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Hyperplanes of the Form $f_1(x, y)z_1 + \cdots + f_k(x, y)z_k + g(x, y)$ Are Variables

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Abstract. The Abhyankar–Sathaye Embedded Hyperplane Problem asks whether any hypersurface of $\mathbb{C}^n$ isomorphic to $\mathbb{C}^{n-1}$ is rectifiable, *i.e.,* equivalent to a linear hyperplane up to an automorphism of $\mathbb{C}^n$. Generalizing the approach adopted by Kaliman, Vénéreau, and Zaidenberg, which consists in using almost nothing but the acyclicity of $\mathbb{C}^{n-1}$, we solve this problem for hypersurfaces given by polynomials of $\mathbb{C}[x, y, z_1, \ldots, z_k]$ as in the title.

The result announced in the title corresponds to the implication (iv) ⇒ (v) in the Main Theorem below. Case $k = 1$ is a well-known result appearing in [Rus76, Sat76, KZ99]; case $k = 2$ can be found in [KVZ04, Th.3.24] or in [KVZ01, Th.2.5]. Before we state this theorem let us clarify the definitions:

- we choose to consider *automorphisms* as invertible endomorphisms of the $\mathbb{C}$-algebras of polynomials $\mathbb{C}[x, y, z_1, \ldots, z_k]$, $\mathbb{C}[x, y]$, *etc.*;
- an *x-automorphism* is an automorphism α such that $\alpha(x) = x$;
- a *variable*, resp. an *x-variable*, is a polynomial v such that $v = \alpha(y)$ for a certain automorphism, resp. x-automorphism, α.

Main Theorem *Let $p = p(x, y, \bar{z}) \in \mathbb{C}[x, y, \bar{z}] = \mathbb{C}[x, y, z_1, \ldots, z_k]$ be a polynomial of degree one in $\bar{z}$,* i.e., *p is of the form*

$$p(x, y, \bar{z}) = f_1(x, y)z_1 + \cdots + f_k(x, y)z_k + g(x, y).$$

Let $X \subset \mathbb{C}^{2+k}_{x,y,\bar{z}}$ be the hypersurface given by the equation $p = 0$. Then the five following assertions are equivalent:

(i) *X is smooth, irreducible and acyclic,* i.e., $\tilde{H}_*(X; \mathbb{Z}) = 0$.

(ii) *Up to an automorphism of $\mathbb{C}[x, y]$ (naturally extended to $\mathbb{C}[x, y, \bar{z}]$), p has the form:*

$$p = h(x)(\tilde{f}_1(x, y)z_1 + \cdots + \tilde{f}_k(x, y)z_k) + g(x, y)$$

where $\bigcap_{i=1}^k \tilde{f}_i^{-1}(0)$ is a finite subset of the parallel lines $h^{-1}(0)$ and

$$\deg_y(g(x_0, y)) = 1, \ \forall x_0 \in h^{-1}(0)$$

(where $h^{-1}(0)$ is first considered as a subset of $\mathbb{C}^2_{x,y}$ and, secondly, as a subset of $\mathbb{C}_x$).

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(iii) *Up to an automorphism of $\mathbb{C}[x,y]$, p is an x-variable.*
(iv) *The polynomial p is a hyperplane or, equivalently, X is isomorphic to $\mathbb{C}^{k+1}$.*
(v) *The polynomial p is a variable or, equivalently, X is rectifiable.*

Remark 1 In the Main Theorem above, the notation $\mathbb{C}[x,y,z_1,\dots,z_k]$ and the assumption that p has degree one in $\bar{z}$ imply that $k \geq 1$ and the f_i are not all zero. However it is worth noticing that whenever $k=0$ or all the f_i are zero, the assertions (i), (iv) and (v) still make sense and are still equivalent, provided that $p(x,y,z_1,\dots,z_k)=g(x,y)$ is irreducible (a usual precaution due to the fact that $g^{-1}(0)$ can be irreducible while $g=h^n$ is not, which turns out unnecessary in the theorem since $p=f_1z_1+\cdots+f_kz_k+g$ is clearly not a power of another polynomial). Indeed, in this special case the canonical projection $X\simeq\mathbb{C}^k\times g^{-1}(0)\to g^{-1}(0)$ is clearly a homotopy equivalence. Hence X is smooth irreducible and acyclic if and only if $g^{-1}(0)$ is; it is a well-known result that $g(x,y)$ is then a line and by the Abhyankar–Moh–Suzuki theorem [AM75, Sat76] $g(x,y)$ is a variable(of $\mathbb{C}[x,y]$), Hence $p(x,y,z_1,\dots,z_k)=g(x,y)$ is a variable (of $\mathbb{C}[x,y,z_1,\dots,z_k]$).

We now turn to the proof of the Main Theorem; the implications (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (i) being obvious, the rest of the article is dedicated to the proof of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

The injection:

$$\text{(1)}\qquad \mathbb{C}[x,y]\hookrightarrow\mathbb{C}[X]={}^{\mathbb{C}[x,y][z_1,\dots,z_k]}\Big/_{(f_1z_1+\cdots+f_kz_k+g)}$$

corresponds to a morphism $\sigma\colon X\to\mathbb{C}^2$ with general fibers

$$\sigma^{-1}(x_0,y_0)=\{z_1,\dots,z_k|f_1(x_0,y_0)z_1+\cdots+f_k(x_0,y_0)z_k+g(x_0,y_0)=0\}$$

isomorphic to $\mathbb{C}^{k-1}$ ($\dim X=k+1$). Clearly, we have an isomorphism, for all $i=1,\dots,k$ such that $f_i\neq0$ (such an f_i exists, as was noticed in Remark 1 above):

$${}^{\mathbb{C}[x,y]_{f_i}[z_1,\dots,z_k]}\Big/_{(f_1z_1+\cdots+f_kz_k+g)}\simeq\mathbb{C}[x,y]_{f_i}{}^{[k-1]}.$$

