphic maps of C^n into $P^N(C)$. If there are $N+2$ hyperplanes in general position such that*

$$\nu(f, H_i) = \nu(g, H_i) = 0$$

for $i = 1, 2, \cdots, N+1$ and

$$\min(\nu(f, H_{N+2}), N) = \min(\nu(g, H_{N+2}), N) ,$$

then f and g are related as $L \cdot g = f$ with a projective linear transformation L of $P^N(C)$ which permutes hyperplanes $H_1, \cdots, H_{N+1}$ and leaves H_{N+2} fixed.

In § 2, we shall give a combinatorial lemma which plays an essential role in this paper. In § 3, we shall recall some classical results in the value distribution theory for holomorphic maps of C into $P^N(C)$ and obtain a new result from them. Theorems I and II are completely proved in § 4.

§ 2. Main Lemma

For later use, we shall give in this section a graph-theoretic combinatorial lemma. We consider a set $A = \{a_{ij}; 1 \leqq i \leqq n, 1 \leqq j \leqq n\}$ consisting of n^2 elements abstractly. Let non-empty subsets C of A and Γ of $C \times C$ be given in some manner. For any $a_{ij}, a_{k\ell}$ in C, we write

$$(i, j) \leftrightarrow (k, \ell) , \quad \text{or} \quad (i, j) \not\leftrightarrow (k, \ell)$$

if $(a_{ij}, a_{k\ell}) \in \Gamma$, or $(a_{ij}, a_{k\ell}) \notin \Gamma$ respectively. We assume that

(A_0) for any $a_{ij}, a_{k\ell}$ in C $(i, j) \not\leftrightarrow (i, j)$ and $(i, j) \leftrightarrow (k, \ell)$ whenever $(k, \ell) \leftrightarrow (i, j)$,

(A_1) if $a_{i_1 j_1}$, $a_{i_1 j_2}$ and $a_{i_2 j_1}$ are in $A - C$ $(1 \leqq i_1, j_1, i_2, j_2 \leqq n)$, then $a_{i_2 j_2}$ are also in $A - C$,

(A_2) for any $a_{ij} \in C$ there exists some $a_{k\ell} \in C$ such that $(i, j) \leftrightarrow (k, \ell)$,

(A_3) if $a_{i_\sigma j_\sigma}, a_{k_\sigma \ell_\sigma} \in C$ $(1 \leqq \sigma \leqq s)$ satisfy the conditions

$$(i_1, j_1) \leftrightarrow (k_1, \ell_1), (i_2, j_2) \leftrightarrow (k_2, \ell_2), \cdots, (i_s, j_s) \leftrightarrow (k_s, \ell_s) ,$$

then $\{i_1, i_2, \cdots, i_s\} = \{k_1, k_2, \cdots, k_s\}$ occurs when and only when $\{j_1, j_2, \cdots, j_s\}$

$= \{\ell_1, \ell_2, \cdots, \ell_s\}$, where some indices may appear repeatedly in $\{i_1, \cdots, i_s\}$ etc. and the equalities mean in this case that they appear the same times in both sides.

In this situation, we give

MAIN LEMMA. *By changing indices i and j of a_{ij}'s individually, it holds that*

(i) *there is a partition of indices*

$$\{1, 2, \cdots, n\} = \{1, 2, \cdots, m_1\} \cup \{m_1 + 1, \cdots, m_2\} \cup \cdots \cup \{m_{t-1} + 1, \cdots, n\}$$

such that $a_{ij} \notin C$ if and only if i and j are in the same class $\{m_{\tau-1} + 1, \cdots, m_\tau\}$ for some $\tau (1 \leqq \tau \leqq t)$, where $m_0 := 0$, $m_t := n$ and $t \geqq 2$,

(ii) *for any $a_{ij}, a_{k\ell}$ in C, $(i, j) \leftrightarrow (k, \ell)$ if and only if $i = \ell$ and $j = k$.*

For the proof, we need some preparations.

LEMMA 2.1. *For any i $(1 \leqq i \leqq n)$, there exist some j_1 and j_2 such that a_{ij_1} and $a_{j_2 i}$ are in $A - C$.*

Proof. Assume that $a_{ij} \in C$ for any j $(1 \leqq j \leqq n)$. By the assumption (A_2), we can take some k_j, ℓ_j such that $(i, j) \leftrightarrow (k_j, \ell_j)$ for each j. Here, $j \neq \ell_j$. In fact, if not, $i \neq k_j$, which contradicts the assumption (A_3). And, $i \neq k_j$ by the same reason. Since $\{1, 2, \cdots, i-1, i+1, \cdots, n\}$ cannot contain n distinct elements, we have indices j', j'' such that $k_{j'} = k_{j''}$ and $j' \neq j''$. Then, for the relations

$$(i, j') \leftrightarrow (k_{j'}, \ell_{j'}), (k_{j''}, \ell_{j''}) \leftrightarrow (i, j'') ,$$

$\{i, k_{j''}\} = \{k_{j'}, i\}$ but $\{j', \ell_{j''}\} \neq \{\ell_{j'}, j''\}$. This contradicts the assumption (A_3). Thus, there exists some j_1 such that $a_{ij_1} \notin C$. The existence of j_2 with $a_{j_2 i} \notin C$ is shown similarly.

We introduce here a provisional notation. For integers k, ℓ with $k \leqq \ell$, we denote by $[k, \ell]$ the set of all integers i with $k \leqq i \leqq \ell$.

