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THE n-TH DERIVATIVE CHARACTERISATION OF MÖBIUS INVARIANT DIRICHLET SPACE

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In this paper we give the n-th derivative criterion for functions belonging to recently defined function spaces Q_p and $Q_{p,0}$. For a special parameter value $p = 1$ this criterion is applied to BMOA and VMOA, and for $p > 1$ it is applied to the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$. Further, a Carleson measure characterisation is given to Q_p, and in the last section the multiplier space from H^q into Q_p is considered.

1. INTRODUCTION AND SOME AUXILIARY RESULTS

Let $\mathcal{D} = \{ z : |z| < 1 \}$ be the unit disk in the complex plane. Let $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ be a Möbius transformation of $\mathcal{D}$. An analytic function f is said to be a Bloch function, denoted by $f \in \mathcal{B}$ (see [**1**]), if

$$\sup_{z\in\mathcal{D}} \left(1 - |z|^2\right)|f'(z)| < \infty.$$

For $0 < p < \infty$, we say that $f \in Q_p$ if f is analytic and

$$\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} |f'(z)|^2 g^p(z,a)\, d\sigma_z < \infty, \tag{1}$$

where $g(z,a)$ is the Green's function $\log|(1 - \overline{a}z)/(z - a)|$ with logarithmic singularity at $a \in \mathcal{D}$ and $d\sigma_z$ is the usual area measure $dx\,dy$ on $\mathcal{D}$. These spaces were introduced by the first author and his collaborators and have been studied in [**4**], [**6**] and elsewhere. For $1 < p < \infty$ the spaces Q_p are all the same and equal to the Bloch space $\mathcal{B}$ (see [**4**, Theorem 1] and also [**15**, Corollary 2.4]). If $p = 1$, we know by definition that $Q_1 = \text{BMOA}$ (the space of analytic functions of bounded mean oscillation) [**8**]. For $0 < p_1 < p_2 \leqslant 1$ we have $Q_{p_1} \subsetneq Q_{p_2} \subset \text{BMOA}$ (see [**6**, Theorem 2]). An important property that is common to these spaces Q_p is that they are all invariant under Möbius transformations, that is, if $f \in Q_p$, then $f \circ \varphi_a \in Q_p$. This is well known in case of $\mathcal{B}$ and BMOA (see [**2**]). We note that in [**10**] a characterisation of boundary values

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for functions in Q_p $(0 < p < 1)$ is given. In this paper we shall derive a criterion for functions in Q_p and $Q_{p,0}$ in terms of their n-th derivatives. Further, we give a criterion for functions f to belong to Q_p and $Q_{p,0}$ by p-Carleson measures. Also a sufficient condition for a function f to belong to the multiplier space (H^q, Q_p) is obtained. This last result should be compared with [**11**, Proposition 1 and Theorem 1]. First we need the following lemma:

Lemma 1. *Let f be an analytic function in $\mathcal{D}$. Then there exist positive constants c_1, c_2 such that*

$$(2)\qquad \begin{aligned} c_1\Big(\left|f^{(n)}(0)\right|^2 + \iint_{\mathcal{D}} \left|f^{(n+1)}(z)\right|^2 (1-|z|)^{\alpha+2}\, d\sigma_z\Big) &\\ \leqslant \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 (1-|z|)^{\alpha}\, d\sigma_z &\\ \leqslant c_2\Big(\left|f^{(n)}(0)\right|^2 + \iint_{\mathcal{D}} \left|f^{(n+1)}(z)\right|^2 (1-|z|)^{\alpha+2}\, d\sigma_z\Big)& \end{aligned}$$

for $0 < \alpha < \infty$.

Proof: Setting $f^{(n)}(z) = \sum_{n=0}^{\infty} a_n z^n$ we have $f^{(n+1)}(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Using Parseval's formula we get

$$(3)\qquad \begin{aligned} &\iint_{\mathcal{D}} |f^{(n)}(z)|^2 (1-|z|)^{\alpha}\, d\sigma_z \\ &\qquad = 2\pi \left|f^{(n)}(0)\right|^2 B(2, \alpha+1) + 2\pi \sum_{n=1}^{\infty} |a_n|^2 B(2n+2, \alpha+1), \end{aligned}$$

where we have the beta function $B(2n+2, \alpha+1) = \int_0^1 t^{2n+1}(1-t)^{\alpha}\, dt$. On the other hand,

$$(4)\qquad \iint_{\mathcal{D}} |f^{(n+1)}(z)|^2 (1-|z|)^{\alpha+2}\, d\sigma_z = 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 B(2n, \alpha+3).$$

