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# KÄHLER SURFACES WITH QUASI CONSTANT HOLOMORPHIC CURVATURE

WLODZIMIERZ JELONEK
*Institute of Mathematics, Cracow University of Technology, Warszawska 24, 31-155 Krakòw, Poland*
*e-mail: wjelon@pk.edu.pl*

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**Abstract.** In the paper we describe Kahler QCH surfaces. We prove that any Calabi type and orthotoric Kahler surfaces are QCH Kahler surfaces. We also classify locally homogeneous QCH surfaces.

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**1. Introduction.** The aim of the present paper is to describe connected Kähler surfaces $(M, g, J)$ admitting a global, two-dimensional, $J$-invariant distribution $\mathcal{D}$ having the following property: The holomorphic curvature

$$K(\pi) = R(X, JX, JX, X),$$

of any $J$-invariant 2-plane $\pi \subset T_xM$, where $X \in \pi$ and $g(X, X) = 1$, depends only on the point $x$ and the number $|X_{\mathcal{D}}| = \sqrt{g(X_{\mathcal{D}}, X_{\mathcal{D}})}$, where $X_{\mathcal{D}}$ is an orthogonal projection of $X$ on $\mathcal{D}$. In this case we have

$$R(X, JX, JX, X) = \phi(x, |X_{\mathcal{D}}|),$$

where $\phi(x, t) = a(x) + b(x)t^2 + c(x)t^4$ and $a, b, c$ are smooth functions on $M$. Also $R = a\Pi + b\Phi + c\Psi$ for certain curvature tensors $\Pi, \Phi, \Psi \in \bigotimes^4 \mathfrak{X}^*(M)$ of Kähler type. The investigation of such manifolds, called QCH Kähler manifolds, was started by Ganchev and Mihova in [**9**, **10**]. In our paper [**12**] we used their local results to obtain a global classification of such manifolds under the assumption that $\dim M = 2n \geq 6$. By $\mathcal{E}$ we shall denote the two-dimensional distribution which is the orthogonal complement of $\mathcal{D}$ in $TM$. In the present paper, we show that a Kähler surface $(M, g, J)$ is a QCH manifold with respect to a distribution $\mathcal{D}$ if and only if is a QCH manifold with respect to the distribution $\mathcal{E}$. We also prove that $(M, g, J)$ is a QCH Kähler surface if and only if the anti-selfdual Weyl tensor $W^-$ is degenerate and there exist a negative almost complex structure $\overline{J}$ which preserves the Ricci tensor $Ric$ of $(M, g, J)$ i.e. $Ric(\overline{J}., \overline{J}.) = Ric(., .)$ and such that $\overline{\omega} = g(\overline{J}., .)$ is an eigenvector of $W^-$ corresponding to simple eigenvalue of $W^-$. Equivalently $(M, g, J)$ is a QCH Kähler surface iff it admits a negative almost complex structure $\overline{J}$ satisfying the Gray second condition $R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) = R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W)$. In [**3**] Apostolov, Calderbank and Gauduchon have classified weakly selfdual Kähler surfaces, extending the result of Bryant who classified selfdual Kähler surfaces [1]. Weakly selfdual Kähler surfaces turned out to be of Calabi type and of orthotoric type or surfaces with parallel Ricci tensor.

We show that any Calabi type Kähler surface and every orthotoric Kähler surface is a QCH manifold. In both cases the opposite complex structure $\overline{J}$ is conformally Kähler. We also classify locally homogeneous QCH Kähler surfaces.

**2. Almost complex structure $\overline{J}$.** Let $(M, g, J)$ be a four-dimensional Kähler manifold with a two-dimensional $J$-invariant distribution $\mathcal{D}$. Let $\mathfrak{X}(M)$ denote the algebra of all differentiable vector fields on $M$ and $\Gamma(\mathcal{D})$ denote the set of local sections of the distribution $\mathcal{D}$. By $\omega$ we shall denote the Kähler form of $(M, g, J)$ i.e. $\omega(X, Y) = g(JX, Y)$. Let $(M, g, J)$ be a QCH Kähler surface with respect to $J-invariant$ two-dimensional distribution $\mathcal{D}$. Let us denote by $\mathcal{E}$ the distribution $\mathcal{D}^{\perp}$, which is a two-dimensional, $J$-invariant distribution. By $h, m$ respectively we shall denote the tensors $h = g \circ (p_{\mathcal{D}} \times p_{\mathcal{D}}), m = g \circ (p_{\mathcal{E}} \times p_{\mathcal{E}})$, where $p_{\mathcal{D}}, p_{\mathcal{E}}$ are the orthogonal projections on $\mathcal{D}, \mathcal{E}$ respectively. It follows that $g = h + m$. Let us define almost complex structure $\overline{J}$ by $\overline{J}_{|\mathcal{E}} = -J_{|\mathcal{E}}$ and $\overline{J}_{|\mathcal{D}} = J_{|\mathcal{D}}$. For every almost Hermitian manifold $(M, g, J)$ the selfdual Weyl tensor $W^+$ decomposes under the action of the unitary group $U(2)$. We have $\bigwedge^* M = \mathbb{R} \oplus LM$ where $LM = [[\bigwedge^{(0,2)} M]]$ and we can write $W^+$ as a matrix with respect to this block decomposition