By a suitable change of indices, we may assume $a_{i1} \notin C$ for any $i \in [1, m]$ and $a_{j1} \in C$ for any $j \in [m+1, n]$, where $1 \leqq m \leqq n-1$ by Lemma 2.1. Then, as is easily seen by the assumption (A_1), if $a_{i_0 k_0} \notin C$ for some $k_0 \in [2, n]$ and $i_0 \in [1, m]$, then $a_{ik_0} \notin C$ for any $i \in [1, m]$ and $a_{jk_0} \in C$ for any $j \in [m+1, n]$. By this reason, choosing indices suitably, we may assume that $a_{ij} \notin C$ if $i \in [1, m]$, $j \in [1, m']$ and $a_{ij} \in C$ if $i \in$

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$[m+1, n]$, $j \in [1, m']$, or $i \in [1, m]$, $j \in [m'+1, n]$, where $1 \leqq m' \leqq n-1$. Moreover, it may be assumed that

(2.2) there are indices $m_1, \cdots, m_{t-1}, m'_1, \cdots, m'_{t-1}$ with

$$m =: m_1 < m_2 < \cdots < m_{t-1} < m_t := n$$
$$m' =: m'_1 < m'_2 < \cdots < m'_{t-1} < m'_t := n$$

such that $a_{ij} \notin C$ if and only if $i \in [m_{\tau-1}+1, m_\tau]$ and $j \in [m'_{\tau-1}+1, m'_\tau]$ for some $\tau \in [1, t]$, where we put $m_0 = m'_0 = 0$.

Later, $m_\tau = m'_\tau$ $(1 \leqq \tau \leqq t)$ will be shown. We assume $m' \leqq m$ for a while by exchanging the situations of indices i and j of a_{ij} if necessary.

For each j in $[m'+1, n]$, we define an index I_j as follows.

(2.3) If $(1, j) \not\leftrightarrow (i, \ell)$ for any $i \in [1, m]$ and $\ell \in [m'+1, n]$, we put $I_j = 1$. Otherwise, choose indices $i_1, i_2, \cdots, i_a$ in $[1, m]$ and $\ell_1, \ell_2, \cdots, \ell_a$ in $[m'+1, n]$ such that

$$(1, j) \leftrightarrow (i_1, \ell_1), (i_1, j) \leftrightarrow (i_2, \ell_2), \cdots, (i_{a-1}, j) \leftrightarrow (i_a, \ell_a)$$

and $(i_a, j) \not\leftrightarrow (i, \ell)$ for any $i \in [1, m]$, $\ell \in [m'+1, n]$. And, put $I_j := i_a$.

These choices are certainly possible. Indeed, if we cannot choose the above i_a, then there are infinitely many $i_\beta \in [1, m]$, $\ell_\beta \in [m'+1, n]$ $(\beta = 1, 2, \cdots)$ such that $(i_\beta, j) \leftrightarrow (i_{\beta+1}, \ell_{\beta+1})$. We have necessarily $i_\beta = i_{\beta'}$, for some β, β' with $\beta + 2 \leqq \beta'$ and relations

$$(i_\beta, j) \leftrightarrow (i_{\beta+1}, \ell_{\beta+1}), (i_{\beta+1}, j) \leftrightarrow (i_{\beta+2}, \ell_{\beta+2}), \cdots, (i_{\beta'-1}, j) \leftrightarrow (i_{\beta'}, \ell_{\beta'}) .$$

This contradicts the assumption (A_3), because

$$\{i_\beta, \cdots, i_{\beta'-1}\} = \{i_{\beta+1}, \cdots, i_{\beta'}\}$$

but

$$\{j, \cdots, j\} \neq \{\ell_{\beta+1}, \cdots, \ell_{\beta'}\} .$$

LEMMA 2.4. *If there are indices $k_0 \in [m+1, n]$, ℓ_0, ℓ'_0 in $[1, n]$ such that*

$$(*)_1 \qquad (I_j, j) \leftrightarrow (k_0, \ell_0), (k_0, \ell'_0) \leftrightarrow (I_{j'}, j') ,$$

then $j = j'$.

Proof. As in (2.3), we can take indices $i_1, \cdots, i_{a-1}, i'_1, \cdots, i'_{a'-1}$ in $[1, m]$ and $\ell_1, \cdots, \ell_a, \ell'_1, \cdots, \ell'_{a'}$ in $[m'+1, n]$ such that

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$$(*)_2 \qquad \begin{array}{l}(1, j) \leftrightarrow (i_1, \ell_1), (i_1, j) \leftrightarrow (i_2, \ell_2), \cdots, (i_{a-1}, j) \leftrightarrow (I_j, \ell_a) \\ (i'_1, \ell'_1) \leftrightarrow (1, j'), (i'_2, \ell'_2) \leftrightarrow (i'_1, j'), \cdots, (I_{j'}, \ell'_{a'}) \leftrightarrow (i'_{a'-1}, j') .\end{array}$$

For the relations $(*)_1$ and $(*)_2$, we see

$$\begin{aligned}&\{I_j, k_0, 1, i_1, \cdots, i_{a-1}, i'_1, \cdots, i'_{a'-1}, I_{j'}\} \\ &\quad = \{k_0, I_{j'}, i_1, \cdots, i_{a-1}, I_j, 1, i'_1, \cdots, i'_{a'-1}\} .\end{aligned}$$

So, by the assumption (A_3)

$$\{j, \ell'_0, j, \cdots, j, \ell'_1, \cdots, \ell'_{a'}\} = \{\ell_0, j', \ell_1, \cdots, \ell_a, j', \cdots, j'\} .$$

This implies $j = j'$ because $j \neq \ell_0, \ell_1, \cdots, \ell_a, j'$.