By Stirling's formula $B(2n+2, \alpha+1) = \big(\Gamma(2n+2)\Gamma(\alpha+1)\big)/\big(\Gamma(2n+\alpha+3)\big) \approx 1/\big(n^{1+\alpha}\big)$ and $B(2n, \alpha+3) \approx 1/\big(n^{\alpha+3}\big)$. In the above, we use the notation $a \approx b$ to denote comparability of the quantities, that is, there are absolute positive constants c_1, c_2 satisfying $c_1 b \leqslant a \leqslant c_2 b$. Thus the assertion follows from (3) and (4). ∎

We note that in the definition (1) of the space Q_p the Green's function $g(z, a)$ can be replaced by $1 - \left|\varphi_a(z)\right|^2$. Further, by [**14**, Lemma 3] we know that $f \in \mathcal{B}$ if and only if

$$M_f^n = \sup_{z\in\mathcal{D}} \left(1-|z|^2\right)^n \left|f^{(n)}(z)\right| < \infty. \tag{5}$$

Using (2) and (5) and replacing $f(z)$ in (2) by $f_a(z) = f\big(\varphi_a(z)\big) - f(a)$ we get a criterion for functions in Q_p:

Proposition. *If f is a Bloch function, then $f \in Q_p$ if and only if*

$$\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f_a^{(n)}(z)\right|^2 \left(1-|z|\right)^{p+2(n-1)} d\sigma_z < \infty \tag{6}$$

for $0 < p < \infty$.

2. The n-th derivative criteria for Q_p and $Q_{p,0}$

In this section we shall obtain the n-th derivative criteria for Q_p and $Q_{p,0}$. In case of the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$ corresponding criteria have been obtained by Axler [7] and Stroethoff [**14**]. Our results will generalise these to some other function spaces and, for example, for $p=1$ we have got the n-th derivative criterion for BMOA $(= Q_1)$ and VMOA $(= Q_{1,0})$. The main result of this section is the following

Theorem 1. *Let $n \geqslant 1$ and let $0 < p < \infty$. Then, for an analytic function f in $\mathcal{D}$, the following conditions are equivalent:*

(i) $f \in Q_p$,

(ii) $\displaystyle\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p \left(1-|z|^2\right)^{2n-2} d\sigma_z < \infty$,

(iii) $\displaystyle\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n)}\big(\varphi_a(z)\big)\right|^2 \left|\varphi_a'(z)\right|^{2n} \left(1-|z|^2\right)^{p+2n-2} d\sigma_z < \infty$,

(iv) $\displaystyle\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 g^p(z,a) \left(1-|z|^2\right)^{2n-2} d\sigma_z < \infty$.

Proof: By change of variables we have

$$\begin{aligned} &\iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p \left(1-|z|^2\right)^{2n-2} d\sigma_z \\ &\qquad = \iint_{\mathcal{D}} \left|f^{(n)}\big(\varphi_a(z)\big)\right|^2 \left|\varphi_a'(z)\right|^{2n} \left(1-|z|^2\right)^{p+2n-2} d\sigma_z \end{aligned} \tag{7}$$

and thus (ii) is equivalent to (iii). Next we shall prove (i) is equivalent to (ii).

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(i) $\Longrightarrow$ (ii). We first consider the case $1 < p \leqslant 2$. Constants appearing in the proofs denoted by M are not always the same in each occurrence. Let $f \in Q_p = \mathcal{B}$. Then, by (5),

$$\left|f^{(n)}(\varphi_a(z))\right|\left(1-|\varphi_a(z)|^2\right)^n = \left|f^{(n)}(\varphi_a(z))\right|\left|\varphi_a'(z)\right|^n(1-|z|^2)^n \leqslant M_f^n$$

and thus

$$\iint_{\mathcal{D}} \left|f^{(n)}(\varphi_a(z))\right|^2 |\varphi_a'(z)|^{2n}(1-|z|^2)^{p+2n-2}\,d\sigma_z$$
$$\leqslant \left(M_f^n\right)^2 \iint_{\mathcal{D}} (1-|z|^2)^{p-2}\,d\sigma_z = M < \infty$$

for $1 < p < \infty$. By (7) the assertion is true.

(ii) $\Longrightarrow$ (i). By [**14,** Theorem 1(D)] we know that

$$f \in \mathcal{B} \Longleftrightarrow \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} |f^{(n)}(z)|^2\left(1-|\varphi_a(z)|^2\right)^2(1-|z|^2)^{2n-2}\,d\sigma_z < \infty. \tag{8}$$

Thus, by (8), we have settled the case $1 < p \leqslant 2$.