Letting $D:=V(f_1,\dots,f_k)\subset\mathbb{C}^2$ and $D':=\sigma^{-1}(D)\subset X$, implies that the restriction

$$\sigma_{|X\setminus D'}\colon X\setminus D'\to\mathbb{C}^2\setminus D,$$

is locally trivial in the Zariski topology, *i.e.,* a fiber bundle, with affine space fibers. Observe that $D'\simeq C\times\mathbb{C}^k$ where $C:=V(f_1,\dots,f_k,g)=D\cap g^{-1}(0)\subset D$. We remark that C must be a finite set as soon as X is irreducible. Let $h(x,y)$ be the greatest common divisor of $f_1,\dots,f_k$. One has

$$p=f_1z_1+\cdots+f_kz_k+g=h(\tilde{f}_1z_1+\cdots+\tilde{f}_kz_k)+g$$

where $\tilde{f}_1, \dots, \tilde{f}_k$ have no common divisor. Again we define:

$$\begin{array}{rclcl} \hat{D} & := & h^{-1}(0) \subset & D & \\ \cup & & & \cup & \text{and} \quad \hat{D}' := \sigma^{-1}(\hat{D}) \simeq \hat{C} \times \mathbb{C}^k. \\ \hat{C} & := & \hat{D} \cap g^{-1}(0) \subset & C & \end{array}$$

Let $\hat{D} = \bigcup_{i=1}^{n} D_i$ and $\hat{D}' = \bigcup_{j=1}^{n'} D'_j$ be the decomposition into irreducible components regarded as Cartier divisors. Letting

$$\sigma^*(D_i) = \sum_{j=1}^{n'} m_{ij} D'_j, \quad i = 1, \dots, n, \tag{2}$$

we consider the $n \times n'$ *multiplicity matrix* $M_\sigma = (m_{ij})$ with non-negative integer entries. The first step in the proof of (i) $\Rightarrow$ (ii) is the following generalization of [KVZ04, Prop. 1.5(a)]:

Lemma 2 *If X is as in* (i), i.e., *X is smooth, irreducible and acyclic, then $n = n'$ and M_σ is unimodular.*

Proof By [Fuj82, 1.18–1.20] (see also [Kal94, 3.2]) the algebra $\mathbb{C}[X]$ is a UFD and its invertible elements are constants (and the same is true for $\mathbb{C}[x, y]$). Hence there are irreducible elements $h_1, \dots, h_n \in \mathbb{C}[x, y]$ and $h'_1, \dots, h'_n \in \mathbb{C}[X]$ such that $D_i = h_i^{-1}(0)$, $i = 1, \dots, n$ and $D'_j = h_j'^{-1}(0)$, $j = 1, \dots, n'$. In view of the injection (1), one can identify elements of $\mathbb{C}[x, y]$ and their images in $\mathbb{C}[X]$ and they then have two different decompositions, as seen as in $\mathbb{C}[x, y]$ or as in $\mathbb{C}[X]$. To sum up one has:

$$\hat{D} = \bigcup_{i=1}^{n} D_i \text{ is given by } h = \prod_{i=1}^{n} h_i^{a_i} \text{ in } \mathbb{C}[x, y];$$

$$\hat{D}' = \bigcup_{j=1}^{n'} D'_j \text{ is given by } h = \prod_{j=1}^{n'} h_j'^{a'_j} \text{ in } \mathbb{C}[X]$$

and $\forall i = 1, \dots, n$,

$$\sigma^*(D_i) = \sum_{j=1}^{n'} m_{ij} D'_j \text{ is given by } h_i = \lambda_i \prod_{j=1}^{n'} h_j'^{m_{ij}} \text{ in } \mathbb{C}[X]$$

(where $\lambda_i \in \mathbb{C}^*$).

There exists at least one $\tilde{f}_i$ coprime with h. Without loss of generality one can assume that $\tilde{f}_1$ is so. Now we note that we have another injection,

$$\mathbb{C}[x, y, z_2, \dots, z_k] \hookrightarrow \mathbb{C}[X] = {}^{\mathbb{C}[x, y, z_2, \dots, z_k][z_1]} \big/ {}_{(f_1 z_1 + \cdots + f_k z_k + g)}$$

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actually $\mathbb{C}[X]$ can be regarded as a *simple birational extension* (see [KVZ01, KVZ04]) of the algebra $A := \mathbb{C}[x, y, z_2, \ldots, z_k]$:

$$\mathbb{C}[X] = \mathbb{C}[x, y, z_2, \ldots, z_k][z_1] \Big/ (f_1 z_1 + \cdots + f_k z_k + g) \simeq A\Big[\frac{r}{q}\Big] \subset A_q$$

where

$$\begin{cases} q = f_1 = h\tilde{f}_1, \\ r = h(\tilde{f}_2 z_2 + \cdots f_k z_k) + g. \end{cases}$$