LEMMA 2.5. *For any $k \in [m+1, n]$ there is one and only one $j \in [m'+1, n]$ such that $(I_j, j) \leftrightarrow (k, \ell)$ for some $\ell \in [1, n]$.*

Proof. The uniqueness of the desired index is a result of Lemma 2.4. On the other hand, by the assumption (A_2), there are indices $k_{m'+1}, \cdots, k_n, \ell_{m'+1}, \cdots, \ell_n$ such that

$$(I_{m'+1}, m'+1) \leftrightarrow (k_{m'+1}, \ell_{m'+1}), \cdots, (I_n, n) \leftrightarrow (k_n, \ell_n) ,$$

where $m+1 \leqq k_{m'+1}, \cdots, k_n \leqq n$ by the property (2.3) of I_j's. Then, $k_{m+1}, \cdots, k_n$ are distinct with each other because of Lemma 2.4. Therefore,

$$n - m' \leqq n - m$$

and so $m \leqq m'$. Since $m' \leqq m$ is assumed, we have $m = m'$ and $\{k_{m+1}, \cdots, k_n\} = \{m+1, \cdots, n\}$. The index j with $k_j = k$ is the desired one.

LEMMA 2.6. *$m_\tau = m'_\tau$ $(1 \leqq \tau \leqq t)$ for the numbers defined as in* (2.2).

Proof. As in the proof of Lemma 2.5, we have $m\ (= m_1) = m'\ (= m'_1)$. The same arguments are available for the other τ. So, we obtain Lemma 2.6.

LEMMA 2.7. *For any $i \in [m+1, n]$ and $j \in [1, m]$ there exist some $k \in [1, m]$ and $\ell \in [m+1, n]$ such that $(i, j) \leftrightarrow (k, \ell)$.*

Proof. Assume the contrary. According to the assumtion (A_2), we choose indices $k_0, \ell_0 \in [1, n]$ such that

$$(\sharp)_1 \qquad (i, j) \leftrightarrow (k_0, \ell_0) .$$

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By the assumption, $m+1 \leqq k_0 \leqq n$. On the other hand, there are indices j_0, j'_0 in $[m+1, n]$ and ℓ'_0, ℓ''_0 in $[1, n]$ such that

$$(\sharp)_2 \qquad (I_{j_0}, j_0) \leftrightarrow (i, \ell'_0), (k_0, \ell''_0) \leftrightarrow (I_{j'_0}, j'_0)$$

because of Lemma 2.5. Moreover, by the property (2.3) of I_j's, we have

$$(\sharp)_3 \qquad \begin{array}{l} (1, j_0) \leftrightarrow (i_1, \ell_1), (i_1, j_0) \leftrightarrow (i_2, \ell_2), \cdots, (i_{a-1}, j_0) \leftrightarrow (I_{j_0}, \ell_a) \\ (i'_1, \ell'_1) \leftrightarrow (1, j'_0), (i'_2, \ell'_2) \leftrightarrow (i'_1, j'_0), \cdots, (I_{j'_0}, \ell'_{a'}) \leftrightarrow (i'_{a'-1}, j'_0) \end{array}$$

for some $i_1, \cdots, i_{a-1}, i'_1, \cdots, i'_{a'-1} \in [1, m]$ and $\ell_1, \cdots, \ell_a, \ell'_1, \cdots, \ell'_{a'}$ in $[m+1, n]$. Observe the indices of the relations $(\sharp)_1$, $(\sharp)_2$ and $(\sharp)_3$. It is easily seen that they contradict the assumption (A_3). Thus, we have Lemma 2.7.

LEMMA 2.8. *By a suitable change of indices i's of a_{ij} among $1, 2, \cdots, m$, there is some index ℓ_{ij} for each $i \in [m+1, n]$ and $j \in [1, m]$ such that $(i, j) \leftrightarrow (j, \ell_{ij})$, where $m+1 \leqq \ell_{ij} \leqq n$.*

Proof. We take $k_1, \cdots, k_m$ in $[1, m]$ and $\ell_1, \cdots, \ell_m$ in $[m+1, n]$ such that

$$(m+1, 1) \leftrightarrow (k_1, \ell_1), \cdots, (m+1, m) \leftrightarrow (k_m, \ell_m)$$

by the use of Lemma 2.7. As is easily seen by the assumption (A_0) and (A_3), we have $\{k_1, \cdots, k_m\} = \{1, \cdots, m\}$. By a change of indices, we may assume that $k_1 = 1, \cdots, k_m = m$. For any $i \in [m+1, n]$, we choose $k'_1, \cdots, k'_m$ in $[1, m]$ and $\ell'_1, \cdots \ell'_m$ in $[m+1, n]$ so that

$$(i, 1) \leftrightarrow (k'_1, \ell'_1), \cdots, (i, m) \leftrightarrow (k'_m, \ell'_m) .$$

By the same reason as the above, $\{k'_1, \cdots, k'_m\} = \{1, 2, \cdots, m\}$. Assume that $k'_j \neq j$ for some j and take the index j' with $k'_{j'} = j$. We observe the relations

$$(i, j) \leftrightarrow (k'_j, \ell'_j), (k'_j, \ell_{k'_j}) \leftrightarrow (m+1, k'_j), (m+1, j) \leftrightarrow (j, \ell_j), (j, \ell'_{j'}) \leftrightarrow (i, j') .$$

As is easily seen by the facts $j \neq \ell'_j, k'_j, \ell_j, j'$, this contradicts the assumption (A_3). Therefore, $k'_j = j$ for any j and we have Lemma 2.8.