$$W^+ = \begin{pmatrix} \frac{\kappa}{6} & W_2^+ \\ (W_2^+)^* & W_3^+ - \frac{\kappa}{12} Id_{|LM} \end{pmatrix},$$

where $\kappa$ is the conformal scalar curvature of $(M, g, J)$ (see [2]). The selfdual Weyl tensor $W^+$ of $(M, g, J)$ is called degenerate if $W_2 = 0$, $W_3 = 0$. It means that the selfdual Weyl tensor of four-manifold $(M, g)$ has at most two eigenvalues as an endomorphism $W^+ : \bigwedge^+ M \to \bigwedge^+ M$. We say that an almost Hermitian structure $J$ satisfies the second Gray curvature condition if

$$R(X, Y, Z, W) - R(JX, JY, Z, W) = R(JX, Y, JZ, W)$$

$$+R(JX, Y, Z, JW),$$

which is equivalent to $Ric(J, J) = Ric$ and $W_2^+ = W_3^+ = 0$. Hence $(M, g, J)$ satisfies the second Gray condition if $J$ preserves the Ricci tensor and $W^+$ is degenerate. We shall denote by $Ric_0$ and $\rho_0$ the trace free part of the Ricci tensor $Ric$ and the Ricci form $\rho$ respectively. An ambiKähler structure on a real four-manifold consists of a pair of Kähler metrics $(g_+, J_+, \omega_+)$ and $(g_-, J_-, \omega_-)$ such that $g_+$ and $g_-$ are conformal metrics and $J_+$ gives an opposite orientation to that given by $J_-$ (i.e the volume elements $\frac{1}{2}\omega_+ \wedge \omega_+$ and $\frac{1}{2}\omega_- \wedge \omega_-$ have opposite signs).

**3. Curvature tensor of a QCH Kähler surface.** We shall recall some results from [9]. Let

$$R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z, \quad (1)$$

and let us write

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

If $R$ is the curvature tensor of a QCH Kähler manifold $(M, g, J)$, then there exist functions $a, b, c \in C^\infty(M)$ such that

$$R = a\Pi + b\Phi + c\Psi, \tag{2}$$

where $\Pi$ is the standard Kähler tensor of constant holomorphic curvature i.e.

$$\Pi(X, Y, Z, U) = \frac{1}{4}(g(Y, Z)g(X, U) - g(X, Z)g(Y, U) \tag{3}$$

$$+g(JY, Z)g(JX, U) - g(JX, Z)g(JY, U) - 2g(JX, Y)g(JZ, U)),$$

the tensor $\Phi$ is defined by the following relation

$$\Phi(X, Y, Z, U) = \frac{1}{8}(g(Y, Z)h(X, U) - g(X, Z)h(Y, U) \tag{4}$$

$$+g(X, U)h(Y, Z) - g(Y, U)h(X, Z) + g(JY, Z)h(JX, U)$$

$$-g(JX, Z)h(JY, U) + g(JX, U)h(JY, Z) - g(JY, U)h(JX, Z)$$

$$-2g(JX, Y)h(JZ, U) - 2g(JZ, U)h(JX, Y)),$$

and finally

$$\Psi(X, Y, Z, U) = -h(JX, Y)h(JZ, U) = -(h_J \otimes h_J)(X, Y, Z, U). \tag{5}$$