Next we suppose $2 < p < \infty$. By using (5) and [**18,** Lemma 4.2.2] the implication (i) $\Longrightarrow$ (ii) is trivial for these values of p. In the opposite direction the assertion is true since we have $\left(1-|\varphi_a(z)|^2\right)^p > (1-r^2)^p$ for $z \in \mathcal{D}(a,r)$, and thus

$$\infty > \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} |f^{(n)}(z)|^2\left(1-|\varphi_a(z)|^2\right)^p(1-|z|^2)^{2n-2}\,d\sigma_z$$
$$\geqslant (1-r^2)^p \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}(a,r)} |f^{(n)}(z)|^2(1-|z|^2)^{2n-2}\,d\sigma_z.$$

Hence, by [**14,** Theorem 1], $f \in \mathcal{B} = Q_p$ if (ii) is satisfied.

Finally we consider the case $0 < p \leqslant 1$. For $n = 1$ (i) $\Longrightarrow$ (ii) is true by [**6,** Proposition 1].

Suppose now that (i) $\Longrightarrow$ (ii) holds for some fixed n. We know that if g is an analytic function in $\mathcal{D}$ then, by (2) in Lemma 1,

$$\iint_{\mathcal{D}} |g'(z)|^2(1-|z|^2)^{\alpha+2}\,d\sigma_z \leqslant M \iint_{\mathcal{D}} |g(z)|^2(1-|z|^2)^{\alpha}\,d\sigma_z \tag{9}$$

for $0 < \alpha < \infty$. If we apply (9) to the function $g(z) = f^{(n)}(z)/(1-\overline{a}z)^p$ and $\alpha = 2n-2+p$ and multiply both sides of the inequality (9) by $\left(1-|a|^2\right)^p$ we obtain

$$\begin{aligned}(10)\qquad &\iint\limits_{\mathcal{D}} \left(\frac{\left|f^{(n+1)}(z)\right|^2}{|1-\overline{a}z|^{2p}} + \frac{p^2\,|a|^2\left|f^{(n)}(z)\right|^2}{|1-\overline{a}z|^{2p+2}} \right.\\ &\qquad \left. + 2\,\mathrm{Re}\,\frac{paf^{(n+1)}(z)\overline{f^{(n)}(z)}}{(1-\overline{a}z)^p(1-a\overline{z})^{p+1}} \right)(1-|z|^2)^{2n+p}(1-|a|^2)^p\,d\sigma_z \\ &\leqslant M \iint\limits_{\mathcal{D}} |f^{(n)}(z)|^2 \frac{(1-|a|^2)^p(1-|z|^2)^{p+2n-2}}{|1-\overline{a}z|^{2p}}\,d\sigma_z \\ &= M\iint\limits_{\mathcal{D}} |f^{(n)}(z)|^2\left(1-|\varphi_a(z)|^2\right)^p\left(1-|z|^2\right)^{2n-2}d\sigma_z.\end{aligned}$$

By the assumption we know that in (10) the supremum of the upper bound is finite. Since $f \in \mathcal{B}$, we get by using (5) and [**18,** Lemma 4.2.2]

$$\begin{aligned}&\iint\limits_{\mathcal{D}} \frac{\left|f^{(n)}(z)f^{(n+1)}(z)\right|}{|1-\overline{a}z|^{2p+1}}(1-|z|^2)^{2n+p}(1-|a|^2)^p\,d\sigma_z \\ &\qquad \leqslant M_f^n M_f^{n+1}\iint\limits_{\mathcal{D}} \frac{(1-|z|^2)^{p-1}(1-|a|^2)^p}{|1-\overline{a}z|^{2p+1}}\,d\sigma_z \leqslant M < \infty\end{aligned}$$

for all $a \in \mathcal{D}$. Moreover,

$$\begin{aligned}&\iint\limits_{\mathcal{D}} \frac{\left|f^{(n)}(z)\right|^2}{|1-\overline{a}z|^{2p+2}}(1-|z|^2)^{2n+p}(1-|a|^2)^p\,d\sigma_z \\ &\qquad \leqslant 4\iint\limits_{\mathcal{D}} |f^{(n)}(z)|^2\frac{(1-|z|^2)^{p+2n-2}(1-|a|^2)^p}{|1-\overline{a}z|^{2p}}\,d\sigma_z \\ &\qquad = 4\iint\limits_{\mathcal{D}} |f^{(n)}(z)|^2\left(1-|\varphi_a(z)|^2\right)^p(1-|z|^2)^{2n-2}\,d\sigma_z\end{aligned}$$

and again the upper bound is finite. Hence, in view of (10), if $f \in Q_p$ (even, in fact, if $f \in \mathcal{B}$) then

$$\sup_{a\in\mathcal{D}}\iint\limits_{\mathcal{D}} \left|f^{(n)}(z)\right|^2\left(1-|\varphi_a(z)|^2\right)^p\left(1-|z|^2\right)^{2n-2}d\sigma_z < \infty$$

implies

$$\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n+1)}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p \left(1-|z|^2\right)^{2n} d\sigma_z < \infty.$$

Thus, by induction on n, (i) implies (ii) for all $n \geqslant 1$.