Here again, in view of the injection $A \hookrightarrow \mathbb{C}[X]$, one can decompose q in A and then in $\mathbb{C}[X]$:

$$h\tilde{f}_1 = q = \prod_{i=1}^{m} q_i^{a_i} = \prod_{i=1}^{n} h_i^{a_i} \prod_{i=n+1}^{m} q_i^{a_i} \text{ in } (\mathbb{C}[x, y] \subset)A \text{ where } q_i = h_i,\ \forall i = 1, \ldots, n$$

$$q = \prod_{j=1}^{m'} {q'}_j^{a'_j} = \prod_{j=1}^{n'} {h'}_j^{a'_j} \prod_{j=n'+1}^{m'} {q'}_j^{a'_j} \text{ in } \mathbb{C}[X] \text{ where } q'_j = h'_j,\ \forall j = 1, \ldots, n'$$

and hence, for every $i = 1, \ldots, m$, there exist non-negative integers $m_{i1}, \ldots, m_{im'}$ such that

$$q_i = \lambda_i \prod_{j=1}^{m'} {q'}_j^{m_{ij}} \quad (\lambda_i \in \mathbb{C}^*). \tag{3}$$

The matrix M_σ is a submatrix of the $m \times m'$ matrix $M_1 := (m_{ij})$, *i.e.*,

$$M_1 = \left[\begin{array}{c|c} M_\sigma & * \\ \hline * & * \end{array}\right].$$

Now, identifying $\mathbb{C}[X]$ and $A\big[\frac{r}{q}\big] \subset A_q$ one has

$$\forall j = 1, \ldots, m', \quad q'_j = \frac{s_j}{q^N} \text{ with } s_j \in A,\ N \in \mathbb{N}$$

and, by (3),

$$q_i = \lambda_i \prod_{j=1}^{m'} \Big(\frac{s_j}{q^N}\Big)^{m_{ij}}.$$

Multiplying the last equality by a sufficiently large power of q, one obtains an equality in $(\mathbb{C}[x, y] \subset)A$ which implies that for every $j = 1, \ldots, m'$, there exist integers $m'_{j1}, \ldots, m'_{jn} \in \mathbb{Z}$ such that

$$q'_j = \lambda'_j \prod_{i=1}^{m} q_i^{m'_{ji}} \quad (\lambda'_j \in \mathbb{C}^*). \tag{4}$$

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Let M_1' be the $m' \times m$ matrix $M_1' := (m'_{ij})$. Plugging (3) into (4) and (4) into (3) we obtain that $M_1'M_1 = I_{m'}$ and $M_1M_1' = I_m$ where I_ω denotes the identity matrix of order ω. Hence $m = m'$ and M_1 is unimodular.

Now let us look at the injection $A \hookrightarrow \mathbb{C}[X]$ from the geometrical point of view; it corresponds to a birational morphism $\mu\colon X \to \mathbb{C}^{2+k-1}$ with exceptional divisor $F' := \mu^{-1}(F)$ with $F := q^{-1}(0) \subset \mathbb{C}^{2+k-1}$, the restriction

$$\mu_{|X\setminus F'}\colon X \setminus F' \to \mathbb{C}^{2+k-1} \setminus F$$

being an isomorphism. Here again,

$$F = \bigcup_{i=1}^{m} F_i \text{ is given by } q = \prod_{i=1}^{m} q_i^{a_i} \text{ in } A;$$

$$F' = \bigcup_{j=1}^{m'} F'_j \text{ is given by } q = \prod_{j=1}^{m'} {q'_j}^{a'_j} \text{ in } \mathbb{C}[X]$$

and $\forall i = 1, \dots, m$,

$$\mu^*(F_i) = \sum_{j=1}^{m'} m_{ij} F'_j \text{ is given by } q_i = \lambda_i \prod_{j=1}^{m'} {q'_j}^{m_{ij}} \text{ in } \mathbb{C}[X].$$

We have also that $F' \simeq E \times \mathbb{C}$ where $E := V(q, r) = F \cap r^{-1}(0) \subset F$. Observe also that F is a cylinder above a curve, indeed, $F = \Gamma \times \mathbb{C}^{k-1}$ where Γ is the curve in $\mathbb{C}^2$ defined by $q = 0$ (with q seen as a polynomial in $\mathbb{C}[x, y]$). Therefore, one can define a morphism $\pi\colon E \to \Gamma$ as the restriction to $E \subset \Gamma \times \mathbb{C}^{k-1}$ of the canonical projection $\Gamma \times \mathbb{C}^{k-1} \to \Gamma$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} F' \simeq E \times \mathbb{C} & \xrightarrow{\mu} & \Gamma \times \mathbb{C}^{k-1} \simeq F \\ \downarrow & & \downarrow \\ E & \xrightarrow{\pi} & \Gamma \end{array}$$

Remark that E, resp. Γ, must have the corresponding decomposition, *i.e.*, $E = \bigcup_{j=1}^{m'=m} E_j$, resp. $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$ with $F'_j \simeq E_j \times \mathbb{C}$, resp. $F_i = \Gamma_i \times \mathbb{C}^{k-1}$.

Remark 3 Clearly, $m_{ij} > 0 \Leftrightarrow \mu(F'_j) \subset F_i \Leftrightarrow E_j \subset F_i = \Gamma_i \times \mathbb{C}^{k-1} \Leftrightarrow \pi(E_j) \subset \Gamma_i \Leftrightarrow E_j \subset \pi^{-1}(\Gamma_i)$.

Notice that for each $j = 1, \dots, m$ there exists $i \in \{1, \dots, m\}$ such that $\pi(E_j) \subseteq \Gamma_i$, and this index $i = i(j)$ is unique unless $\pi_{|E_j} = \text{const}$.

We call an irreducible component E_j of E *vertical*[1] if $\pi|_{E_j} = \text{const}$ (*i.e.*, $\deg(\pi|_{E_j}) = 0$) and *non-vertical* otherwise (thus the vertical components of E are disjoint and each of them is isomorphic to $\mathbb{C}^{k-1}$).

