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EXPLICIT ZERO-COUNTING THEOREM FOR HECKE–LANDAU ZETA-FUNCTIONS

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Abstract

We give an explicit upper bound for the number of zeros of Hecke–Landau zeta-functions in a rectangle.

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1. Introduction

Let K denote any fixed totally imaginary field with discriminant $\Delta = \Delta(K)$ and degree $[K : \mathbb{Q}] = 2r_2$, where $2r_2$ is the number of complex-conjugate fields of K. Denote by $\mathfrak{f}$ a given nonzero integral ideal of the ring of algebraic integers O_K and by $H \pmod{\mathfrak{f}}$ any ideal class mod $\mathfrak{f}$ in the 'narrow' sense. Let $\chi(H)$ be a character of the abelian group of ideal classes $H \pmod{\mathfrak{f}}$ and let $\chi(\mathfrak{a})$ be the usual extension of $\chi(H)$. Let $s = \sigma + it$. The Hecke–Landau zeta-functions associated to χ are defined by

$$\zeta(s,\chi) = \sum_{\mathfrak{a}\in O_K} \frac{\chi(\mathfrak{a})}{(N\mathfrak{a})^s}, \quad \sigma > 1,$$

where $\mathfrak{a}$ runs through integral ideals and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. Throughout, χ_0 denotes the principal character modulo $\mathfrak{f}$. Let $N_\chi(T)$ denote the number of zeros of $\zeta(s,\chi)$ in the rectangle $0 \le \sigma \le 1$, $|t| \le T$. The aim of this paper is to prove the following explicit estimate for $N_\chi(T)$.

Theorem 1.1. *Let $T \ge 1$ and $\chi \ne \chi_0$ be a primitive character modulo $\mathfrak{f}$. Then*

$$\left|N_\chi(T) - \frac{T}{\pi}\log\left(\left(\frac{T}{2\pi e}\right)^{2r_2}|\Delta|N\mathfrak{f}\right)\right| \le 2r_2(A_1\log T + A_2\log\log(T+5)) + 2r_2(A_3\log(|\Delta|N\mathfrak{f}) + A_4\log\log(|\Delta|N\mathfrak{f})^{1/2r_2} + A_5),$$

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where

$$A_1 = \frac{1}{2\pi\log 2}, \quad A_2 = \frac{2}{\log 2}, \quad A_3 = \frac{1}{4\pi\log 2}, \quad A_4 = \frac{1}{\log 2},$$

$$A_5 = A_5(T, \Delta, \mathfrak{f}) = \frac{1}{\pi\log 2}\Big(1 + \frac{3}{\log(T+3)} + \frac{\log(|\Delta|N\mathfrak{f})^{1/2r_2}}{\log(T+3)}\Big)^{-1}$$
$$+ \frac{1}{2\pi\log 2}(1+2\eta(T))\log\Big(1+\frac{5}{T}\Big) + \frac{1}{4\pi\log 2}\eta(T)\log(|\Delta|N\mathfrak{f})$$
$$+ \frac{3r_2}{2\log 2}\log\Big(\Big(1+\frac{4}{\log((|\Delta|N\mathfrak{f})^{1/2r_2}(T+3))}\Big)\Big(1+\frac{\log((|\Delta|N\mathfrak{f})^{1/2r_2})}{\log(T+3))}\Big)\Big)$$
$$+ 3.347190$$

and

$$\eta(T) = (\log(e^3(|\Delta|N\mathfrak{f})^{1/2r_2}(T+3)))^{-1} \le \tfrac{1}{4}.$$

For the Riemann zeta-function $\zeta(s)$, estimates of this kind were obtained by Backlund [1] and Rosser [9] and improved by Trudgian [10, 11]. Estimates for the Dirichlet L-functions $L(s,\chi)$ and for the Dedekind zeta-functions $\zeta_K(s)$ are due to McCurley [7] and Kadiri and Ng [5], respectively. Recently, Trudgian [12] made significant improvements to the results for the Dirichlet L-functions and the Dedekind zeta-functions by a generalisation of the method introduced by Backlund [1] for the Riemann zeta-function. Theorems of this type for the Hecke–Landau zeta-functions exist in the literature, but do not have explicit constants. Our proof is similar in spirit to that of Trudgian [12], although the explicit expressions for the numerical constants do not contain the parameter $0 < \eta \le \frac{1}{2}$ which appears in [12].