LEMMA 2.9. *After a suitable change of indices j's of a_{ij} among $m+1, \cdots, n$, it holds that $(i, j) \leftrightarrow (j, i)$ for any $j \in [m+1, n]$ and $j \in [1, m]$.*

Proof. As a consequence of Lemma 2.8, we may assume that

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$$(m+1,1) \leftrightarrow (1, \ell_{m+1}), \cdots, (n,1) \leftrightarrow (1, \ell_n) ,$$

where $\{\ell_{m+1}, \cdots, \ell_n\} = \{m+1, \cdots, n\}$ by the assumption (A_3). Changing indices if necessary, we have $(\ell, 1) \leftrightarrow (1, \ell)$ for any $\ell \in [m+1, n]$. Assume that for some $i_0 \in [1, m]$ and $j_0 \in [m+1, n]$ $(i_0, j_0) \not\leftrightarrow (j_0, i_0)$. Then, by Lemma 2.8, there is some $\ell_0 \in [m+1, n]$ $(j_0, i_0) \leftrightarrow (i_0, \ell_0)$ such that $\ell_0 \neq j_0$. If we choose $k_{m+1}, \cdots, k_n$ in $[m+1, n]$ such that $(j, i_0) \leftrightarrow (i_0, k_j)$ for each $j \in [m+1, n]$, it is easily seen that $\{k_{m+1}, \cdots, k_n\} = \{m+1, \cdots, n\}$. Therefore, there are an index k_0 such that $(k_0, i_0) \leftrightarrow (i_0, j_0)$, where $k_0 \neq j_0$ by the assumption. We observe the relations

$$(i_0, \ell_0) \leftrightarrow (j_0, i_0), (k_0, i_0) \leftrightarrow (i_0, j_0), (1, k_0) \leftrightarrow (k_0, 1), (j_0, 1) \leftrightarrow (1, j_0) .$$

Obviously, these indices do not satisfy the assumption (A_3). Thus, we get Lemma 2.9.

Proof of Main Lemma. By Lemma 2.9, we may assume that $(i, j) \leftrightarrow (j, i)$ for any $i \in [m+1, n]$ and $j \in [1, m]$. The conclusion (i) of Main Lemma is a direct result of Lemma 2.6 because Lemma 2.6 is available for the above choice of indices. We shall prove the conclusion (ii). There are indices k, ℓ with $(i, j) \leftrightarrow (k, \ell)$ for any i, j with $a_{ij} \in C$ by the assumption (A_2). So, we have only to show that $k = j$ and $\ell = i$ whenever $(i, j) \leftrightarrow (k, \ell)$. By virtue of the assumption (A_0), it suffices to study the following three cases.

1°) $m+1 \leqq i \leqq n$, $1 \leqq j \leqq m$, $1 \leqq k \leqq m$ and $m+1 \leqq \ell \leqq n$.

2°) $m+1 \leqq i \leqq n$, $1 \leqq j \leqq m$, $m+1 \leqq k \leqq n$ and $1 \leqq \ell \leqq n$.

3°) $m+1 \leqq i, j, k, \ell \leqq n$.

Observe the relations

$$(i, j) \leftrightarrow (k, \ell), (j, i) \leftrightarrow (i, j), (k, \ell) \leftrightarrow (\ell, k), (\ell, j) \leftrightarrow (j, \ell)$$

for the case 1°) and

$$(i, j) \leftrightarrow (k, \ell), (1, i) \leftrightarrow (i, 1), (k, 1) \leftrightarrow (1, k) ,$$

for the cases 2°) and 3°) respectively. In any case, indices in the relations do not satisfy the assumption (A_3) except the case $(k, \ell) = (j, i)$. Thus, Main Lemma is completely proved.

§ 3. A result from the value distribution theory

We shall introduce some definitions and notations. For a domain

D in the complex plane C, a divisor $\nu(z)$ on D is defined as an integer-valued function on D such that $\{z \in D; \nu(z) \neq 0\}$ has no accumulation point in D. Let us take a divisor ν on $\{z \in C; |z| < R\}$ $(0 < R \leqq +\infty)$ with $\nu(0) = 0$. We put

$$n(r, \nu) := \sum_{|z| \leqq r} \nu(z)$$

$$N(r, \nu) := \int_0^r \frac{n(t, \nu)}{t} dt = \sum_{|z| \leqq r} \nu(z) \log \frac{r}{|z|} ,$$

where $0 \leqq r \leqq R$.

Let f be a non-constant meromorphic function on C. We define $\nu_f(a) = n, = 0$ and $= -m$ if $f(z)$ has a zero of order n at $z = a$, if $f(a) \neq 0$ and if $f(z)$ has a pole of order m at $z = a$, respectively. And, put $N(r, f) = N(r, \nu_f)$. Then, the well-known Jensen's formula is given as follows.

(3.1) If $f(0) \neq 0, \infty$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = N(r, f) + \log |f(0)| \quad (r > 0) .$$

Now, let us take a holomorphic map f of C into $P^N(C)$. For an arbitrarily fixed homogeneous coordinates $w_1 : \cdots : w_{N+1}$, we can take holomorphic functions $f_1, \cdots, f_{N+1}$ such that $f = f_1 : \cdots : f_{N+1}$ and f_i $(1 \leqq i \leqq N + 1)$ have no common zeros. In the following, we shall call such a representation of f a reduced representation. For a reduced representation $f = f_1 : f_2 : \cdots : f_{N+1}$, we put

$$u(z) := \max_{1 \leqq i \leqq N+1} \log |f_i(z)|$$

and, following H. Cartan [2], define the characteristic function of f as

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta - u(0) ,$$

which is determined independently of any choice of a reduced representation of f.

Assume that f is non-degenerate, i.e., $f(C)$ is not contained in any hyperplane of $P^N(C)$. For a hyperplane

$$H : a^1 w_1 + a^2 w_2 + \cdots + a^{N+1} w_{N+1} = 0$$

and a reduced representation $f = f_1 : f_2 : \cdots : f_{N+1}$, we consider a holomorphic function

$$F := a^1 f_1 + a^2 f_2 + \cdots + a^{N+1} f_{N+1}$$

and define $\nu(f, H) := \nu_F$.