where $h_J(X, Y) = h(JX, Y)$. Let $V = (V, g, J)$ be a real $2n$ dimensional vector space with complex structure $J$ which is skew-symmetric with respect to the scalar product $g$ on $V$. Let assume further that $V = D \oplus E$ where $D$ is a two-dimensional, $J$-invariant subspace of $V$, $E$ denotes its orthogonal complement in $V$. Note that the tensors $\Pi, \Phi, \Psi$ given above are of Kähler type. It is easy to check that for a unit vector $X \in V$ $\Pi(X, JX, JX, X) = 1$, $\Phi(X, JX, JX, X) = |X_D|^2$, $\Psi(X, JX, JX, X) = |X_D|^4$, where $X_D$ means an orthogonal projection of a vector $X$ on the subspace $D$ and $|X_D| = \sqrt{g(X_D, X_D)}$. It follows that for a tensor (2) defined on $V$ we have

$$R(X, JX, JX, X) = \phi(|X_D|),$$

where $\phi(t) = a + bt^2 + ct^4$.

Let $J, \overline{J}$ be Hermitian, opposite orthogonal structures on a Riemannian four-manifold $(M, g)$ such that $J$ is a positive almost complex structure. Let $\mathcal{E} = ker(J\overline{J} - Id)$, $\mathcal{D} = ker(J\overline{J} + Id)$ and let the tensors $\Pi, \Phi, \Psi$ be defined as above where $h = g(p_{\mathcal{D}}, p_{\mathcal{D}})$. Let us define a tensor $K = \frac{1}{6}\Pi - \Phi + \Psi$. Then $K$ is a curvature tensor, $b(K) = 0$, $c(K) = 0$ where $b$ is Bianchi operator and $c$ is the Ricci contraction. Define the endomorphism $K : \bigwedge^2 M \to \bigwedge^2 M$ by the formula $g(K\phi, \psi) = -K(\phi, \psi)$ (see (1) and note that we use convention $R(X, Y, Z, W) = g(R(X, Y)Z, W))$. Then we have

LEMMA 3.1. *The tensor $K$ satisfies $K(\bigwedge^+ M) = 0$. Let $\phi, \psi \in \bigwedge^- M$ be the local forms orthogonal to $\overline{\omega}$ such that $g(\phi, \psi) = g(\psi, \psi) = 2$ and $g(\phi, \psi) = 0$. Then $K(\overline{\omega}) = \frac{1}{3}\overline{\omega}$, $K(\phi) = -\frac{1}{6}\phi$, $K(\psi) = -\frac{1}{6}\psi$.*

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*Proof.* A straightforward computation. □

In the special case of a Kähler surface $(M, g, J)$ we get for a QCH manifold $(M, g, J)$

PROPOSITION 3.1. *Let $(M, g, J)$ be a Kähler surface which is a QCH manifold with respect to the distribution $\mathcal{D}$. Then $(M, g, J)$ is also QCH manifold with respect to the distribution $\mathcal{E} = \mathcal{D}^{\perp}$ and if $\Phi', \Psi'$ are the above tensors with respect to $\mathcal{E}$ then*

$$R = (a + b + c)\Pi - (b + 2c)\Phi' + c\Psi'. \tag{6}$$

*Proof.* Let us assume that

$$X \in TM, |X| = 1.$$

Then if $\alpha = |X_{\mathcal{D}}|, \beta = |X_{\mathcal{E}}|$ then $1 = \alpha^2 + \beta^2$. Hence $R(X, JX, JX, X) = a + b\alpha^2 + c\alpha^4 = a + b(1 - \beta^2) + c(1 - \beta^2)^2 = a + b + c - (b + 2c)\beta^2 + c\beta^4$. □

If $(M, g, J)$ is a QCH Kähler surface then one can show that the Ricci tensor $Ric$ of $(M, g, J)$ satisfies the equation

$$Ric(X, Y) = \lambda m(X, Y) + \mu h(X, Y), \tag{7}$$

where $\lambda = \frac{3}{2}a + \frac{b}{4}, \mu = \frac{3}{2}a + \frac{5}{4}b + c$ are eigenvalues of $\rho$ (see **[9]**, Corollary 2.1 and Remark 2.1.) In particular the distributions $\mathcal{E}, \mathcal{D}$ are eigendistributions of the tensor $\rho$ corresponding to the eigenvalues $\lambda, \mu$ of $\rho$. The Kulkarni–Nomizu product of two symmetric $(2, 0)$-tensors $h, k \in \bigotimes^2 TM^*$ we call a tensor $h \oslash k$ defined as follows:

$$h \oslash k(X, Y, Z, T) = h(X, Z)k(Y, T) + h(Y, T)k(X, Z)$$

$$-h(X, T)k(Y, Z) - h(Y, Z)k(X, T).$$

Similarly, we define the Kulkarni–Nomizu product of two 2-forms $\omega, \eta$

$$\omega \oslash \eta(X, Y, Z, T) = \omega(X, Z)\eta(Y, T) + \omega(Y, T)\eta(X, Z)$$

$$-\omega(X, T)\eta(Y, Z) - \omega(Y, Z)\eta(X, T).$$

Then $b(\omega \oslash \eta) = -\frac{2}{3}\omega \wedge \eta$ where $b$ is the Bianchi operator. Note that

$$\Pi = -\frac{1}{4}\left(\frac{1}{2}(g \oslash g + \omega \oslash \omega) + 2\omega \otimes \omega\right), \tag{8}$$

$$\Phi = -\frac{1}{8}(h \oslash g + h_J \oslash \omega + 2\omega \otimes h_J + 2h_J \otimes \omega), \tag{9}$$

$$\Psi = -h_J \otimes h_J, \tag{10}$$

where $\omega = g(J., .)$ is the Kähler form. Note that $b(\Psi) = \frac{1}{3}h_J \wedge h_J = 0$ since $h_J = e_1 \wedge e_2$ is primitive, where $e_1, e_2$ is an orthonormal basis in $\mathcal{D}$.

THEOREM 3.1. *Let $(M, g, J)$ be a Kähler surface. If $(M, g, J)$ is a QCH manifold then $W^- = c(\frac{1}{6}\Pi - \Phi + \Psi)$ and $W^-$ is degenerate. The 2-form $\overline{\omega}$ is an eigenvector of $W^-$ corresponding to a simple eigenvalue of $W^-$ and $\overline{J}$ preserves the Ricci tensor. On the other hand, let us assume that $(M, g, J)$ admits a negative almost complex structure $\overline{J}$ such that $Ric(\overline{J}, \overline{J}) = Ric$. Let $\mathcal{E} = ker(J\overline{J} - Id)$, $\mathcal{D} = ker(J\overline{J} + Id)$. If $W^- = \frac{\kappa}{2}(\frac{1}{6}\Pi - \Phi + \Psi)$ or equivalently if the half-Weyl tensor $W^-$ is degenerate and $\overline{\omega}$ is an eigenvector of $W^-$ corresponding to a simple eigenvalue of $W^-$ then $(M, g, J)$ is a QCH manifold.*

*Proof.* Note that for a Kähler surface $(M, g, J)$ the Bochner tensor coincides with $W^-$ and we have

$$R = -\frac{\tau}{12}\left(\frac{1}{4}(g \oslash g + \omega \oslash \omega) + \omega \otimes \omega\right)$$

$$-\frac{1}{4}\left(\frac{1}{2}(Ric_0 \oslash g + \rho_0 \oslash \omega) + \rho_0 \otimes \omega + \omega \otimes \rho_0\right) + W^-.$$

If $(M, g, J)$ is a QCH Kähler surface then $Ric = \lambda m + \mu h$ where $\lambda = \frac{3}{2}a + \frac{b}{4}$, $\mu = \frac{3}{2}a + \frac{5}{4}b + c$. Consequently $Ric_0 = -\frac{b+c}{2}m + \frac{b+c}{2}h = \delta h - \delta m$ where $\delta = \frac{b+c}{2}$. Hence $Ric_0 = 2\delta h - \delta g$. Hence we have

$$R = -\frac{\tau}{12}\left(\frac{1}{4}(g \oslash g + \omega \oslash \omega) + \omega \otimes \omega\right)$$

$$-\frac{1}{4}(\frac{1}{2}((2\delta h - \delta g) \oslash g + (2\delta h_J - \delta\omega) \oslash \omega) + (2\delta h_J - \delta\omega) \otimes \omega +$$

$$\omega \otimes (2\delta h_J - \delta\omega)) + W^-.$$

Consequently

$$R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi + W^- = \left(a - \frac{c}{6}\right)\Pi + (b + c)\Phi + W^-,$$

and $a\Pi + b\Phi + c\Psi = (a - \frac{c}{6})\Pi + (b + c)\Phi + W^-$ hence $W^- = c(\frac{1}{6}\Pi - \Phi + \Psi)$. It follows that $W^-$ is degenerate and $\overline{\omega}$ is an eigenvalue of $W^-$ corresponding to the simple eigenvalue of $W^-$. It is also clear that $Ric(\overline{J}, \overline{J}) = Ric$.