[1]This term comes from the picture obtained with $\mathbb{C}^2_{x,y} \times \mathbb{C}^{k-1}$ visualized as a horizontal plane $\mathbb{R}^2_{x,y} \times$ vertical line $\mathbb{R}_z$.

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Remark 4 The uniqueness of the index $i = i(j)$ for a non-vertical component E_j and the unimodularity of M_1 imply that the jth column of the matrix M_1 is the ith vector of the standard basis $(\bar{e}_1, \dots \bar{e}_n)$ in $\mathbb{R}^n$, and two different non-vertical components E_j and $E_{\underline{j}}$ of E project into two different irreducible components Γ_i and $\Gamma_{\underline{i}}$ of Γ.

We remark that

$$\begin{aligned}\bigcup_{j=1}^{m} E_j = E = V(q, r) &= V(h\tilde{f}_1, f_2z_2 + \cdots + f_kz_k + g)\\ &= V\big(h\tilde{f}_1, h(\tilde{f}_2z_2 + \cdots + \tilde{f}_kz_k) + g\big)\\ &= V(h, g) \cup \bigcup_{j=n'+1}^{m} E_j\\ &= \hat{C} \times \mathbb{C}^{k-1} \cup \bigcup_{j=n'+1}^{m} E_j\end{aligned}$$

where $\hat{C}(\subset C)$ is a finite set of points $\{P_1, \dots, P_{n'}\} := \hat{C}$. Actually one has $\forall j = 1, \dots, n'$, $\pi|_{E_j} = \text{const} = P_j$ and thus those components $E_j = \{P_j\} \times \mathbb{C}^{k-1}$ are vertical. Remember also that

$$\mathbb{C}^2 \supset \Gamma = V(q) = V(h\tilde{f}_1) = V(h) \cup V(\tilde{f}_1) = \bigcup_{i=1}^{n} D_i \cup \bigcup_{i=n+1}^{m} \Gamma_i.$$

In view of Remark 3, one can understand how E is positioned in $\Gamma \times \mathbb{C}^{k-1}$ by looking at the matrix M_1:

$$\begin{array}{c} \\ \\ \\ D_1 = \Gamma_1 \\ \vdots \\ D_n = \Gamma_n \\ \Gamma_{n+1} \\ \vdots \\ \Gamma_m \end{array} \quad \begin{array}{c} \begin{array}{cccccc} P_1 \times \mathbb{C}^{k-1} & \cdots & P_{n'} \times \mathbb{C}^{k-1} & & & \\ \| & & \| & & & \\ E_1 & \cdots & E_{n'} & E_{n'+1} & \cdots & E_m \end{array} \\ \left[\begin{array}{c|c} M_\sigma & * \\ \hline * & * \end{array}\right] \end{array}$$

For every $i \in \{1, \dots, m\}$,

$$\pi^{-1}(\Gamma_i) = \Gamma_i \times \mathbb{C}^{k-1} \cap V(f_2z_2 + \cdots + f_kz_k + g).$$

Hence

$$\pi\big(\pi^{-1}(\Gamma_i)\big) \supset \Gamma_i \setminus V(f_2, \dots, f_k).$$

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If $i \geq n+1$, *i.e.*, $\Gamma_i \not\subset h^{-1}(0)$, then $\Gamma_i \setminus V(f_2, \dots, f_k)$ is a dense subset of Γ_i and hence $\pi^{-1}(\Gamma_i)$ must contain at least one non-vertical component. By Remark 4, $\pi^{-1}(\Gamma_i)$ contains exactly one non-vertical component E_j; moreover j must be greater than n', since otherwise we would have:

$$\left[\begin{array}{c} m_{1j} \\ \vdots \\ m_{nj} \\ \hline m_{n+1\,j} \\ \vdots \\ \cdot \\ \vdots \\ m_{mj} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array}\right]$$

with

$$\begin{bmatrix} m_{1j} \\ \vdots \\ m_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

being a column vector of M_σ which is impossible, by definition of M_σ. Since this is true for every $i \geq n+1$ and since, by Remark 3, for two distinct components Γ_i and $\Gamma_{\underline{i}}$, $\mu^{-1}(\Gamma_i)$ and $\mu^{-1}(\Gamma_{\underline{i}})$ can not contain the same non-vertical component E_j, we have that $m - n = m' - n' = m - n'$ and, up to reordering $E_{n+1}, \dots, E_m$:

$$M_1 = \left[\begin{array}{c|c} M_\sigma & 0 \\ \hline * & I_{m-n} \end{array}\right].$$

Hence $n = n'$ and M_σ is unimodular (because M_1 is so). ∎

As we have seen previously, we have a fiber bundle with affine space fibers:

$$\sigma|_{X \setminus D'} : X \setminus D' \to \mathbb{C}^2 \setminus D. \tag{5}$$

This will allow us to link homologies of D, $D' \simeq C \times \mathbb{C}^k$ and C. Actually we will need to consider

- the one point compactification of $\mathbb{C}^2$: $\dot{\mathbb{C}}^2 = \mathbb{C}^2 \cup \{\infty\}$;
- the one point compactification of X: $\dot{X} = X \cup \{\infty\}$;
- and the corresponding new sets:

$$\dot{D} := D \cup \{\infty\} \subset \dot{\mathbb{C}}^2,$$
$$\dot{C} := C \cup \{\infty\} \subset \dot{D},$$
$$\dot{D}' := D' \cup \{\infty\} \subset \dot{X}.$$

