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LATTICE POINTS IN A CONVEX SET OF GIVEN WIDTH

Dedicated to Professor George Szekeres on the occasion of his 65th birthday

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Let S be a closed bounded convex set in d-dimensional Euclidean space E^d. The width $w(S)$ of S is the minimum distance between supporting hyperplanes of S, and $L(S)$ is the number of integral lattice points in the interior of S.

If a is a positive real number, we define

$$g(a, d) = \min\{L(S) \colon w(S) > a\}.$$

Recently Scott (1973) has proved that

(1) $$g\left(\frac{2+\sqrt{3}}{2}, 2\right) = 1.$$

In Section 2 of this note we prove that

(2) $$g(a, 2) \geqq \left[\frac{2a}{2+\sqrt{3}}\right]^2,$$

where $[q]$ denotes the integral part of q. We also show that

(3) $$g(a, 2) \leqq \left[\frac{a^2}{\sqrt{3}}\right].$$

Earlier Sallee (1969) obtained a sharper result than Scott's (1) for sets of constant width in E^2. A set $W \subset E^d$ is said to have constant width a if the distance between any two parallel supporting hyperplanes of W equals a. From Sallee's result we have

(1*) $$g^*(1.546, 2) = 1,$$

where

$$g^*(a, d) = \min\{L(W) \colon w(W) > a\}.$$

We have the following estimates for g^*:

$$g^*(a, 2) \geqq \left[\frac{a}{1.546}\right]^2 \tag{2*}$$

and

$$g^*(a, 2) \leqq \frac{a^2}{2}(\pi - \sqrt{3}). \tag{3*}$$

In Section 3 we prove an analogue of Minkowski's classical result. Let $K \subset E^d$ be a convex body which is central symmetric about the origin 0. We define as before

$$g_0(a, d) = \min\{L(K) \colon w(K) > a\}.$$

Then

$$g_0(2, d) = 2d + 1 \tag{1_0}$$

and

$$g_0(a, d) \sim \frac{\pi^{d/2} \cdot a^d}{2^d \cdot \Gamma\left(\frac{d+2}{2}\right)} \qquad \text{as } a \to \infty. \tag{2_0}$$

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We now prove (2). Let S be a closed bounded convex set in E^2 with $w(S) > a$. Write $r = [2a/(2+\sqrt{3})]$. If $r = 0$, the result is clear. So suppose $r \geqq 1$ and consider the similarity transformation

$$S \to S' = \frac{1}{r}S = \left\{\frac{1}{r}Y \colon Y \in S\right\}.$$

Obviously,

$$w(S') = \frac{1}{r}w(S) > \frac{a}{r}.$$

Now let $T = (t_1, t_2)$ be a lattice point with $0 \leqq t_1, t_2 \leqq r - 1$ and consider the translate S'' of S' given by

$$S'' = S' - \frac{1}{r}T = \left\{X - \frac{1}{r}T \colon X \in S'\right\}.$$

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Obviously,

$$w(S'') = w(S') > \frac{a}{r} \geqq \frac{2+\sqrt{3}}{2}.$$

By (1), S'' contains a lattice point G. Hence S' contains the point $G + (1/r)T$, and so S contains the point $P = r(G + (1/r)T) = rG + T$. But $T = (t_1, t_2)$ might have been chosen in r^2 different ways, for we could have selected each of t_1, t_2 in r different ways. Therefore we have r^2 distinct lattice points $P = (p_1, p_2)$ in S. These are distinct, since $p_i \equiv t_i \pmod r$ $(i = 1, 2)$ and the t_i are a complete set of residue mod r. Hence $L(S) \geqq r^2$, from which we have (2).

The proof of (2*) is analogous to that of (2), using (1*) instead of (1).

To prove (3) we use the following

LEMMA 1. *Let $R \subset E^2$ be a closed bounded measurable region. Then the minimum number of lattice points in R is always less than the measure of R.*

For the proof, see Theorem 3 in Niven and Zuckerman (1967).

Now the area of an equilateral triangle of width a is $(a^2/\sqrt{3})$, from which (3) follows.

We remark that this bound is the best we can obtain by making use of the lemma, since it is well-known that of all convex sets of a given width, the equilaterial triangle has the smallest area.

Analogously, the area of the Reuleaux triangle of constant width a is $\frac{1}{2}a^2(\pi - \sqrt{3})$, from which (3*) follows.

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Both (1_0) and (2_0) are simple consequences of the following

LEMMA 2. *A central symmetric convex body $K \subset E^d$ centered at the origin, and of width a, contains the d-dimensional ball U centred at the origin, of radius $a/2$.*

Since $bd\, K$ is a closed set, there is a point P of it at a minimum distance m from 0. Then any supporting plane of K at P is normal to $0P$, since otherwise there is a point of $bd\, K$ nearer to 0 than P. By the central symmetry, $w(K) \leqq 2m$ and so $m \geqq \frac{1}{2}a$. The stated result follows.

We now prove (1_0). Suppose that K is a central symmetric convex body centred at 0, of width exceeding 2. By the lemma above, K contains a d-dimensional ball U of radius exceeding 1, centred at 0 and hence K contains, besides the origin, each of the $2d$ points $(0, \cdots, 0, \pm 1, 0, \cdots, 0)$. Hence $g(2, d) \geqq 2d + 1$. On the other hand, a d-dimensional ball of radius s $(1 < s < \sqrt{2})$

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contains, besides the origin, precisely the $2d$ points $(0,\cdots,0,\pm 1,0,\cdots,0)$, whence $g(2,d)\leqq 2d+1$. This proves (1_0).

It is well-known that the number of lattice points inside a d-dimensional ball U of radius r is asymptotically equal to its volume $\pi^{d/2}r^d/\Gamma((d+2)/2))$ as $r\to\infty$. An analogous argument to that just given, proves (2_0).

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