(ii) $\Longrightarrow$ (i). In this case we shall also proceed by induction. If (ii) holds for $0 < p \leqslant 1$, then by [**14**, Theorem 1] $f \in \mathcal{B}$. For $n = 1$ the implication is true by [**6**, Proposition 1]. Suppose now that (ii) $\Longrightarrow$ (i) is true for some fixed n. By Lemma 1 we have for an analytic function g in $\mathcal{D}$,

$$\iint_{\mathcal{D}} |g(z)|^2 \left(1-|z|^2\right)^\alpha d\sigma_z \leqslant M\left(|g(0)|^2 + \iint_{\mathcal{D}} |g'(z)|^2 \left(1-|z|^2\right)^{\alpha+2} d\sigma_z\right), \tag{11}$$

where $0 < \alpha < \infty$. Assume that

$$\sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n+1)}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p \left(1-|z|^2\right)^{2n} d\sigma_z < \infty.$$

In (11) we substitute $g(z) = \left(f^{(n)}(z)\right)/(1-\bar{a}z)^p$, $\alpha = 2n-2+p$ and multiply both sides of (11) by $\left(1-|a|^2\right)^p$. Then we get

$$\begin{aligned}
&\iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p \left(1-|z|^2\right)^{2n-2} d\sigma_z \\
&= \iint_{\mathcal{D}} \left|\frac{f^{(n)}(z)}{(1-\bar{a}z)^p}\right|^2 (1-|z|^2)^{2n-2+p} (1-|a|^2)^p \, d\sigma_z \\
&\leqslant M\left(\left|f^{(n)}(0)\right|^2 (1-|a|^2)^p + \iint_{\mathcal{D}} \left|\frac{d}{dz}\frac{f^{(n)}(z)}{(1-\bar{a}z)^p}\right|^2 (1-|z|^2)^{2n+p}(1-|a|^2)^p \, d\sigma_z\right) \\
&= M\Bigg(\left|f^{(n)}(0)\right|^2 (1-|a|^2)^p \\
&\qquad + \iint_{\mathcal{D}} \left|\frac{f^{(n+1)}(z)}{(1-\bar{a}z)^p} + p\bar{a}\frac{f^{(n)}(z)}{(1-\bar{a}z)^{p+1}}\right|^2 (1-|z|^2)^{2n+p}(1-|a|^2)^p \, d\sigma_z\Bigg) \\
&\leqslant M\Bigg(|f^{(n)}(0)|^2(1-|a|^2)^p + \iint_{\mathcal{D}} \frac{|f^{(n+1)}(z)|^2}{|1-\bar{a}z|^{2p}} (1-|z|^2)^{2n+p}(1-|a|^2)^p \, d\sigma_z \\
&\qquad + p^2|a|^2 \iint_{\mathcal{D}} \frac{|f^{(n)}(z)|^2}{|1-\bar{a}z|^{2p+2}} (1-|z|^2)^{2n+p}(1-|a|^2)^p \, d\sigma_z \\
&\qquad + 2p|a| \iint_{\mathcal{D}} \frac{|f^{(n+1)}(z)f^{(n)}(z)|}{|1-\bar{a}z|^{2p+1}} (1-|z|^2)^{2n+p}(1-|a|^2)^p \, d\sigma_z\Bigg).
\end{aligned}$$

Since $f \in \mathcal{B}$, we have $\left|f^{(n)}(0)\right|^2 \left(1-|a|^2\right)^p \leqslant \left(M_f^n\right)^2$. By assumption,

$$\iint_{\mathcal{D}} \frac{\left|f^{(n+1)}(z)\right|^2}{|1-\overline{a}z|^{2p}} \left(1-|z|^2\right)^{2n+p} \left(1-|a|^2\right)^p d\sigma_z$$
$$= \iint_{\mathcal{D}} \left|f^{(n+1)}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p \left(1-|z|^2\right)^{2n} d\sigma_z \leqslant M < \infty.$$