On the other hand, let us assume that a Kähler surface $(M, g, J)$ admits a negative almost complex structure $\overline{J}$ preserving the Ricci tenor $Ric$ and such that $W^-$ is degenerate with eigenvector $\overline{\omega}$ corresponding to the simple eigenvalue of $W^-$. Equivalently it means that $\overline{J}$ satisfies the second Gray condition of the curvature i.e. $R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) = R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W)$. Then $W^- = \frac{\kappa}{2}((\frac{1}{6}\Pi - \Phi + \Psi)$. If $Ric_0 = \delta(h - m)$ then as above $R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi + W^-$. Consequently $R = (\frac{\tau}{6} - \delta)\Pi + 2\delta\Phi + \frac{\kappa}{2}(\frac{1}{6}\Pi - \Phi + \Psi)$ and

$$R = \left(\frac{\tau}{6} - \delta + \frac{\kappa}{12}\right)\Pi + \left(2\delta - \frac{\kappa}{2}\right)\Phi + \frac{\kappa}{2}\Psi. \tag{11}$$

□

REMARK 3.1. Note that $\kappa$ is the conformal scalar curvature of $(M, g, \overline{J})$. The Bochner tensor of QCH manifold was first identified in [**10**].

COROLLARY 3.1. *A Kähler surface $(M, g, J)$ is a QCH manifold iff it admits a negative almost complex structure $\overline{J}$ satisfying the second Gray condition of the curvature i.e.*

$$R(X, Y, Z, W) - R(\overline{J}X, \overline{J}Y, Z, W) =$$

$$R(\overline{J}X, Y, \overline{J}Z, W) + R(\overline{J}X, Y, Z, \overline{J}W).$$

*The $J$-invariant distribution $\mathcal{D}$ with respect to which $(M, g, J)$ is a QCH manifold is given by $\mathcal{D} = ker(J\overline{J} - Id)$ or by $\mathcal{D} = ker(J\overline{J} + Id)$.*

THEOREM 3.2. *Let us assume that $(M, g, J)$ is a Kähler surface admitting a negative Hermitian structure $\overline{J}$ such that $Ric(\overline{J}, \overline{J}) = Ric$. Then $(M, g, J)$ is a QCH manifold.*

*Proof.* If a Hermitian manifold $(M, g, J)$ has a J-invariant Ricci tensor $Ric$ then the tensor $W^+$ is degenerate (see [**5**]). □

REMARK 3.2. If a Kähler surface $(M, g, J)$ is compact and admits a negative Hermitian structure $\overline{J}$ as above then $(M, g, \overline{J})$ is locally conformally Kähler and hence globally conformally Kähler if $b_1(M)$ is even. Thus $(M, g, J)$ is ambiKähler since $b_1(M)$ is even.

Now we give examples of QCH Kähler surfaces. First we give (see [**3**])

DEFINITION 3.1. A Kähler surface $(M, g, J)$ is said to be of Calabi type if it admits a non-vanishing Hamiltonian Killing vector field $\xi$ such that the almost Hermitian pair $(g, I)$ – with $I$ equal to $J$ on the distribution spanned by $\xi$ and $J\xi$ and $-J$ on the orthogonal distribution – is conformally Kähler.

It is known that for a Kähler surface of Calabi type we have $\rho_0 = \delta\omega_I$ where $\omega_I$ is the Kähler form of $(M, g, I)$ (see [**3**]) and consequently $Ric(I, I) = Ric$. Hence we have

THEOREM 3.3. *Every Kähler surface of Calabi type is a QCH Kähler surface.*

DEFINITION 3.2. A Kähler surface $(M, g, J)$ is orthotoric if it admits two independent Hamiltonian Killing vector fields with Poisson commuting momentum maps $\xi\eta$ and $\xi + \eta$ such that $d\xi$ and $d\eta$ are orthogonal.

An explicit classification of orthotoric Kähler metrics is given in [A-C-G]. Every orthotoric surface admits a negative Hermitian structure $I$ which is conformally Kähler. We also have $\rho_0 = \delta\omega_I$ where $\omega_I$ is the Kähler form of $(M, g, I)$(see [**3**]).