DEFINITION 3.2. For a positive integer p, we define

$$N_p(r, f, H) := N(r, \min(p, \nu(f, H)))$$
$$N(r, f, H) := N(r, \nu(f, H)) .$$

We can conclude from (3.1)

$$N_p(r, f, H) \leqq N(r, f, H) \leqq T(r, f) + K , \tag{3.3}$$

where K is a constant not depending on r.

We recall here the second fundamental theorem in the value distribution theory given by H. Cartan in [2], which is essentially used in the followings.

THEOREM 3.4. *Let f be a non-degenerate holomorphic map of $\boldsymbol{C}$ into $P^N(\boldsymbol{C})$ and H_i $(1 \leqq i \leqq q)$ be hyperplanes in general position with $f(0) \notin \cup_i H_i$. Then,*

$$(q - N - 1)T(r, f) \leqq \sum_{1 \leqq i \leqq q} N_N(r, f, H_i) + S(r) ,$$

where

$$S(r) = O(\log r) + O(\log T(r, f)) \qquad \|$$

and "$\|$" means that this holds outside an open set E in $\boldsymbol{R}$ such that $\int_E \frac{dt}{t} < +\infty$.

Remark. In Theorem 3.4, if f is rational, i.e., represented as $f = f_1 : f_2 : \cdots : f_{N+1}$ with polynomials f_i, then $S(r) = O(1)$.

Now, let us consider two non-degenerate holomorphic maps f, g of $\boldsymbol{C}$ into $P^N(\boldsymbol{C})$ and $N + 2$ hyperplanes $H_1, \cdots, H_{N+2}$ in general position. We assume that

$$\nu(f, H_i) = \nu(g, H_i) = 0 \tag{3.5}$$

i.e., $f(\boldsymbol{C}) \cap H_i = g(\boldsymbol{C}) \cap H_i = \phi$ for $i = 1, 2, \cdots, N + 1$ and

$$\min(\nu(f, H_{N+2}), N) = \min(\nu(g, H_{N+2}), N) . \tag{3.6}$$

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We choose homogeneous coordinates $w_1 : w_2 : \cdots : w_{N+1}$ on $P^N(C)$ such that H_i are represented as

$$H_i : w_i = 0 \qquad 1 \leqq i \leqq N+1 ,$$
$$H_{N+2} : w_1 + w_2 + \cdots + w_{N+1} = 0 .$$

In this situation, we can prove the following

PROPOSITION 3.7. *Take reduced representations $f = f_1 : f_2 : \cdots : f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$. Then there exists some constants $c_1, c_2, \cdots, c_{N+1}, d_1, d_2, \cdots, d_{N+1}$ such that $c_i - d_j \neq 0$ for some i, j and*

$$\sum_{1 \leqq i, j \leqq N+1} (c_i - d_j) f_i g_j = 0 . \tag{3.8}$$

To prove this, we need some preparations. For brevity, we denote H_{N+2} by H and define

$$N'(r, f) := N(r, \nu(f, H) - \min(\nu(f, H), \nu(g, H)))$$
$$N'(r, g) := N(r, \nu(g, H) - \min(\nu(f, H), \nu(g, H)))$$

for each positive number r.

LEMMA 3.9. *It holds that*

$$N'(r, f) + N'(r, g) \leqq N(r, f, H) - N_N(r, f, H) + N(r, g, H) - N_N(r, g, H) .$$

Proof. According to the assumption (3.6), we see easily

$$\begin{aligned}(\nu(f, H) &- \min(\nu(f, H), \nu(g, H))) + (\nu(g, H) - \min(\nu(f, H), \nu(g, H))) \\ &= |\nu(f, H) - \nu(g, H)| \\ &\leqq |\nu(f, H) - \min(\nu(f, H), N)| + |\nu(g, H) - \min(\nu(f, H), N)| \\ &= (\nu(f, H) - \min(\nu(f, H), N)) + (\nu(g, H) - \min(\nu(g, H), N)) .\end{aligned}$$

By linearlity and monotonicity of integrals, we can conclude Lemma 3.9.

LEMMA 3.10. *It holds that*

$$N'(r, f) + N'(r, g) = O(\log r) + O(\log(T(r, f) + T(r, g))) \quad \| .$$

Here, if f and g are both rational, the right hand side is replaced by $O(1)$.

Proof. Since $N_N(r, f, H_i) = N_N(r, g, H_i) = 0 (1 \leqq i \leqq N+1)$ by the assumption (3.5), Theorem 3.4 implies that

$$T(r,f)-N_N(r,f,H)=O(\log r)+O(\log T(r,f))\quad \|\,,$$
$$T(r,g)-N_N(r,g,H)=O(\log r)+O(\log T(r,g))\quad \|\,.$$

Therefore, by (3.3), we see

$$N(r,f,H)-N_N(r,f,H)=O(\log r)+O(\log T(r,f))\quad \|\,,$$
$$N(r,g,H)-N_N(r,f,H)=O(\log r)+O(\log T(r,g))\quad \|\,.$$

By virtue of Lemma 3.9, we can conclude

$$\begin{aligned} &N'(r,f)+N'(r,g)\\ &\quad\leqq O(\log r)+O(\log T(t,f)T(r,g)) \quad \|\\ &\quad\leqq O(\log r)+O(\log (T(r,f)+T(r,g)))\quad \|\,. \end{aligned}$$

The latter half of Lemma 3.10 is due to Remark to Theorem 3.4.

