Bull. Aust. Math. Soc. **89** (2014), 460–465
doi:10.1017/S0004972713000762

UNIQUE REPRESENTATION BI-BASIS FOR THE INTEGERS

RAN XIONG and MIN TANG ✉

(Received 27 June 2013; accepted 7 July 2013; first published online 12 September 2013)

Abstract

For $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$, let $r_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 + a_2, a_1 \le a_2\}$ and $\delta_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 - a_2\}$. We call A a unique representation bi-basis if $r_A(n) = 1$ for all $n \in \mathbb{Z}$ and $\delta_A(n) = 1$ for all $n \in \mathbb{Z} \setminus \{0\}$. In this paper, we construct a unique representation bi-basis of $\mathbb{Z}$ whose growth is logarithmic.

2010 *Mathematics subject classification*: primary 11B34.
Keywords and phrases: bi-basis, representation function.

1. Introduction

For sets A, B of integers and any integer c, we define the sets

$$A + B = \{a + b : a \in A, b \in B\}, \quad A - B = \{a - b : a \in A, b \in B\}$$

and the translations

$$c + A = \{c + a : a \in A\}, \quad c - A = \{c - a : a \in A\}.$$

For $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$, let

$$r_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 + a_2, a_1 \le a_2\},$$
$$\delta_A(n) = \#\{(a_1, a_2) \in A^2 : n = a_1 - a_2\}.$$

The counting function for the set A is $A(y, x) = \#\{a \in A : y \le a \le x\}$.

In 2003, Nathanson [4] constructed a family of arbitrarily sparse sets $A \subseteq \mathbb{Z}$ satisfying $r_A(n) = 1$ for all $n \in \mathbb{Z}$. In 2011, Tang *et al.* [6] proved that there exists a family of sets $A \subseteq \mathbb{Z}$ satisfying $\delta_A(n) = 1$ for all nonzero integers n. We call A a bi-basis of $\mathbb{Z}$ if $r_A(n) \ge 1$ for all $n \in \mathbb{Z}$ and $\delta_A(n) \ge 1$ for all $n \in \mathbb{Z} \setminus \{0\}$. In particular, we call A a unique representation bi-basis of $\mathbb{Z}$ if $r_A(n) = 1$ for all $n \in \mathbb{Z}$ and $\delta_A(n) = 1$ for all $n \in \mathbb{Z} \setminus \{0\}$. For other related problems, see [1–3, 5].

This work was supported by the National Natural Science Foundation of China, Grant No. 10901002 and Anhui Provincial Natural Science Foundation, Grant No. 1208085QA02.

In this paper, we obtain the following results.

Theorem 1.1. *Let $\varphi(x)$ be a positive function such that $\lim_{x\to\infty}\varphi(x) = +\infty$. Then there exists a set $A \in \mathbb{Z}$ such that*

$$r_A(n) = 1 \quad \text{for all } n \in \mathbb{Z},$$
$$\delta_A(n) = 1 \quad \text{for all } n \in \mathbb{Z} \setminus \{0\}$$

and

$$A(-x, x) \le \varphi(x)$$

for all $x > 1$.

Theorem 1.2. *There exists a unique representation bi-basis A of $\mathbb{Z}$ such that*

$$\frac{4(\log x - \log 2)}{\log 15} - 1 < A(-x, x) < \frac{4(\log x - \log 2)}{\log 3} + 7$$

for all $x > 1$.

2. Proof of Theorem 1.1

We will construct an ascending sequence of finite sets $A_1 \subseteq A_2 \subseteq \cdots$ such that the following three conditions are satisfied:

(i) $\#A_k = 4k - 1$;
(ii) $r_{A_k}(n) \le 1$ for all $n \in \mathbb{Z}$, $\delta_{A_k}(n) \le 1$ for all $n \in \mathbb{Z} \setminus \{0\}$;
(iii) $r_{A_{2k}}(n) = 1$ for $n \in [-k-1, k+1]$, $\delta_{A_k}(n) = 1$ for all $n \in [-k-2, k+2] \setminus \{0\}$.