In particular the Hermitian structure $I$ preserves Ricci tensor $Ric$. Hence we get

THEOREM 3.4. *Every orthotoric Kähler surface is a QCH Kähler surface.*

Note that both Calabi type and orthotoric Kähler surfaces are ambiKähler. On the other hand we have

THEOREM 3.5. *Let $(M, g, J)$ be ambiKähler surface which is a QCH manifold. Then locally $(M, g, J)$ is orthotoric or of Calabi type or a product of two Riemannian surfaces or is an anti-selfdual Einstein–Kähler surface.*

*Proof.* (We follow [**4**]). Let us denote by $g_-$ the second Kähler metric. Let us assume that $g_- \neq g$. Then $g = \phi^{-2}g_-$ and the field $X = grad_{\omega_-}\phi$ is a Killing vector field $L_X g = L_X g_- = 0$ and is holomorphic with respect to $\overline{J}$. We shall show that $X$ is also holomorphic with respect to $J$. In fact $Ric_0 = \delta g(J\overline{J}, .)$ and $L_X Ric = 0$, $L_X\delta = 0$. Hence $0 = \delta g((L_X J)\overline{J}, .)$ and consequently $L_X J = 0$ in $U = \{x : Ric_0(x) \neq 0\}$. If $(M, g)$ is Einstein then $W^+ \neq 0$ everywhere or $(M, g, J)$ is anti-selfdual. In the first case $X$ preserves the simple eigenspace of $W^+$ and hence $\omega$, consequently $L_X J = 0$. Note that $X = \overline{J} grad_g \psi$ where $\psi = -\frac{1}{\phi}$. Since $L_X\omega = 0$ we have $dX\lrcorner\omega = 0$ and consequently the 1-form $J\overline{J}d\psi$ is closed and locally equals $\frac{1}{2}d\sigma$. Thus the two form $\Omega = \frac{3}{2}\sigma\omega + \psi^3\omega_-$, where $\omega_-$ is the Kähler form of $(M, g_-, \overline{J})$, is a Hamiltonian form in the sense of [**3**] and the result follows from the classification in [**3**]. This form is defined globally if $H^1(M) = 0$. □

REMARK 3.3. Note that in the compact case every Killing vector field on a Kähler surface is holomorphic. If $(M, g, J)$ is an Einstein Kähler anti-selfdual then in the case where it is not conformally flat the manifold $(M, g, \overline{J})$ is a selfdual Einstein Hermitian conformal to selfdual Kähler metric. Such a metric must be either orthotoric or of Calabi type. Thus $(M, g, J)$ is of Calabi type if $(M, g, \overline{J})$ is of Calabi type, however $(M, g, J)$ cannot be orthotoric if $(M, g, \overline{J})$ is orthotoric.

Now we shall investigate Einstein QCH Kähler surfaces.

THEOREM 3.6. *Let $(M, g, J)$ be a Kähler–Einstein surface. Then $(M, g, J)$ is a QCH Kähler surface if and only if it admits a negative Hermitian structure $\overline{J}$ or it has constant holomorphic curvature and admits any negative almost complex structure. If $(M, g, J)$ is QCH and the second case does not hold then $\overline{J}$ is conformally Kähler hence $(M, g, J)$ is ambiKähler.*

*Proof.* If an Einstein four-manifold $(M, g)$ admits a degenerate tensor $W^-$ then $W^- = 0$ or $W^- \neq 0$ on the whole of $M$. In the second case by the result of Derdzinski it admits a Hermitian structure $\overline{J}$ which is conformally Kähler and the metric $(g(W^-, W^-))^{\frac{1}{3}}g$ is a Kähler metric with respect to $\overline{J}$. □

REMARK 3.4 *Compare [4].* If $(M, g, J)$ is a QCH Kähler Einstein surface which is not anti-selfdual then in the case $H^1(M) = 0$ on $(M, g, J)$ there is defined global Hamiltonian two form and on the open and dense subset $U$ of $M$ the metric $g$ is:

(a) A Kähler product metric of two Riemannian surfaces of the same Gauss curvature.
(b) Kähler Einstein metric of Calabi type over a Riemannian surface $(\Sigma, g_\Sigma)$ of constant Gauss curvature $k$ of the form $g = zg_\Sigma + \frac{z}{V(z)}dz^2 + \frac{V(z)}{z}(dt + \alpha)^2$ where $V(z) = a_1z^3 + kz^2 + a_2$.
(c) Kähler Einstein ambitoric metric of parabolic type (see [**4**], section 5.4.).