Proof of Proposition 3.7. We take a holomorphic function h on C such that $\nu_h=\min(\nu(f,H),\nu(g,H))$. And, we consider a holomorphic map Φ of C into $P^{2N}(C)$ defined as

$$\Phi=f_1\tilde{g}:f_2\tilde{g}:\cdots:f_{N+1}\tilde{g}:-g_1\tilde{f}:\cdots:-g_N\tilde{f}\,, \tag{3.11}$$

for some fixed homogeneous coordinates on $P^{2N}(C)$, where $\tilde{f}:=f_1+\cdots+f_{N+1}/h$ and $\tilde{g}:=g_1+\cdots+g_{N+1}/h$. Since f_i and g_i $(1\leqq i\leqq N+1)$ have no zeros and $\tilde{f}$ and $\tilde{g}$ have no common zeros, (3.11) is a reduced representation of Φ. For the proof of Proposition 3.7, we have only to show that Φ is degenerate. In fact, in this case, there exist some constants $c_1,\cdots,c_{N+1},d_1,d_2,\cdots,d_N$, at least one of which is not zero, such that

$$\sum_{1\leq i\leq N+1}c_if_i(g_1+\cdots+g_{N+1})-\sum_{1\leqq j\leqq N}d_jg_j(f_1+\cdots+f_{N+1})=0\,.$$

Here, at least one of c_i's is not zero, because g is non-degenerate. Putting $d_{N+1}=0$, we have the desired relation (3.8).

Now, let us assume that Φ is non-degenerate. We denote by $u_1:u_2:\cdots:u_{2N+1}$ the above fixed homogeneous coordinates on $P^{2N}(C)$ and consider $2N+2$ hyperplanes

$$\tilde{H}_i:u_i=0\qquad 1\leqq i\leqq 2N+1\,,$$
$$\tilde{H}_{2N+2}:u_1+u_2+\cdots+u_{2N+1}=0$$

in $P^{2N}(C)$, which are located in general position. Then,

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$$\begin{aligned}\nu(\Phi, H_i) = \nu_{\tilde{g}} &= \nu(g, H) - \min(\nu(f, H), \nu(g, H)) \qquad \text{if } 1 \leqq i \leqq N+1\,,\\ = \nu_{\tilde{f}} &= \nu(f, H) - \min(\nu(f, H), \nu(g, H)) \qquad \text{if } N+2 \leqq i \leqq 2N+1\,.\end{aligned}$$

Moreover, since $\sum_{1\leqq i\leqq N+1}(\tilde{g}f_i - \tilde{f}g_i) = 0$,

$$\nu(\Phi, H_{2N+2}) = \nu_{\tilde{f}g_{N+1}} = \nu_{\tilde{f}}\,.$$

We apply here Theorem 3.4 to a holomorphic map Φ of C into $P^{2N}(C)$ and hyperplanes $H_1, \cdots, H_{2N+2}$. We have

$$(3.12)\qquad \begin{aligned}T(r, \Phi) &\leqq \sum_{1\leqq j\leqq 2N+2} N_{2N}(r, \Phi, \tilde{H}_j) + O(\log rT(r, \Phi)) \quad \|\\ &\leqq (N+1)(N'(r, f) + N'(r, g)) + O(\log rT(r, \Phi)) \quad \|\,.\end{aligned}$$

Put

$$\begin{aligned}u_\Phi &:= \max(\log|f_1\tilde{g}|, \cdots, \log|f_{N+1}\tilde{g}|, \log|g_1\tilde{f}|, \cdots, \log|g_N\tilde{f}|)\\ u_f &:= \max(\log|f_1|, \log|f_2|, \cdots, \log|f_{N+1}|)\\ u_g &:= \max(\log|g_1|, \cdots, , \log|g_N|) = \max(\log|g_1|, \cdots, \log|g_{N+1}|)\,,\end{aligned}$$

where we used a reduced representation of g with $g_{N+1} \equiv 1$. Then,

$$u_\Phi(z) \geqq \begin{cases} u_f(z) + \log|\tilde{g}(z)| \\ u_g(z) + \log|\tilde{f}(z)|\,. \end{cases}$$

Taking the mean value of each term on $\{z \in C; |z| = r\}$, we obtain by (3.1)

$$\begin{aligned}&T(r, \Phi) + u_\Phi(0)\\ &\qquad \geqq \begin{cases} T(r, f) + u_f(0) + N(r, \tilde{g}) + \log|\tilde{g}(0)| \\ T(r, g) + u_g(0) + N(r, \tilde{f}) + \log|\tilde{f}(0)|\,. \end{cases}\end{aligned}$$

Here, $N(r, \tilde{g}) = N'(r, f)$ and $N(r, \tilde{f}) = N'(r, g)$. So, by (3.12),

$$(3.13)\qquad \begin{aligned}T(r, f) + T(r, g) &\leqq 2T(r, \Phi) - N'(r, f) - N'(r, g) + O(1)\\ &\leqq (2N+1)(N'(r, f) + N'(r, g)) + O(\log rT(r, \Phi)) \quad \|\,.\end{aligned}$$

On the other hand, since

$$\max(|f_1\tilde{g}|, \cdots, |f_{N+1}\tilde{g}|, |g_1\tilde{f}|, \cdots, |g_N\tilde{f}|) \leqq (N+1)(\max_i|g_i|)\times(\max_j|f_j|)/|h|\,,$$

we have

$$u_\Phi(z) \leqq u_f(z) + u_g(z) - \log|h| + \log(N+1)$$

and by (3.1)

$$\frac{1}{2\pi}\int_0^{2\pi}\log|h(re^{i\theta})|\,d\theta = N(r, h) + \log|h(0)|\,.$$

Therefore,

$$\begin{aligned} T(r, \Phi) &\leqq T(r, f) + T(r, g) - N(r, h) + O(1) \\ &\leqq T(r, f) + T(r, g) + O(1)\,. \end{aligned}$$

By (3.13) and Lemma 3.10, we can conclude

$$T(r, f) + T(r, g) \leqq O(\log r) + O(\log (T(r, f) + T(r, g))) \quad \|\,.$$

If f or g is transcendental, then

$$\lim_{r\to\infty}\frac{\log r}{T(r, f) + T(r, g)} = 0\,.$$

Factoring each term of the above inequality by $T(r, f) + T(r, g)$ and tending r to the infinity, we have an absurd inequality. In the case that f and g are both rational, the remaining terms of the obtained inequalities in the above arguments can be replaced by $O(1)$. We have a contradiction in this case too. Therefore, Φ is degenerate and hence Proposition 3.7 is completely proved.