The other terms involving integrals can be estimated as follows (see [**18**, Lemma 4.2.2]):

$$\iint_{\mathcal{D}} \frac{\left|f^{(n)}(z)\right|^2}{|1-\overline{a}z|^{2p+2}} \left(1-|z|^2\right)^{2n+p} \left(1-|a|^2\right)^p d\sigma_z$$
$$\leqslant \left(M_f^n\right)^2 \left(1-|a|^2\right)^p \iint_{\mathcal{D}} \frac{\left(1-|z|^2\right)^p}{|1-\overline{a}z|^{2p+2}} d\sigma_z$$
$$\leqslant M\left(M_f^n\right)^2 \left(1-|a|^2\right)^p \frac{1}{\left(1-|a|^2\right)^p} \leqslant M$$

and

$$\iint_{\mathcal{D}} \frac{\left|f^{(n+1)}(z) f^{(n)}(z)\right|}{|1-\overline{a}z|^{2p+1}} \left(1-|z|^2\right)^{2n+p} \left(1-|a|^2\right)^p d\sigma_z$$
$$\leqslant M_f^n M_f^{n+1} \left(1-|a|^2\right)^p \iint_{\mathcal{D}} \frac{\left(1-|z|^2\right)^{p-1}}{|1-\overline{a}z|^{2p+1}} d\sigma_z$$
$$\leqslant M M_f^n M_f^{n+1} \left(1-|a|^2\right)^p \frac{1}{\left(1-|a|^2\right)^p} \leqslant M.$$

By assumption we have $f \in Q_p$ and thus, by induction, we have proved (ii) $\Longrightarrow$ (i).

(iv) $\Longrightarrow$ (ii). This is obvious from the inequality $1-\left|\varphi_a(z)\right|^2 \leqslant 2g(z,a)$ for all $z,\ a \in \mathcal{D}$.

(ii) $\Longrightarrow$ (iv). Let

$$I(a) = \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 g^p(z,a) \left(1-|z|^2\right)^{2n-2} d\sigma_z$$
$$= \iint_{\mathcal{D}(a,1/4)} \left|f^{(n)}(z)\right|^2 g^p(z,a) \left(1-|z|^2\right)^{2n-2} d\sigma_z$$
$$+ \iint_{\mathcal{D}\setminus\mathcal{D}(a,1/4)} \left|f^{(n)}(z)\right|^2 g^p(z,a) \left(1-|z|^2\right)^{2n-2} d\sigma_z = I_1(a) + I_2(a),$$

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where $\mathcal{D}(a,1/4) = \{ z \in \mathcal{D} \mid |\varphi_a(z)| < 1/4 \}$. Since

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|} \begin{cases} \geqslant \log 4 > 1, & z \in \mathcal{D}(a,1/4), \\ \leqslant 4\left(1 - |\varphi_a(z)|^2\right), & z \in \mathcal{D} \setminus \mathcal{D}(a,1/4), \end{cases}$$

we obtain, for $p_0 = \max(p,2)$, that

$$I_1(a) \leqslant \iint\limits_{\mathcal{D}(a,1/4)} \left|f^{(n)}(z)\right|^2 g^{p_0}(z,a)\left(1-|z|^2\right)^{2n-2} d\sigma_z$$

and

$$I_2(a) \leqslant 4^p \iint\limits_{\mathcal{D}\setminus\mathcal{D}(a,1/4)} \left|f^{(n)}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p \left(1-|z|^2\right)^{2n-2} d\sigma_z.$$

Since we have proved that (ii) implies $f \in Q_p \subset \mathcal{B}$, we get that (5) is satisfied, and so, from $p_0 \geqslant 2$, we get

$$\begin{aligned} \sup_{a\in\mathcal{D}} I_1(a) &\leqslant \left(M_f^n\right)^2 \sup_{a\in\mathcal{D}} \iint\limits_{\mathcal{D}(a,1/4)} g^{p_0}(z,a)\left(1-|z|^2\right)^{-2} d\sigma_z \\ &= \left(M_f^n\right)^2 \iint\limits_{\mathcal{D}(0,1/4)} \left(\log\frac{1}{|w|}\right)^{p_0} \left(1-|w|^2\right)^{-2} d\sigma_w < \infty. \end{aligned}$$

By (ii), $\sup_{a\in\mathcal{D}} I_2(a) < \infty$. Thus

$$\sup_{a\in\mathcal{D}} I(a) = \sup_{a\in\mathcal{D}}\left(I_1(a) + I_2(a)\right) < \infty,$$

or (iv) is satisfied. The proof is completed. □

Contained in the Bloch space is the little Bloch space $\mathcal{B}_0$, which is by definition the set of all analytic functions f in $\mathcal{D}$ for which $\left(1-|z|^2\right)|f'(z)| \to 0$ as $|z| \to 1$. For $0 < p < \infty$, we say that $f \in Q_{p,0}$ if f is analytic and

$$\lim_{|a|\to 1} \iint\limits_{\mathcal{D}} |f'(z)|^2 g^p(z,a)\, d\sigma_z = 0.$$