Conditions (ii) and (iii) imply that the infinite set

$$A = \bigcup_{k=1}^{\infty} A_k$$

is a unique representation bi-basis for $\mathbb{Z}$.

We construct the sets A_k by induction. Let $A_1 = \{1, -1, 2\}$, so that

$$A_1 + A_1 = \{0, 1, 2, -2, 3, 4\}, \quad A_1 - A_1 = \{0, \pm 1, \pm 2, \pm 3\}.$$

Suppose that, for some integer $k \ge 1$, we have constructed a set A_k satisfying (i) and (ii).

For $k \ge 1$, define

$$d_k = \max\{|a| : a \in A_k\}.$$

Then

$$A_k \subseteq [-d_k, d_k].$$

If both d_k and $-d_k$ belong to A_k, then we have the two representations of 0 in the sumset $A_k + A_k$:

$$0 = 1 + (-1) = d_k + (-d_k).$$

https://doi.org/10.1017/S0004972713000762 Published online by Cambridge University Press

That is, only one of the two numbers d_k and $-d_k$ belongs to A_k. Then we know that if $-d_k \in A_k$, then $A_k + A_k \subseteq [-2d_k, 2d_k - 2]$. Otherwise, $A_k + A_k \subseteq [-2d_k + 2, 2d_k]$. Moreover, in either case, $A_k - A_k \subseteq [-2d_k + 1, 2d_k - 1]$.

For $k \geq 1$, let

$$u_k = \min\{|n| : n \notin A_k + A_k\}, \qquad v_k = \min\{n > 0 : n \notin A_k - A_k\}.$$

We know that

$$1 \leq u_k \leq 2d_k - 1, \quad 4 \leq v_k \leq 2d_k - 1.$$

Choose integers $x_k \geq 3d_k + 1$, $y_k \geq 3x_k + 2u_k$.

Case 1: $u_k \notin A_k + A_k$. Put

$$A_{k+1} = A_k \cup \{u_k + x_k, -x_k, y_k, v_k + y_k\}.$$

Then

$$\begin{aligned} A_{k+1} + A_{k+1} &= S \cup (A_k + A_k) \cup (u_k + x_k + A_k) \cup (-x_k + A_k) \cup (y_k + A_k) \\ &\quad \cup (v_k + y_k + A_k), \\ A_{k+1} - A_{k+1} &= T \cup (A_k - A_k) \cup \pm(u_k + x_k - A_k) \cup \pm(x_k + A_k) \\ &\quad \cup \pm(y_k - A_k) \cup \pm(v_k + y_k - A_k), \end{aligned}$$

where

$$\begin{aligned} S &= \{2(y_k + v_k), 2y_k + v_k, 2y_k, u_k + v_k + x_k + y_k, u_k + x_k + y_k, \\ &\qquad v_k + y_k - x_k, y_k - x_k, 2(u_k + x_k), u_k, -2x_k\}, \\ T &= \{\pm(v_k + x_k + y_k), \pm(x_k + y_k), \pm(y_k - x_k + v_k - u_k), \\ &\qquad \pm(y_k - x_k - u_k), \pm(u_k + 2x_k), \pm v_k\}. \end{aligned}$$

We know that

$$\begin{gathered} u_k + x_k + A_k \subseteq [2d_k + 3, x_k + 3d_k - 1], \qquad -x_k + A_k \subseteq [-x_k - d_k, -2d_k - 1], \\ y_k + A_k \subseteq [y_k - d_k, y_k + d_k], \qquad v_k + y_k + A_k \subseteq [y_k - d_k, y_k + 3d_k - 1], \\ x_k + 3d_k < 2(u_k + x_k) < v_k + y_k - x_k < y_k - d_k. \end{gathered}$$

Moreover, $(y_k + A_k) \cap (v_k + y_k + A_k) = \emptyset$. In fact, if $(y_k + A_k) \cap (v_k + y_k + A_k) \neq \emptyset$, then there are $a, a' \in A_k$ such that $y_k + a = v_k + y_k + a'$, so $v_k = a - a'$, which is impossible. Hence

$$S, A_k + A_k, u_k + x_k + A_k, -x_k + A_k, v_k + y_k + A_k, y_k + A_k$$

are pairwise disjoint.