THEOREM 3.7. *Let $(M, g, J)$ be a selfdual Kähler surface with $Ric_0 \neq 0$ everywhere on $M$. Then $(M, g, J)$ is a QCH Kähler surface with Hermitian complex structure $\overline{J}$.*

*Proof.* We show as in Theorem 1 that $R = \frac{\tau}{6}\Pi + 2\delta\Phi - \delta\Pi$ where $\rho_0 = \delta\overline{\omega}$. Note that in $U = \{x : Ric_0 \neq 0\}$ the negative structure $\overline{J}$ is uniquely determined and is Hermitian in $U$ (see Proposition 4 in [**5**]). □

REMARK 3.5. Note that a selfdual Kähler surface $(M, g, J)$ is QCH if admits any negative almost complex structure $\overline{J}$ preserving the Ricci tensor $Ric$. For example $\mathbb{CP}^2$

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with standard Fubini–Studi metric is selfdual however is not QCH since it does not admit any negative almost complex structure. However the manifold $M = \mathbb{CP}^2 - \{p_0\}$ for any point $p_0 \in \mathbb{CP}^2$ is QCH and admits a negative Hermitian complex structure (see [**13**]). In [**8**] there are constructed many examples of selfdual Kähler surfaces with $Ric_0 \neq 0$ hence QCH Kähler selfdual surfaces. Every selfdual Kähler metric is weakly selfdual. Selfdual metrics were classified by Bryant in [**1**]. From [**3**] it follows that selfdual Kähler metrics with non-parallel Ricci tensor are orthotoric or of Calabi type and in fact are ambiKähler.

LEMMA 3.2. *Let $M$ be a connected QCH Kähler surface which is not Einstein. Then the following conditions are equivalent:*

($a$) *The scalar curvature $\tau$ of $(M, g, J)$ is constant and $\overline{J}$ is almost Kähler*

($b$) *The eigenvalues $\lambda, \mu$ of Ric are constant.*

*Proof.* (a)⇒(b) Note that $\rho = \lambda\omega_1 + \mu\omega_2$ where $\lambda, \mu$ are eigenvalues of $Ric$ and $\omega_2 = h_J, \omega_1 = m_J$. Note that $d\omega_1 + d\omega_2 = 0$ and

$$(\mu - \lambda)d\omega_1 = d\lambda \wedge \omega_1 + d\mu \wedge \omega_2. \tag{12}$$

Note that $\overline{J}$ is almost Kähler if and only if $d\omega_1 = 0$. Hence from (12) we get $p_{\mathcal{D}}(\nabla\lambda) = 0, p_{\mathcal{E}}(\nabla\mu) = 0$. Since $\tau$ is constant we get $\nabla\lambda = -\nabla\mu$ in an open set $U = \{x : \lambda(x) \neq \mu(x)\}$. Thus $\nabla\lambda = \nabla\mu = 0$ in $U$ and consequently $U = M$ and $\lambda, \mu$ are constant.

($b$) ⇒ ($a$) This implication is trivial. □

Now we give a classification of locally homogeneous QCH Kähler surfaces.

PROPOSITION 3.2. *Let $(M, g, J)$ be a QCH locally homogeneous manifold. Then the following cases occur:*

($a$) *$(M, g, J)$ has constant holomorphic curvature (hence is locally symmetric and selfdual).*

($b$) *$(M, g, J)$ is locally a product of two Riemannian surfaces of constant scalar curvature.*

($c$) *$(M, g, J)$ is locally isometric to a unique four-dimensional proper three-symmetric space.*

*Proof.* If $(M, g)$ is Einstein locally homogeneous four-manifold then is locally symmetric (see [**14**]). A locally irreducible locally symmetric Kähler surface is selfdual.(see [**7**]). If $(M, g)$ is not Einstein then using Lemma we see that $(M, g, \overline{J})$ is an almost Kähler manifold satisfying the Gray condition $G_2$. Hence $||\nabla\overline{J}||$ is constant on $M$ and in the case $||\nabla\overline{J}|| \neq 0$ it is strictly almost Kähler manifold satisfying $G_2$. Such manifolds are classified in [**2**] and are locally isometric to a proper three-symmetric space (see [**16**]). Note that they are Kähler in an opposite orientation. If $||\nabla\overline{J}|| = 0$ then the case (b) holds. □

PROPOSITION 3.3. *Let $(M, g, J)$ be a QCH Kähler surface. If $(M, g)$ is conformally Einstein then the almost Hermitian structure $\overline{J}$ is Hermitian or $(M, g, J)$ is selfdual.*