By [**4**, Corollary 2] we know that $Q_{p,0} = \mathcal{B}_0$ for $1 < p < \infty$ (see also [**16**]). On the other hand, if $p = 1$ we have that $Q_{p,0} =$ VMOA (the space of analytic functions of vanishing mean oscillation) [**12**]. If $0 < p_1 < p_2 \leqslant 1$, then $Q_{p_1,0} \subsetneq Q_{p_2,0}$ (see [**6**]). By the above proof, Theorem 2 and [**14**, Lemma 4] we get the corresponding theorem in the limit case:

Theorem 2. *Let $n \geqslant 1$ and let $0 < p < \infty$. Then, for an analytic function f in $\mathcal{D}$, the following conditions are equivalent:*

(i) $f \in Q_{p,0}$,

(ii) $\displaystyle \lim_{|a|\to 1} \iint_{\mathcal{D}} |f^{(n)}(z)|^2 \left(1 - |\varphi_a(z)|^2\right)^p \left(1 - |z|^2\right)^{2n-2} d\sigma_z = 0$,

(iii) $\displaystyle \lim_{|a|\to 1} \iint_{\mathcal{D}} \left|f^{(n)}(\varphi_a(z))\right|^2 |\varphi_a'(z)|^{2n} \left(1 - |z|^2\right)^{p+2n-2} d\sigma_z = 0$,

(iv) $\displaystyle \lim_{|a|\to 1} \iint_{\mathcal{D}} |f^{(n)}(z)|^2 g^p(z,a) \left(1 - |z|^2\right)^{2n-2} d\sigma_z = 0$.

3. Q_p AND ENTIRE FUNCTIONS

In this section we shall generalise Theorem 1 by replacing the weight factor by an infinite series of weight factors. For $1 < p < \infty$ and $n = 1$ this case was considered in [**3**] when criteria for the Bloch space were established.

Theorem 3. *Let $0 < p < \infty$, let $n \geqslant 1$ be an integer, and let $E(\rho) = \sum\limits_{k=0}^{\infty} b_k \rho^k$ be an entire function with $b_k \geqslant 0$ and $b_0 > 0$. If*

$$\overline{\lim_{k\to\infty}}\, k \sqrt[k]{b_k} < 2e, \tag{12}$$

then, for an analytic function f in $\mathcal{D}$, the following conditions are equivalent:

(i) $f \in Q_p$,

(ii) $\displaystyle \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 \left(1 - |z|^2\right)^{2n-2} g^p(z,a) E\big(g(z,a)\big)\, d\sigma_z < \infty$,

(iii) $\displaystyle \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 \left(1 - |z|^2\right)^{2n-2} \left(1 - |\varphi_a(z)|^2\right)^p E\big(g(z,a)\big)\, d\sigma_z < \infty$.

Proof: (i) $\Longrightarrow$ (ii). Let $E_1(\rho) = E(\rho) - b_0 = \sum\limits_{k=1}^{\infty} b_k \rho^k$. Since $f \in Q_p$, we get by Theorem 1,

$$b_0 \sup_{a\in\mathcal{D}} \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 \left(1 - |z|^2\right)^{2n-2} g^p(z,a)\, d\sigma_z < \infty. \tag{13}$$

Since $f \in Q_p \subset \mathcal{B}$, we have by (5),

$$M_f^n = \sup_{z\in\mathcal{D}} \left|f^{(n)}(z)\right| \left(1 - |z|^2\right)^n < \infty.$$

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Thus

$$\begin{aligned}
&\sup_{a\in\mathcal{D}}\iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2\left(1-|z|^2\right)^{2n-2} g^p(z,a)E_1\big(g(z,a)\big)\,d\sigma_z\\
&\quad\leqslant \sum_{k=1}^{\infty} b_k\left(M_f^n\right)^2 \sup_{a\in\mathcal{D}}\iint_{\mathcal{D}}\left(1-|z|^2\right)^{-2} g^{k+p}(z,a)\,d\sigma_z\\
&\quad= \sum_{k=1}^{\infty} b_k\left(M_f^n\right)^2 \iint_{\mathcal{D}}\left(1-|w|^2\right)^{-2}\left(\log\frac{1}{|w|}\right)^{k+p} d\sigma_w\\
&\quad= \left(M_f^n\right)^2 \sum_{k=1}^{\infty} b_k J(k+p),
\end{aligned}$$

where $J(k+p)=\iint_{\mathcal{D}}\left(1-|w|^2\right)^{-2}\left(\log 1/|w|\right)^{k+p} d\sigma_w$. By **[17,** Lemma 3.3] we see that (12) implies $\sum_{k=1}^{\infty} b_k J(k+p)<\infty$. Thus

$$\sup_{a\in\mathcal{D}}\iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2\left(1-|z|^2\right)^{2n-2} g^p(z,a)E_1\big(g(z,a)\big)\,d\sigma_z<\infty. \tag{14}$$

Combining (13) and (14) we see that (ii) is true.