We are going to prove the following

Lemma 5 *If X is as in* (i), *then C and D have the same Euler characteristic and there are isomorphisms between the reduced cohomology groups:*

$$\tilde{H}^*(\dot{D};\mathbb{Z}) \simeq \tilde{H}^{*-2}(\dot{C};\mathbb{Z}).$$

Proof The fiber bundle with affine space fibers (5) is between quasi-affine complex varieties. Using the natural homeomorphism $\mathbb{C}^d \approx \mathbb{R}^{2d}$, we obtain a fiber bundle between quasi-affine real varieties but quasi-affine real varieties are actually all affine varieties[2]. Then one can consider the coordinate ring R, resp. R', of the real affine variety homeomorphic to $\mathbb{C}^2 \setminus D$, resp. $X \setminus D'$. In algebraic terms we have that R' is a locally polynomial R-algebra and by the main result of [BCW77] R' is isomorphic to the symmetric algebra of a finitely generated projective R-module. Geometrically, it means that the fiber bundle $\sigma|_{X\setminus D'}$ is equivalent to a real vector bundle (by a morphism between real varieties). But any real vector bundle is homotopy-equivalent to its 0 section; hence $X \setminus D'$ and $\mathbb{C}^2 \setminus D$ are homotopy-equivalent and consequently

$$H_*(X \setminus D';\mathbb{Z}) \simeq H_*(\mathbb{C}^2 \setminus D;\mathbb{Z}). \tag{6}$$

In particular $e(X \setminus D') = e(\mathbb{C}^2 \setminus D)$ where e stands for the Euler characteristic, and, by the additivity of the Euler characteristic (see [Dur87]), $e(X) - e(D') = e(\mathbb{C}^2) - e(D)$. The hypersurface X and the affine plane $\mathbb{C}^2$ being both acyclic, one has $e(X) = e(\mathbb{C}^2) = 1$, hence $e(D) = e(D')$. Moreover C is isomorphic to $C \times (0,\ldots,0)$ which is a deformation retract of $C \times \mathbb{C}^k \simeq D'$. Hence $e(D') = e(C) = e(D)$.

Remark 6 We have $\dot{X} \setminus \dot{D}' = X \setminus D'$ and $\dot{\mathbb{C}}^2 \setminus \dot{D} = \mathbb{C}^2 \setminus D$.

By [KVZ04, Proposition 1.12], $\dot{X}$ is a homology $2(k+1)$-sphere and Alexander duality holds for $\dot{X}$:

$$\tilde{H}_*(\dot{X} \setminus \dot{D}';\mathbb{Z}) \simeq \tilde{H}^{2(k+1)-1-*}(\dot{D}';\mathbb{Z}).$$

Of course the same argument is valid for $\mathbb{C}^2$:

$$\tilde{H}_*(\dot{\mathbb{C}}^2 \setminus \dot{D};\mathbb{Z}) \simeq \tilde{H}^{2\cdot 2-1-*}(\dot{D};\mathbb{Z}).$$

Using isomorphism (6) and Remark 6 one obtains:

$$\begin{array}{ccccc} \tilde{H}_*(X \setminus D';\mathbb{Z}) & \simeq & \tilde{H}_*(\dot{X} \setminus \dot{D}';\mathbb{Z}) & \simeq & \tilde{H}^{2(k+1)-1-*}(\dot{D}';\mathbb{Z}) \\ \wr\| & & & & \\ \tilde{H}_*(\mathbb{C}^2 \setminus D;\mathbb{Z}) & \simeq & \tilde{H}_*(\dot{\mathbb{C}}^2 \setminus \dot{D};\mathbb{Z}) & \simeq & \tilde{H}^{2\cdot 2-1-*}(\dot{D};\mathbb{Z})\ . \end{array}$$

Hence

$$\tilde{H}^*(\dot{D};\mathbb{Z}) \simeq \tilde{H}^{*+2(k-1)}(\dot{D}';\mathbb{Z}). \tag{7}$$

[2]Indeed, $\mathbb{R}^d_{\bar{x}} \supset V\big(p_1(\bar{x}),\ldots,p_m(\bar{x})\big) \setminus V\big(q_1(\bar{x}),\ldots,q_n(\bar{x})\big) \simeq V\big(p_1(\bar{x}),\ldots,p_m(\bar{x}),1-z\sum q_i(\bar{x})^2\big) \subset \mathbb{R}^{d+1}_{\bar{x},z}$.

From $D' \simeq C \times \mathbb{C}^k$ one deduces that $\dot{D}'$ is homeomorphic to $\dot{C} \times \dot{\mathbb{R}^{2k}} = \dot{C} \times \mathbb{S}^{2k}$ quotiented by $\dot{C} \vee \mathbb{S}^{2k} := \dot{C} \times \{\infty\} \cup \{\infty\} \times \mathbb{S}^{2k}$, *i.e.*,

$$\dot{D}' \approx {}^{\dot{C} \times \mathbb{S}^{2k}} \Big/ {}_{\dot{C} \vee \mathbb{S}^{2k}}$$