*Proof.* Let us assume that $(M, g_1)$ is an Einstein manifold where $g_1 = f^2 g$. Then $(M, g_1)$ is an Einstein manifold with degenerate half-Weyl tensor $W^-$. Consequently $W^- = 0$ or $W^- \neq 0$ everywhere. In the second case the metric

$$(g_1(W^-, W^-))^{\frac{1}{3}} g_1,$$

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is a Kähler metric with respect to $\overline{J}$. Thus $\overline{J}$ is Hermitian and conformally Kähler. □

REMARK 3.6. Every QCH Kähler surface is a holomorphically pseudosymmetric Kähler manifold. (see [**17**, **11**] ). In fact from [**11**] it follows that $R.R = (a + \frac{b}{2})\Pi.R$. Hence in the case of QCH Kähler surfaces we have

$$R.R = \frac{1}{6}(\tau - \kappa)\Pi.R, \tag{13}$$

where $\tau$ is the scalar curvature of $(M, g, J)$ and $\kappa$ is the conformal scalar curvature of $(M, g, \overline{J})$. Note that (13) is the obstruction for a Kähler surface to have a negative almost complex $\overline{J}$ structure satisfying the Gray condition $(G_2)$. In an extremal situation where $(M, g, \overline{J})$ is Kähler we have $R.R = 0$.

Now we classify QCH Kähler surfaces for which $a, b, c$ are all constant. Then $\lambda, \mu$ are constant and if $(M, g)$ is not Einstein the almost complex structure $\overline{J}$ is almost Kähler. Hence $(M, g, \overline{J})$ is a $G_2$ almost Kähler manifold. Consequently $|\nabla\overline{\omega}|$ is constant and $(M, g, J)$ is a product of two Riemannian surfaces of constant scalar curvature or is a proper three-symmetric space. If $(M, g)$ is Einstein then $\kappa = 2c$ is constant and $|W^-|^2 = \frac{1}{24}\kappa^2$ is constant. Thus $\kappa = 0$ and $(M, g, J)$ has constant holomorphic curvature (is a real space form) or by [**7**] the manifold $(M, g, \overline{J})$ is Kähler hence $(M, g, J)$ is a product of two Riemannian surfaces of constant scalar curvature. Note that for a proper 3-symmetric space we have $\delta = \frac{\kappa}{4}$ for the distribution $\mathcal{D}$ perpendicular to the Kähler nullity of $\overline{J}$ (see [**2**]), thus $b = 2\delta - \frac{\kappa}{2} = 0$ and $a = \frac{1}{6}(\tau - \kappa) = -\frac{1}{2}|\nabla\overline{\omega}|^2$. Since $\mu = 0$ $c = -\frac{3}{2}a$ and $\tau = -\kappa$ where $\kappa = \frac{3}{2}|\nabla\overline{\omega}|^2$. Hence

$$R.R = -\frac{\kappa}{3}\Pi.R, \tag{14}$$

where $\kappa = \frac{3}{2}|\nabla\overline{\omega}|^2$ is constant. Summarizing we have proved

PROPOSITION 3.4. *Let us assume that $(M, g, J)$ is a QCH Kähler surface with constant $a, b, c$. Then the following cases occur:*

(*a*) *$(M, g, J)$ has constant holomorphic curvature (hence is locally symmetric and selfdual).*

(*b*) *$(M, g, J)$ is locally a product of two Riemannian surfaces of constant scalar curvature.*

(*c*) *$(M, g, J)$ is locally isometric to a unique four-dimensional proper three-symmetric space and $a = -\frac{1}{3}\kappa$, $b = 0$, $c = \frac{1}{2}\kappa$ where $\kappa = \frac{3}{2}|\nabla\overline{\omega}|^2$ is constant scalar curvature of $(M, g, \overline{J})$, consequently $R = -\frac{1}{3}\kappa\Pi + \frac{1}{2}\kappa\Psi$.*

REMARK 3.7. We consider above the proper three-symmetric space as a QCH manifold with respect to the distribution $\mathcal{D}$ perpendicular to the Kähler nullity of $\overline{J}$. If we consider it as a QCH manifold with respect to the distribution $\mathcal{E} = \mathcal{D}^\perp$ then $R = \frac{1}{6}\kappa\Pi - \kappa\Phi' + \frac{1}{2}\kappa\Psi'$ (see Proposition 3.1.).

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