ON CERTAIN CYCLES IN GRAPHS

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1. Introduction

We show that every simple graph of order $2r$ and minimum degree $\geqq 4r/3$ has the property that for any partition of its vertex set into 2-subsets, there is a cycle which contains exactly one vertex from each 2-subset. We show that the bound $4r/3$ cannot be lowered to r, but conjecture that it can be lowered to $r+1$.

2. Definitions

Throughout this paper n will denote a positive integer and r an integer $\geqq 3$. Given a real number x, $\lceil x \rceil$ will denote the least integer $\geqq x$. Our basic graph-theoretic terminology and notation is that of Bondy and Murty (**2**), save that we use the word "graph" to mean "simple graph".

We shall require the following notation. Let G be a graph, let $\xi, \eta \in V(G)$, let $W \subset V(G)$ and let $H \subset G$. Then $\tilde{W}$ denotes $V(G) \backslash W$, $\tilde{H}$ denotes $G[V(G) \backslash V(H)]$, $e(\xi, W)$ denotes the number of edges incident with ξ whose other end lies in W, $e(\xi, \eta)$ denotes $e(\xi, \{\eta\})$ and $W\delta$ denotes the set of edges exactly one of whose ends lies in W. If $\xi \in V(H)$, then the *H-degree* of ξ is the degree of ξ in the graph H.

Let G be a graph of order nr and let $\Pi = \{V_1, V_2, \ldots, V_r\}$ be a partition of $V(G)$ into n-subsets. We say that Π is an *n-partition* of G. If H is a cycle of G such that $|V(H) \cap V_i| = 1$ for $i = 1, 2, \ldots, r$, then we say that H is a Π-*cycle* of G. If there exists a Π-cycle of G, then we say that G is Π-*round*. If G is Π-*round* for all n-partitions Π of G, then we say that G is *n-round*.

Our aim is to seek bounds for the least integer $p = h(n, r)$ so that if $|V(G)| = nr$ and $\delta(G) \geqq p$, then G is n-round. Most of our attention will be devoted to the case $n = 2$.

Since a graph is 1-round if and only if it has a hamiltonian cycle, a well-known theorem of Dirac (**4**) determines that $h(1, r) = \lceil r/2 \rceil$. An unpublished result of Graver (see (**1**)) implies that $h(n, 3) = 2n$. In the next section we shall consider the function $h(2, r)$.

3. 2-round graphs

The theorem of Dirac mentioned in Section 2 implies that $h(2, r) \leqq \lceil 3r/2 \rceil$; if $|V(G)| = 2r$ and $\delta(G) \geqq 3r/2$, then any r vertices of G induce a subgraph with minimum degree $\geqq r/2$, which has, by Dirac's Theorem, a hamiltonian cycle. If G_r is the union of two complete graphs of order $r+1$ which share just two vertices ξ and η, then G_r is not

Π-round for any 2-partition Π of G_r which contains $\{\xi, \eta\}$. Since $\delta(G_r) = r$, this implies $h(2, r) \geqq r+1$. We conjecture that in fact $h(2, r) = r+1$. In the remainder of this section we prove that $h(2, r) \leqq \lceil 4r/3 \rceil$.

Theorem 1. *If* $|V(G)| = 2r$, Π *is a 2-partition of* G *and* $d(\xi)+d(\eta) \geqq (8r-1)/3$ *whenever* $\{\xi, \eta\} \in \Pi$, *then* G *is* Π*-round.*

Proof. For each $\xi \in V(G)$, let ξ' denote the other vertex of G in the same cell of Π as ξ. Let W be a transversal of Π such that $|W\delta|$ is as small as possible and let $H = G[W]$. Let $\xi \in W$. By the minimality of $|W\delta|$,

$$\begin{aligned} 0 &\leqq |[(W \setminus \{\xi\}) \cup \{\xi'\}]\delta| - |W\delta| \\ &= [e(\xi, W) + e(\xi', \tilde{W}) + e(\xi, \xi')] - [e(\xi, \tilde{W}) + e(\xi', W) - e(\xi, \xi')] \\ &= e(\xi, W) + e(\xi', \tilde{W}) + 2e(\xi, \xi') - [d(\xi) - e(\xi, W) + d(\xi') - e(\xi', \tilde{W})] \\ &= 2[e(\xi, W) + e(\xi', \tilde{W}) + e(\xi, \xi')] - [d(\xi) + d(\xi')] \\ &\leqq 2[e(\xi, W) + e(\xi', \tilde{W}) + e(\xi, \xi')] - (8r-1)/3 \end{aligned}$$

and so

$$e(\xi, W) + e(\xi', \tilde{W}) \geqq (8r-1)/6 - e(\xi, \xi') \geqq (8r-7)/6.$$

Hence

$$d_H(\xi) + d_{\tilde{H}}(\xi') \geqq (8r-7)/6. \tag{1}$$

Since ξ was an arbitrary vertex in W, it follows that there are either $\geqq r/2$ vertices of H-degree $\geqq (8r-7)/12$ or $\geqq r/2$ vertices of $\tilde{H}$-degree $\geqq (8r-7)/12$. In addition, from (1) it follows that $\delta(H) \geqq (2r-1)/6$ and $\delta(\tilde{H}) \geqq (2r-1)/6$. By a theorem of Chvátal (**3**) it follows that either H or $\tilde{H}$ has a hamiltonian cycle, which is a Π-cycle of G. Hence G is Π-round.

Corollary 2. $h(2, r) \leqq \lceil 4r/3 \rceil$.

We now know that $r+1 \leqq h(2, r) \leqq \lceil 4r/3 \rceil$. The upper and lower bounds coincide for $r = 3$. We can show that $h(2, 4) = 5$ and $h(2, 5) = 6$, confirming our conjecture and improving on Corollary 2 in the cases $r = 4$ and 5.

4. Application to polar graphs

In a series of papers (**5**), (**6**), (**7**), (**8**) Zelinka introduced to the literature the concepts of polar graphs and polarised graphs first defined by F. Zitek. The results of Section 3 can be interpreted in the context of hamiltonian homopolar cycles in polar graphs. The definitions relevant to the present section can be found in the papers of Zelinka.

Let $\theta(r)$ be the least integer y so that if P is a polar graph of order r each of whose poles has degree (see (**8**)) $\geqq y$, then P has a hamiltonian homopolar (see (**6**)) cycle.

Proposition 3. $\theta(r) = h(2, r) - 1$.

Sketch of proof. Given a graph G of order $2r$ with a 2-partition $\Pi = \{\{\xi_1, \eta_1\}, \{\xi_2, \eta_2\}, \ldots, \{\xi_r, \eta_r\}\}$, form a polar graph $P(G, \Pi)$ by merging each ξ_i and η_i into one vertex ζ_i of $P(G, \Pi)$, assigning incidences of edges with ξ_i to one pole of ζ_i and incidences with η_i to the other pole of ζ_i. This construction can be reversed, to produce from a polar graph P a graph $G(P)$ together with a 2-partition $\Pi(P)$ of $G(P)$, where for the purposes of the present investigation we insist on the vertices in a cell of $\Pi(P)$ being adjacent.

A Π-cycle of G corresponds to a hamiltonian homopolar cycle in $P(G, \Pi)$, and a hamiltonian homopolar cycle in P corresponds to a $\Pi(P)$-cycle in $G(P)$. By means of this correspondence it follows that $\theta(r) = h(2, r) - 1$.

Corollary 4. $r \leqq \theta(r) \leqq \lceil 4r/3 \rceil - 1$.

To close we remark that the conjecture of Section 3 is equivalent to $\theta(r) = r$.

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