By [Dol72, V.4.4],

$$\tilde{H}^*(\dot{D}'; \mathbb{Z}) \simeq H^*(\dot{C} \times \mathbb{S}^{2k}, \dot{C} \vee \mathbb{S}^{2k}; \mathbb{Z}),$$

and using the Künneth Theorem for the Cohomology of Product Spaces [Mas80, VIII 11.2] one obtains

$$\tilde{H}^*(\dot{D}'; \mathbb{Z}) \simeq \tilde{H}^{*-2k}(\dot{C}; \mathbb{Z})$$

which, together with (7), yields the conclusion of Lemma 5. ■

Now we assume that (i) holds and prove (ii). From Lemma 5 one can deduce that

(8) $$\tilde{H}_0(\dot{D}; \mathbb{Z}) = 0$$

(using for example the universal coefficient theorem for cohomology groups, see [Mas80, VII 4.3]). Recall that

$$\mathbb{C}^2 \supset D = V(f_1, \dots, f_k) = V(h\tilde{f}_1, \dots, h\tilde{f}_k) = V(h) \cup V(\tilde{f}_1, \dots, \tilde{f}_k),$$
$$D = \hat{D} \amalg D_{\text{fin}}.$$

where D_{fin} is a finite set and $\amalg$ stands for the disjoint union. We have

$$\dot{D} = \dot{\hat{D}} \amalg D_{\text{fin}}$$

where $\dot{\hat{D}} := \hat{D} \cup \{\infty\}$ is a connected subset of $\dot{D}$. Hence

$$\operatorname{rank}\left(\tilde{H}_0(\dot{D}; \mathbb{Z})\right) = 1 + \#D_{\text{fin}} - 1$$

and, by (8), $\#D_{\text{fin}}{=}0$, *i.e.*, $D_{\text{fin}} = \varnothing$. We have proved that

(9) $$V(\tilde{f}_1, \dots, \tilde{f}_k) \subset h^{-1}(0) = \hat{D} = D.$$

We have also

$$\hat{D}' = \sigma^{-1}(\hat{D}) = \sigma^{-1}(D) = D' = \bigcup_{j=1}^{n'} D'_j \simeq \hat{C} \times \mathbb{C}^k$$

and

$$\hat{C} = C = V(h, g) = \{P_1, \dots, P_{n'}\}.$$

Recall that $D = \bigcup_{i=1}^{n} D_i$ and $\sigma^*(D_i) = \sum_{j=1}^{n'} m_{ij} D'_j$, $i = 1, \dots, n$, where $M_\sigma = (m_{ij})$.

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Remark 7 $m_{ij} > 0 \Leftrightarrow P_j \in D_i$.

By Lemma 2, $n = n'$ and, by Lemma 5, $e(C) = e(D)$; hence

$$n = e(D).$$

Let $\mathrm{cc}(D)$ be the set of connected components in D. We have

$$n = e(D) = \sum_{\Delta \in \mathrm{cc}(D)} 1 - \mathrm{rank}\big(H_1(\Delta;\mathbb{Z})\big) \leq \#\mathrm{cc}(D) \leq n.$$

Hence every irreducible component D_i is isolated and acyclic and by Remark 7

$$P_j \in D_i \Leftrightarrow \begin{bmatrix} m_{1j} \\ \vdots \\ \cdot \\ \vdots \\ m_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ m_{ij} > 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By Lemma 2, up to reordering, $M_\sigma = I_n$. It means that if $h = \prod_{i=1}^n h_i$ is the decomposition of h into prime factors in $\mathbb{C}[x,y]$, then every h_i is also prime in $\mathbb{C}[X]$. In other words

$$\mathbb{C}[X] \big/ {(h_i)} \simeq \Big(\mathbb{C}[x,y] \big/ {(h_i, g)}\Big)[\bar{z}]$$

is integral and hence $D_i = h_i^{-1}(0)$ and $g^{-1}(0)$ meet only once and transversally.

If $e(D) = n > 1$ then, by [Zaĭ85][3], up to an automorphism of $\mathbb{C}[x,y]$, $h(x,y)$ is a polynomial in x with n roots.

If $e(D) = n = 1$ then D is an acyclic irreducible curve, *i.e.*, D is homeomorphic to $\mathbb{C}$ and, by [ZL83][3], up to an automorphism of $\mathbb{C}[x,y]$, h is a quasi-homogeneous polynomial: $h(x,y) = x^k - y^l$ with k and l coprime. The fact that $h^{-1}(0)$ and $g^{-1}(0)$ meet only once and transversally implies that the equation $g(t^l, t^k) = 0$ has a unique solution t_0, this solution being different from 0 ($(0,0)$ is the singular point of $x^k - y^l = 0$). We have $g(t^l, t^k) = a(t - t_0)^d$, which is possible only if k or l is equal to 1 since otherwise the derivative of the left-hand side would vanish at $t = 0$. Finally h is equivalent to x up to an automorphism of $\mathbb{C}[x,y]$. We have proved that, up to an automorphism, p has the form:

$$p = h(x)(\tilde{f}_1(x,y)z_1 + \cdots + \tilde{f}_k(x,y)z_k) + g(x,y).$$

The inclusion $\bigcap_{i=1}^k \tilde{f}_i^{-1}(0) \subset h^{-1}(0)$ has already been proved in (9) and

$$\deg_y(g(x_0,y)) = 1, \ \forall x_0 \in h^{-1}(0)$$

[3]Note that this result includes the classical Abhyankar–Moh–Suzuki theorem [AM75, Suz74].

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is just another way to say that every component $x = x_0$ of $h^{-1}(0)$, meets $g^{-1}(0)$ only once and transversally. We have proved the implication (i) ⇒ (ii) in the Main Theorem.

Now let us prove the implication (ii) ⇒ (iii).