§4. The Proofs of Theorems I and II

We shall prove first Theorem II stated in §1 for the case $n = 1$. Let f, g be algebraically non-degenerate holomorphic maps of $\boldsymbol{C}$ into $P^N(\boldsymbol{C})$ such that there are hyperplanes H_i $(1 \leqq i \leqq N + 2)$ in general position satisfying the condition $\nu(f, H_i) = \nu(g, H_i) = 0 (1 \leqq i \leqq N + 1)$ and

$$\min(\nu(f, H_{N+2}), N) = \min(\nu(g, H_{N+2}), N)\,.$$

As was shown in §3, if we choose homogeneous coordinates such that

$$\text{(4.1)} \qquad \begin{aligned} &H_i : w_i = 0 \qquad 1 \leqq i \leqq N + 1 \\ &H_{N+2} : w_1 + \cdots + w_{N+1} = 0 \end{aligned}$$

and reduced representations $f = f_1 : f_2 : \cdots : f_{N+1}$, $g = g_1 : g_2 : \cdots : g_{N+1}$, we have the relation (3.8) for some constants c_i and d_j, where f_i and g_j have no zeros.

We put $a_{ij} = f_i g_j$ and consider the set

$$A := \{a_{ij}\,; 1 \leqq i, j \leqq N + 1\}\,.$$

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And, we define subsets C of A and Γ of $C\times C$ as

$$C := \{a_{ij}\in A\,;\, c_i - d_j \neq 0 \text{ for constants } c_i, d_j \text{ as in (3.8)}\}\,,$$
$$\Gamma := \{(a_{ij}, a_{k\ell})\,;\, a_{ij}/a_{k\ell} \text{ is of constant and } (i,j)\neq(k,\ell)\}$$

respectively. For these sets, we shall show that the assumption (A_0) $\sim(A_3)$ in § 2 are all satisfied. The assumption (A_0) is obviously valid. If $c_{i_1} - d_{j_1} = 0$, $c_{i_2} - d_{j_2} = 0$ and $c_{i_1} - d_{j_2} = 0$ $(1 \leqq i_1, i_2, j_1, j_2 \leqq N+1)$, then

$$c_{i_2} - d_{j_2} = (c_{i_2} - d_{j_1}) + (d_{j_1} - c_{i_1}) + (c_{i_1} - d_{j_2}) = 0\,,$$

whence (A_1) is satisfied.

The assumption (A_2) can be easily seen by the relation (3.8) and the following classical theorem of E. Borel,

THEOREM 4.2 ([1]). *Let $h_1, h_2, \cdots, h_p$ be nowhere vanishing holomorphic functions on* **C** *satisfying the relation*

$$h_1 + h_2 + \cdots + h_p = 0\,.$$

Then, there is a partition of the set of indices $I := \{1, 2, \cdots, p\}$ into the disjoint union of subsets

$$I = I_1 \cup \cdots \cup I_k$$

such that for any $i, j \in I_\kappa$ $h_i/h_j \equiv$ const. and

$$\sum_{i\in I_\kappa} h_i \equiv 0 \qquad (1 \leqq \kappa \leqq k)\,.$$

Particularly, for any $i = 1, \cdots, p$, there is some j such that $i \neq j$ and $h_i/h_j \equiv$ const.

To verify the assumption (A_3), we take $a_{i_\sigma j_\sigma}$ and $a_{k_\sigma \ell_\sigma}$ $(1 \leqq \sigma \leqq s)$ in **C** satisfying the condition

$$(i_1, j_1) \leftrightarrow (k_1, \ell_1), (i_2, j_2) \leftrightarrow (k_2, \ell_2), \cdots, (i_s, j_s) \leftrightarrow (k_s, \ell_s)\,,$$

namely, $f_{i_\sigma} g_{j_\sigma}/f_{k_\sigma} g_{\ell_\sigma} \equiv$ const. $(1 \leqq \sigma \leqq s)$. This implies that

$$f_{i_1} f_{i_2} \cdots f_{i_s} g_{j_1} g_{j_2} \cdots g_{j_s} = c f_{k_1} f_{k_2} \cdots f_{k_s} g_{\ell_1} g_{\ell_2} \cdots g_{\ell_s}$$

for some constant c. If $\{i_1, \cdots, i_s\} = \{k_1, \cdots, k_s\}$, we have a relation

$$g_{j_1} g_{j_2} \cdots g_{j_s} = c g_{\ell_1} g_{\ell_2} \cdots g_{\ell_s}\,.$$

On the other hand, there is no algebraic relation among $g_1, \cdots, g_{N+1}$

because g is assumed to be algebraically non-degenerate. We can conclude $\{j_1, j_2, \cdots, j_s\} = \{\ell_1, \ell_2, \cdots, \ell_s\}$. Similarly, $\{j_1, \cdots, j_s\} = \{\ell_1, \cdots, \ell_s\}$ implies $\{i_1, \cdots, i_s\} = \{k_1, \cdots, k_s\}$. This shows that the assumption (A_3) is also satisfied.