Similarly, we can show that

$$A_k - A_k,\ T,\ \pm(u_k + x_k - A_k),\ \pm(x_k + A_k),\ \pm(y_k - A_k),\ \pm(v_k + y_k - A_k)$$

are pairwise disjoint.

By the hypothesis, if $n \in A_k + A_k$, then $r_{A_{k+1}}(n) = r_{A_k}(n) = 1$, and if $n(\neq 0) \in A_k - A_k$, then $\delta_{A_{k+1}}(n) = \delta_{A_k}(n) = 1$. Moreover,

$$u_k + x_k + A_k,\ -x_k + A_k,\ v_k + y_k + A_k,\ y_k + A_k$$

are translations. If n belongs to one of the above four sets, then $r_{A_{k+1}}(n) = 1$. Similarly, if n belongs to one of the sets

$$\pm(u_k + x_k - A_k),\ \pm(x_k + A_k),\ \pm(y_k - A_k),\ \pm(v_k + y_k - A_k),$$

then $\delta_{A_{k+1}}(n) = 1$. It follows that, for all $k \geq 2$,

$$r_{A_{k+1}}(n) \leq 1 \quad \text{for all } n \in \mathbb{Z},$$

and

$$\delta_{A_{k+1}}(n) \leq 1 \quad \text{for all } n \in \mathbb{Z} \setminus \{0\}.$$

Case 2: $u_k \in A_k + A_k$. Put

$$A_{k+1} = A_k \cup \{-u_k - x_k, x_k, y_k, v_k + y_k\}.$$

As in the proof of Case 1, we know that $r_{A_{k+1}}(n) \leq 1$ for all $n \in \mathbb{Z}$, $\delta_{A_{k+1}}(n) \leq 1$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Now we shall prove that the set A satisfies (iii).

If $u_k \notin A_k + A_k$, then, by the construction of A_{k+1} in Case 1, $u_k \in A_{k+1} + A_{k+1}$. If $-u_k \in A_{k+1} + A_{k+1}$, then, by the definition of u_{k+1}, $u_{k+2} \geq u_{k+1} > u_k$. If $-u_k \notin A_{k+1} + A_{k+1}$, then $u_{k+1} = u_k$. Thus $u_{k+1} = u_k \in A_{k+1} + A_{k+1}$. By the construction of A_{k+2} in Case 2, $-u_{k+1} \in A_{k+2} + A_{k+2}$. Thus $u_{k+2} > u_{k+1} = u_k$.

If $u_k \in A_k + A_k$, then, by the construction of A_{k+1} in Case 2, $-u_k \in A_{k+1} + A_{k+1}$. Moreover, $u_k \in A_k + A_k \subset A_{k+1} + A_{k+1}$, so $u_{k+2} \geq u_{k+1} > u_k$.

By the above discussion, $u_{k+2} > u_k$. By the construction of A_2, $u_2 \geq 3$. Thus $u_{2k} \geq u_2 + k - 1 \geq k + 2$. If there exists an integer n such that $|n| \leq k + 1$ and $n \notin A_{2k} + A_{2k}$, then $u_{2k} \leq k + 1$, which is a contradiction. Hence

$$\{-k-1, -k \cdots -1, 0, 1 \cdots k, k+1\} \subseteq A_{2k} + A_{2k}.$$

Similarly, we can show that $v_k < v_{k+1}$. Combining with $v_1 = 4$, we have $v_k \geq k + 3$. Hence

$$\{-k-2, -k-1 \cdots -1, 0, 1 \cdots k+1, k+2\} \subseteq A_k - A_k.$$

Let $A = \bigcup_{k=1}^{\infty} A_k$. Then $\mathbb{Z} = A + A = A - A$. If $r_A(n) \geq 2$ for some integer n or $\delta_A(m) \geq 2$ for some nonzero integer m, then there exists a positive integer k such that $r_{A_k}(n) \geq 2$ or $\delta_{A_k}(m) \geq 2$, which is a contradiction. So A is a unique bi-basis of $\mathbb{Z}$.

https://doi.org/10.1017/S0004972713000762 Published online by Cambridge University Press