Let p be a polynomial as in (ii). We prove that p is an x-variable by induction on, say, "the total intersection number":

$$\iota = \iota(\tilde{f}_1, \dots, \tilde{f}_k) := \dim_{\mathbb{C}} \mathbb{C}[x, y] \Big/ (\tilde{f}_1, \dots, \tilde{f}_k)$$

First if $\iota = 0$ then there exists k polynomials $r_1, \dots, r_k \in \mathbb{C}[x, y]$ such that $\tilde{f}_1 r_1 + \dots + \tilde{f}_k r_k = 1$ and, by the Quillen–Suslin theorem[4] there is a linear automorphism α of $\mathbb{C}[x, y][z_1, \dots, z_k]$ such that $\alpha(z_1) = \tilde{f}_1 z_1 + \dots + \tilde{f}_k z_k$. By assumption on g in (ii), one has

$$g(x, y) = g_0(x) + g_1(x)y + h_{\text{red}}(x) \sum_{i \geq 2} \tilde{g}_i(x) y^i \quad \text{with } g_1 \text{ prime to } h.$$

The polynomial

$$\alpha^{-1}(p) = h(x)z_1 + g(x, y) = h(x)z_1 + g_0(x) + g_1(x)y + h_{\text{red}}(x) \sum_{i \geq 2} \tilde{g}_i(x) y^i$$

is an x-variable by a result due to Russell [Rus76, 2.2] (see also [Vn01, 8.1] and [EV99] for generalizations). Hence p is an x-variable.

Now suppose that $\iota \geq 1$ and that the result is true for any total intersection number less than or equal to $\iota - 1$. Let (x_0, y_0) be in $V(\tilde{f}_1, \dots, \tilde{f}_k)$. Up to a translation one can assume that $x_0 = 0$. One has

$$\tilde{f}_1(0, y)z_1 + \dots + \tilde{f}_k(0, y)z_k = d(y)(\check{f}_1(y)z_1 + \dots + \check{f}_k(y)z_k)$$

where $d(y)$ is the greatest common divisor of $\tilde{f}_1(0, y), \dots, \tilde{f}_k(0, y)$. Again by the Quillen–Suslin theorem[5] there is a linear automorphism α_0 of $\mathbb{C}[y][z_1, \dots, z_k]$ such that $\alpha_0(z_1) = \check{f}_1(y)z_1 + \dots + \check{f}_k(y)z_k$. Extending α_0 to $\mathbb{C}[x, y][z_1, \dots, z_k]$ one has

$$\alpha_0^{-1}(\tilde{f}_1(x, y)z_1 + \dots + \tilde{f}_k(x, y)z_k) \equiv d(y)z_1 \bmod (x)$$

that is to say

$$\alpha_0^{-1}(\tilde{f}_1(x, y)z_1 + \dots + \tilde{f}_k(x, y)z_k) = \check{f}_1(x, y)z_1 + x\check{f}_2(x, y)z_2 + \dots + x\check{f}_k(x, y)z_k$$

and hence

$$\alpha_0^{-1}(p) = h(x)\big(\check{f}_1(x, y)z_1 + x\check{f}_2(x, y)z_2 + \dots + x\check{f}_k(x, y)z_k\big) + g(x, y). \tag{10}$$

[4] Actually here we need a weaker version of this theorem which was proved by Seshadri [Ses58].

[5] Here we only need the "easy" version over a PID ($\mathbb{C}[y]$): *Projective modules of finite type over a PID are free,* which is another formulation of the result we use here.

Given by α_0, or rather its Jacobi matrix, we have the equality of the ideals

$$(\tilde{f}_1, \ldots, \tilde{f}_k) = (\check{f}_1, x\check{f}_2, \ldots, x\check{f}_k),$$

hence

$$\iota(\tilde{f}_1, \ldots, \tilde{f}_k) = \iota(\check{f}_1, x\check{f}_2, \ldots, x\check{f}_k).$$

Recall that we started with $(x_0, y_0) = (0, y_0) \in V(\tilde{f}_1, \ldots, \tilde{f}_k) = V(\check{f}_1, x\check{f}_2, \ldots, x\check{f}_k)$. Hence $\check{f}_1(0, y_0) = 0$ and $\iota(\check{f}_1, x) \geq 1$, whence we have the inequality

$$\iota(\check{f}_1, \check{f}_2, \ldots, \check{f}_k) < \iota(\check{f}_1, x\check{f}_2, \ldots, x\check{f}_k).$$

We can apply the induction hypothesis to the polynomial

$$\check{p} := h(x)(\check{f}_1(x, y)z_1 + \check{f}_2(x, y)z_2 + \cdots + \check{f}_k(x, y)z_k) + g(x, y)$$

which is then an x-variable. By assumption,

$$\check{p} = g_0(0) + g_1(0)y + x[\dot{g}_2(x, y) + \dot{h}(x)(\check{f}_1(x, y)z_1 + \check{f}_2(x, y)z_2 + \cdots + \check{f}_k(x, y)z_k)]$$

with $g_1(0) \neq 0$. Let γ be an x-automorphism such that $\gamma(y) = \check{p}$, let γ_0 be the automorphism of $\mathbb{C}[y][\bar{z}]$ obtained by fixing $x = 0$ in γ and let ρ be the affine automorphism of $\mathbb{C}[y]$ defined by $\rho(y) = g_0(0) + g_1(0)y = \gamma_0(y)$. Extending γ_0 and ρ to $\mathbb{C}[x, y][\bar{z}]$, one has

$$\gamma\gamma_0^{-1}\rho(y) = \gamma(y) = \check{p}$$

and, $\forall i = (1,)2, \ldots, k$

$$\gamma\gamma_0^{-1}\rho(z_i) = \gamma\gamma_0^{-1}(z_i) = z_i + xr_i(x, y, \bar{z}).$$