By virtue of Main Lemma, we can conclude that, after a suitable change of indices i and j of f_i and g_j individually,

$$f_i g_j / f_k g_\ell \equiv \text{const.}$$

if and only if $(i, j) = (\ell, k)$ for any (i, j) and (k, ℓ) with $c_i - d_j \neq 0$ and $c_k - d_\ell \neq 0$. Moreover, by the relation (3.8) and Theorem 4.2, we have

$$f_i g_j - f_j g_i \equiv 0$$

for any i, j with $a_{ij} \in C$. In particular, as a result of (i) of Main Lemma,

$$f_i g_j = f_j g_i$$

if $m + 1 \leqq i \leqq N + 1, 1 \leqq j \leqq m$ or $1 \leqq i \leqq m, m + 1 \leqq j \leqq N + 1$. Easily we see

$$\frac{f_1}{g_1} = \frac{f_2}{g_2} = \cdots = \frac{f_{N+1}}{g_{N+1}}\,.$$

Going back to the original indices, this shows that there is a permutation $\pi = \begin{pmatrix} 1, 2, \cdots, N+1 \\ \pi_1, \pi_2, \cdots, \pi_{N+1} \end{pmatrix}$ such that

$$\frac{f_1}{g_{\pi_1}} = \frac{f_2}{g_{\pi_2}} = \cdots = \frac{f_{N+1}}{g_{\pi_{N+1}}}\,.$$

Therefore, f and g are related as $L \cdot g = f$ with a projective transformation

$$L : w'_i = w_{\pi_i} \qquad 1 \leqq i \leqq N + 1\,.$$

Let us prove Theorem II for the general case. Let f, g be meromorphic maps which satisfy the conditions as in Theorem II, where we assume $f(0), g(0) \notin H_{N+2}$. Choosing homogeneous coordinates as in (4.1), we take representations $f = f_1 : f_2 : \cdots : f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$ with nowhere zero holomorphic functions $f_1, f_2, \cdots, f_{N+1}, g_1, g_2, \cdots, g_{N+1}$. For any $a = (a_1, a_2, \cdots, a_{N+1}) \in C^{N+1} - \{0\}$, we consider a holomorphic map f_a of C into $P^N(C)$ defined as

$$f_a(z) = f_1(az) : f_2(az) : \cdots : f_{N+1}(az) \quad (z \in \boldsymbol{C})$$

where $az = (a_1 z, a_2 z, \cdots, a_{N+1} z)$. And, we define a map $g_a : \boldsymbol{C} \to P^N(\boldsymbol{C})$ similarly by g. Then, the following fact is valid.

LEMMA 4.3. *Let E be the set of all $a \in \boldsymbol{C}^n - \{0\}$ such that $\nu(f_a, H_{N+2})(z) \neq \nu(f, H_{N+2})(az)$ or $\nu(g_a, H_{N+2})(z) \neq \nu(g, H_{N+2})(az)$ for some z. Then, for the canonical map $\varpi : (z_1, \cdots, z_n) \in \boldsymbol{C}^n - \{0\} \mapsto z_1 : \cdots : z_n \in P^{n-1}(\mathrm{C})$, the set $\varpi(E)$ is nowhere dense in $P^{n-1}(\boldsymbol{C})$.*

For the proof, see e.g., [3], Proposition 2.7, p. 275.

Let S_{N+1} be the set of all permutations of indices $1, 2, \cdots, N+1$. By L_π we denote the projective linear transformation of $P^N(\boldsymbol{C})$ defined as

$$L_\pi : w'_i = w_{\pi_i} \qquad (1 \leqq i \leqq N+1)$$

for each $\pi = \begin{pmatrix} 1, 2, \cdots, N+1 \\ \pi_1, \pi_2, \cdots, \pi_{N+1} \end{pmatrix} \in S_{N+1}$. For any a in $\boldsymbol{C}^n - (E \cup \{0\})$, since f_a and g_a satisfy the assumptions of Theorem II as holomorphic maps of $\boldsymbol{C}$ into $P^N(\boldsymbol{C})$, applying Theorem II for the case $n = 1$, we can conclude that $L_\pi \cdot g_a = f_a$ for some $\pi \in S_{N+1}$. Let F_π be the set of all points a in $\boldsymbol{C}^n - (E \cup \{0\})$ such that $L_\pi \cdot g_a = f_a$. Then, $\boldsymbol{C}^n - (E \cup \{0\}) = \bigcup_{\pi \in S_{N+1}} F_\pi$. Each F_π is an analytic subset of $\boldsymbol{C}^n - (E \cup \{0\})$. In this situation, it can be easily seen that $F_{\pi_0} = \boldsymbol{C}^n - (E \cup \{0\})$ for some π_0. This shows that Theorem II is also true for the case $n \geqq 2$.

We shall prove next Theorem I. Let f, g be algebraically nondegenerate meromorphic maps of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ such that $\nu(f, H_i) = \nu(g, H_i) = 0$ for $i = 1, \cdots, N+1$ and

$$\min(\nu(f, H_j), N) = \min(\nu(g, H_j), N)$$

for $j = N+2, \cdots, 2N+3$. Apply Theorem II to $N+2$ hyperplanes $H_1, H_2, \cdots, H_{N+1}$ and H_i for each $i = N+2, \cdots, 2N+3$. There is a projective linear transformation L_i such that $L_i \cdot g = f$ and L_i permutes hyperplanes $H_1, \cdots, H_{N+1}$ and fixes H_i. By the assumption of nondegeneracy, we have easily $L := L_{N+2} = \cdots = L_{2N+3}$. This implies that L fixes $N+2$ hyperplanes $H_{N+2}, \cdots, H_{2N+3}$ in general position. It follows that $L =$ identity and so $f = g$, which completes the proof of Theorem I.

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