Now we will show that A can be arbitrarily sparse. Given a function $\varphi(x)$ tending to infinity as $x \to \infty$, we use induction to construct a sequence $\{x_k\}_{k=1}^{\infty}$ such that $A(-x, x) \le \varphi(x)$ for all $x > x_1$. We observe that

$$A(-x, x) = A_{k+1}(-x, x) \le 4k + 3 \quad \text{for } d_k \le x < d_{k+1}.$$

We begin by choosing an integer $x_1 \ge 7$ such that

$$\varphi(x) \ge 7 \quad \text{for } x \ge x_1.$$

Then

$$A(-x, x) \le 7 \le \varphi(x) \quad \text{for } x_1 \le x \le d_2.$$

Let $k \ge 2$, and suppose we have selected an integer $x_{k-1} \ge 3d_{k-1} + 1$ such that

$$\varphi(x) \ge 4k - 1 \quad \text{for } x \ge x_{k-1}$$

and

$$A(-x, x) \le \varphi(x) \quad \text{for } x_{k-1} \le x \le d_k.$$

There exists an integer $x_k \ge 3d_k + 1$ such that

$$\varphi(x) \ge 4k + 3 \quad \text{for } x \ge x_k.$$

Then

$$A(-x, x) \le 4k + 3 \le \varphi(x) \quad \text{for } x_k \le x \le d_{k+1},$$

so

$$A(-x, x) \le \varphi(x) \quad \text{for } x_1 \le x \le d_{k+1}.$$

It follows that

$$A(-x, x) \le \varphi(x) \quad \text{for all } x \ge x_1.$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

We apply the method of Theorem 1.1 with

$$x_k = 3d_k + 1, \quad y_k = 3x_k + 2u_k \quad \text{for all } k \ge 2.$$

Note that

$$3d_k < x_k < u_k + x_k < v_k + y_k < 15d_k,$$

that is,

$$3d_k < d_{k+1} < 15d_k.$$

Since $d_1 = 2$,

$$2 \cdot 3^{k-1} < d_k < 2 \cdot 15^{k-1}.$$

https://doi.org/10.1017/S0004972713000762 Published online by Cambridge University Press

For $d_k < x \le d_{k+1}$,

$$2 \cdot 3^{k-1} < x < 2 \cdot 15^k.$$

Then

$$\frac{\log x - \log 2}{\log 15} < k < \frac{\log x - \log 2}{\log 3} + 1.$$

It is easy to see that

$$4k - 1 \le A(-x, x) \le 4k + 3 \quad \text{for } d_k < x \le d_{k+1}.$$

Hence

$$\frac{4(\log x - \log 2)}{\log 15} - 1 < A(-x, x) < \frac{4(\log x - \log 2)}{\log 3} + 7.$$

This completes the proof of Theorem 1.2.

Acknowledgement

We sincerely thank the referee for valuable comments.

References

[1] Y. G. Chen, 'The difference basis and bi-basis of $\mathbb{Z}_m$', *J. Number Theory* **130** (2010), 716–726.
[2] J. Cilleruelo and M. B. Nathanson, 'Perfect difference sets constructed from Sidon sets', *Combinatorica* **28** (2008), 401–414.
[3] J. Lee, 'Infinitely often dense bases for the integers with a prescribed representation function', *Integers* **10** (2010), 299–307.
[4] M. B. Nathanson, 'Unique representation bases for integers', *Acta Arith.* **108** (2003), 1–8.
[5] M. Tang, 'Dense sets of integers with a prescribed representation function', *Bull. Aust. Math. Soc.* **84** (2011), 40–43.
[6] C. W. Tang, M. Tang and L. Wu, 'Unique difference bases of $\mathbb{Z}$', *J. Integer Seq.* **14** (2011), Article 11.1.8.

RAN XIONG, School of Mathematics and Computer Science,
Anhui Normal University, Wuhu 241003, China
e-mail: ranxiong2012@163.com

MIN TANG, School of Mathematics and Computer Science,
Anhui Normal University, Wuhu 241003, China
e-mail: tmzzz2000@163.com

https://doi.org/10.1017/S0004972713000762 Published online by Cambridge University Press