Let σ be the automorphism of $\mathbb{C}(x)[y][\bar{z}]$ given by

$$\sigma(z_i) = xz_i, \ \forall i = 2, \ldots, k.$$

Of course σ is not an automorphism of $\mathbb{C}[x][y, \bar{z}]$, but the composition $\sigma\gamma\gamma_0^{-1}\rho\sigma^{-1}$ will be. Indeed, let us compute

$$\begin{aligned}
\sigma\gamma\gamma_0^{-1}\rho\sigma^{-1}(y) &= \sigma\gamma\gamma_0^{-1}\rho(y) = \sigma(\check{p}) \\
&= \sigma\big(h(\check{f}_1 z_1 + \check{f}_2 z_2 + \cdots + \check{f}_k z_k) + g\big) \\
&= h(\check{f}_1 z_1 + x\check{f}_2 z_2 + \cdots + x\check{f}_k z_k) + g \\
&= \alpha_0^{-1}(p) && (\text{ see (10)}),
\end{aligned}$$

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$$\sigma\gamma\gamma_0^{-1}\rho\sigma^{-1}(z_1) = \sigma\gamma\gamma_0^{-1}(z_1)$$

and $\forall i = 2, \ldots, k$

$$\begin{aligned}\sigma\gamma\gamma_0^{-1}\rho\sigma^{-1}(z_i) = \sigma\gamma\gamma_0^{-1}\rho\Big(\frac{z_i}{x}\Big) &= \frac{\sigma\gamma\gamma_0^{-1}\rho(z_i)}{x}\\ &= \frac{\sigma\big(z_i + xr_i(x, y, z_1, z_2, \ldots, z_k)\big)}{x}\\ &= \frac{xz_i + xr_i(x, y, z_1, xz_2, \ldots, xz_k)}{x}\\ &= z_i + r_i(x, y, z_1, xz_2, \ldots, xz_k).\end{aligned}$$

The images by $\sigma\gamma\gamma_0^{-1}\rho\sigma^{-1}$ of all the coordinates $y, z_1, \ldots, z_k$ are in $\mathbb{C}[x][y, \bar{z}]$ and a similar computation would show the same for its inverse $\sigma\rho^{-1}\gamma_0\gamma^{-1}\sigma^{-1}$. Hence $\sigma\gamma\gamma_0^{-1}\rho\sigma^{-1}$ is an x-automorphism of $\mathbb{C}[x][y, \bar{z}]$. Finally, the polynomial

$$\sigma\gamma\gamma_0^{-1}\rho\sigma^{-1}(y) = \alpha_0^{-1}(p)$$

is an x-variable and so is p.

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References

[AM75] S. S. Abhyankar and T. T. Moh, *Embeddings of the line in the plane.* J. Reine Angew. Math. **276**(1975), 148–166.

[BCW77] H. Bass, E. H. Connell, and D. L. Wright, *Locally polynomial algebras are symmetric algebras.* Invent. Math. **38**(1976/77), 279–299.

[Dol72] A. Dold, *Lectures on Algebraic Topology*. Die Grundlehren der mathematischen Wissenschaften 200, Springer-Verlag, New York, 1972.

[Dur87] A. H. Durfee, *Algebraic varieties which are a disjoint union of subvarieties.* In: Geometry and Topology, Lecture Notes in Pure and Appl. Math. 105, Dekker, New York, 1987, pp. 99–102.

[EV99] E. Edo and S. Vénéreau, *Length 2 variables of $A[x, y]$ and transfer.* Ann. Polin. Math. **76**(2001), 67–76.

[Fuj82] T. Fujita, *On the topology of noncomplete algebraic surfaces.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29**(1982), 503–566.

[Kal94] Sh. Kaliman, *Exotic analytic structures and Eisenman intrinsic measures.* Israel J. Math. **88**(1994), 411–423.

[KVZ01] Sh. Kaliman, S. Vénéreau, and M. Zaidenberg, *Extensions birationnelles simples de l'anneau de polynmes* $\mathbb{C}^3$. C. R. Acad. Sci. Paris Sér. I Math. **333**(2001), 319–322.

[KVZ04] ———, *Simple birational extensions of the polynomial ring* $\mathbb{C}^{[3]}$. Trans. Amer. Math. Soc. **356**(2004), 509–555.

[KZ99] Sh. Kaliman and M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group.* Transform. Groups **4**(1999), 53–95.

[Mas80] W. S. Massey, *Singular Homology Theory*. Graduate Texts in Mathematics 70, Springer-Verlag, New York, 1980.

[Rus76] P. Russell, *Simple birational extensions of two dimensional affine rational domains.* Compositio Math. **33**(1976), 197–208.

[Sat76] A. Sathaye, *On linear planes.* Proc. Amer. Math. Soc. **56**(1976), 1–7.

[Ses58] C. S. Seshadri, *Triviality of vector bundles over the affine space* k^2. Proc. Natl. Acad. Sci. U.S.A. **44**(1958), 456–458.

[Suz74] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace* $\mathbb{C}^2$. J. Math. Soc. Japan **26**(1974), 241–257.

[Vn01] S. Vénéreau, *Automorphismes et variables de l'anneau de polynômes* $A[y_1, \ldots, y_n]$. Ph.D. thesis, Université Grenoble I, Institut Fourier, 2001.

[Zaĭ85] M. G. Zaĭdenberg, *Rational actions of the group* $\mathbb{C}^*$ *on* $\mathbb{C}^2$*, their quasi-invariants and algebraic curves in* $\mathbb{C}^2$ *with Euler characteristic* 1. Dokl. Akad. Nauk SSSR **280**(1985), 277–280, (Russian), Soviet Math. Dokl. **31**(1985), 57–60.

[ZL83] M. G. Zaĭdenberg and V. Ya. Lin, *An irreducible, simply connected algebraic curve in* $\mathbb{C}^2$ *is equivalent to a quasihomogeneous curve.* Dokl. Akad. Nauk SSSR **271**(1983), 1048–1052, (Russian), Soviet Math. Dokl. **28**(1983), 200–204.

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