(ii) $\Longrightarrow$ (iii). This is obvious by $1-|\varphi_a(z)|^2\leqslant 2g(z,a)$ for z, $a\in\mathcal{D}$.

(iii) $\Longrightarrow$ (i). Since $b_0>0$ we get from (iii) that

$$b_0\sup_{a\in\mathcal{D}}\iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2\left(1-|z|^2\right)^{2n-2}\left(1-\left|\varphi_a(z)\right|^2\right)^p d\sigma_z<\infty$$

and so, by Theorem 1, we see that $f\in Q_p$. ∎

Theorem 3 is critical in the following sense:

THEOREM 4. *Let $0<p<\infty$, let $n\geqslant 1$ be an integer, and let $E(\rho)=\sum_{k=0}^{\infty} b_k\rho^k$ be an entire function with $b_k\geqslant 0$ and $b_0>0$. Suppose that*

$$\varlimsup_{k\to\infty} k\sqrt[k]{b_k}>2e,$$

and for an analytic function f on $\mathcal{D}$, one of the following conditions is satisfied:

(i) $\sup_{a\in\mathcal{D}}\iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2\left(1-|z|^2\right)^{2n-2} g^p(z,a)E\big(g(z,a)\big)\,d\sigma_z<\infty,$

(ii) $\sup_{a\in\mathcal{D}}\iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2\left(1-|z|^2\right)^{2n-2}\left(1-|\varphi_a(z)|^2\right)^p E\big(g(z,a)\big)\,d\sigma_z<\infty.$

Then f is a polynomial, whose degree is less than n, or a constant.

We need the following lemma which can be proved in much the same way as in [**17**, Lemma 2.9].

Lemma 2. *Let $0 < p < \infty$, let $0 < r < 1$, and let $n \geqslant 1$ be an integer. Then, for an analytic function f on $\mathcal{D}$ and $a \in \mathcal{D}$,*

$$\left|f^{(n)}(a)\right|\left(1-|a|^2\right)^n \leqslant \frac{16}{\pi c(n) r^2 (\log 1/r)^p} \iint_{\mathcal{D}} \left|f^{(n)}(z)\right|^2 \left(1-|z|^2\right)^{2n-2} g^p(z,a)\, d\sigma_z,$$

where $c(n)$ is a constant depending only on n.

By means of Lemma 2, the proof of Theorem 4 is same as in [**17**, Theorem 3.10]. We omit it here.

4. Q_p and Carleson measure

Let I be a subarc on the unit circle and let

$$S(I) = \{z : z/|z| \in I,\ 1-|I| \leqslant |z| < 1\},$$

where $|I|$ denotes the arc length of I. A positive measure μ on $\mathcal{D}$ is a bounded p-Carleson measure, $0 < p < \infty$, if

$$\mu(S(I)) = O(|I|^p). \tag{15}$$

If the right-hand side of (15) is $o\left(|I|^p\right)$ then we say that μ is a compact p-Carleson measure.

It has been proved by Stegenga [**13**] (see also [**9**] for the case of the unit ball in $\mathbb{C}^n$) that, for $1 \leqslant p < \infty$, μ is a bounded p-Carleson measure if and only if

$$\iint_{\mathcal{D}} |f(z)|^2 d\mu(z) \leqslant C\left(\iint_{\mathcal{D}} |f'(z)|^2 \left(1-|z|^2\right)^p d\sigma_z + |f(0)|^2\right) \tag{16}$$

for all analytic functions f on $\mathcal{D}$ for which the integral on the right-hand side of the inequality (16) is finite.

If $0 < p < 1$ and inequality (16) holds then μ is a bounded p-Carleson measure. However, in this case the implication in the opposite direction is not true [**13**].

In view of [**5**, Lemma 2.1] Theorems 1 and 2 give immediately

Theorem 5. *Let $n \geqslant 1$ be a natural integer and let $0 < p < \infty$. Then, for an analytic function f in $\mathcal{D}$, we have*

(i) $f \in Q_p$ *if and only if* $d\mu = \left|f^{(n)}(z)\right|^2 \left(1-|z|^2\right)^{2n-2+p} d\sigma_z$ *is a bounded p-Carleson measure,*

(ii) $f \in Q_{p,0}$ *if and only if* $d\mu = \left|f^{(n)}(z)\right|^2 \left(1-|z|^2\right)^{2n-2+p} d\sigma_z$ *is a compact p-Carleson measure.*

5. A sufficient condition for multipliers from H^q into Q_p

For $0 < q \leqslant \infty$, by H^q we denote the space of functions f, analytic in $\mathcal{D}$, for which

$$M_q^q(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^q \, d\theta$$

or

$$M_\infty(r,f) = \max_{0\leqslant\theta<2\pi} \left|f\left(re^{i\theta}\right)\right|$$

remains bounded as $r \to 1$.

Let A and B be two vector spaces of sequences. A sequence $\lambda = \{\lambda_n\}$ is said to be a multiplier from A to B if $\{\lambda_n\alpha_n\} \in B$ whenever $\{\alpha_n\} \in A$. The set of all multipliers from A to B will be denoted by (A,B). In this section we regard spaces of analytic functions in $\mathcal{D}$ as sequence spaces by identifying a function with its sequence of Taylor coefficients.

From [**11**, Proposition 1] we get the following result:

Theorem MP. *If $1 < p \leqslant 2$ then a necessary and sufficient condition that $g \in (H^p, \mathrm{BMOA})$ is that*

$$M_q(r,g') \leqslant c/(1-r), \; 0 < r < 1,$$

where $1/p + 1/q = 1$ and $\left(H^1, \mathrm{BMOA}\right) = \mathcal{B}$.

We shall need the multiplier transformation $D^s g$ of g, $g(z) = \sum\limits_{n=0}^{\infty} \widehat{g}(n) z^n$, which is defined by

$$D^s g(z) = \sum_{n=0}^{\infty} (n+1)^s \widehat{g}(n) z^n, \; s \text{ any real number.}$$

Now we are ready to prove

Theorem 6. *If $1 \leqslant q \leqslant 2$, $0 < p \leqslant 1$ then a sufficient condition that $g \in (H^q, Q_p)$ is that*

$$M_{q'}(r,g') = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|g'\left(re^{i\theta}\right)\right|^{q'} d\theta\right)^{1/q'} \leqslant \frac{c}{(1-r)^{(1+p)/2}},$$

where $1/q + 1/q' = 1$. In particular, $g \in \left(H^1, Q_p\right)$ if

$$M_\infty(g',r) = \max_{|z|=r} |g'(z)| \leqslant \frac{c}{(1-r)^{(1+p)/2}}.$$

Proof: Let $f \in H^q$ and let

$$h(z) = f \star g(z) = \sum_{n=0}^{\infty} \widehat{f}(n)\widehat{g}(n) z^n.$$

Then

$$\begin{aligned}\left|r^2 D^2 h(r^2 e^{it})\right|^2 &= \left|\frac{1}{2\pi}\int_0^{2\pi} D^1 f(re^{i\theta}) D^1 g\left(re^{i(t-\theta)}\right) d\theta\right|^2 \\ &\leqslant M_{q'}^2(r, g') M_q^2(r, f') \leqslant \frac{c}{(1-r)^{1+p}} M_q^2(r, f').\end{aligned}$$

Hence, by [**11**, Lemma 1],

$$(1-r)^{2+p} M_\infty^2(r^2, h'') \leqslant c(1-r) M_q^2(r, f')$$

and, by Lemma HL1 in [**11**],

$$\int_0^1 (1-r)^{2+p} M_\infty^2(r^2, h'')\, dr \leqslant c \int_0^1 (1-r) M_q^2(r, f')\, dr < \infty.$$

We shall show that if

$$\int_0^1 (1-r)^{2+p} M_\infty^2(r^2, h'')\, dr < \infty$$

then $h \in Q_p$.

By [**6**, Lemma 4], we have

$$\sup_{a\in\mathcal{D}} \int_0^{2\pi} \frac{\left(1-|a|^2\right)^p}{|1-\bar{a}re^{it}|^{2p}}\, dt < \infty$$

and thus

$$\begin{aligned}&\sup_{a\in\mathcal{D}} \int_{\mathcal{D}} |h''(z)|^2 \left(1-|z|^2\right)^{p+2} \frac{\left(1-|a|^2\right)^p}{|1-\bar{a}z|^{2p}}\, d\sigma_z \\ &\qquad \leqslant \sup_{a\in\mathcal{D}} \int_0^1 (1-r)^{2+p} M_\infty^2(r, h'') \int_0^{2\pi} \frac{\left(1-|a|^2\right)^p}{|1-\bar{a}re^{it}|^{2p}}\, dt dr \\ &\qquad \leqslant c \int_0^1 (1-r)^{2+p} M_\infty^2(r, h'')\, dr < \infty\end{aligned}$$

which, by our Theorem 1, implies $h \in Q_p$ for $n = 2$.

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