Explicit results of this kind are useful for estimating the computational complexity of an algorithm which generates special primes [4]. Such primes are needed to construct an elliptic curve over a prime field using complex multiplication. In order to calculate exactly the running time of the algorithm, one needs an explicit bound for the number of special primes from the interval $[x, 2x]$ and this involves estimating sums over the zeros of $\zeta(s,\chi)$. Because of this application, we consider only Hecke–Landau zeta-functions of a totally imaginary algebraic number field.

2. Lemmas used in the proof of the main theorem

The function $\zeta(s,\chi_0)$ is regular in the whole complex plane, except for a simple pole at $s = 1$. For $\chi \ne \chi_0$, $\zeta(s,\chi)$ is regular in the whole complex plane. If $\chi \ne \chi_0$ is a primitive character modulo $\mathfrak{f}$, then $\zeta(s,\chi)$ satisfies the functional equation [6, Satz LXI, page 100]

$$\Phi(s,\chi) = W(\chi)\Phi(1-s,\overline{\chi}), \tag{2.1}$$

where $\Phi(s,\chi) = A(\mathfrak{f})^s\Gamma(s)^{r_2}\zeta(s,\chi)$, $A(\mathfrak{f}) = (2\pi)^{-r_2}\sqrt{|\Delta|N\mathfrak{f}}$ and $|W(\chi)| = 1$.

The Γ-function, $\Gamma(s)$, can be extended over the whole complex plane as a meromorphic function with simple poles at the negative integers and zero. We need an explicit version of Stirling's formula (for the first part, compare [8, page 294]).

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LEMMA 2.1. *For* $|\arg s| \le \frac{1}{2}\pi$,

$$\log\Gamma(s) = \Big(s-\frac{1}{2}\Big)\log s - s + \frac{1}{2}\log 2\pi + R_2(s), \quad |R_2(s)| \le \frac{C_1}{|s|},$$

$$\log\Gamma(s) = \Big(s-\frac{1}{2}\Big)\log s - s + \frac{1}{12s} - \frac{1}{360s^2} + \frac{1}{2}\log 2\pi + R_4(s), \quad |R_4(s)| \le \frac{C_2}{|s|^3},$$

where $C_1 = \frac{1}{12}$ *and* $C_2 = 0.0065$.

PROOF. The proof of Lemma 2.1 follows [3]. Let the 'split plane' be the set of all complex numbers other than the negative reals and zero. Throughout the split plane,

$$\Gamma(s) = \sqrt{2\pi}s^{(s-1/2)}\exp\Big(-s + \frac{B_2}{2s} + \frac{B_4}{4\cdot 3\cdot s^3} + \cdots + \frac{B_{2n}}{2n(2n-1)s^{2n-1}} + R_{2n}\Big),$$

where

$$R_{2n} = -\int_0^\infty \frac{\overline{B_{2n}}(x)\,dx}{2n(s+x)^{2n}} = -\int_0^\infty \frac{\overline{B_{2n+1}}(x)\,dx}{(2n+1)(s+x)^{2n+1}}$$

and

$$\overline{B_n}(x) = B_n(x-[x]), \quad B_n = B_n(0)$$

and $B_n(x)$ are the Bernoulli polynomials. For $|\arg s| \le \frac{1}{2}\pi$,

$$\log\Gamma(s) = \Big(s-\frac{1}{2}\Big)\log s - s + \frac{1}{2}\log 2\pi + \frac{B_2}{2s} + \cdots + \frac{B_{2n}}{2n(2n-1)s^{2n-1}} + R_{2n}.$$

We use the above formula with $n = 1, 2$. For $x \in [0, 1]$,

$$B_{2n}(x) = (-1)^{n+1}\frac{2\cdot(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\frac{\cos(2m\pi x)}{m^{2n}}.$$

Hence,

$$|\overline{B_2}(x)| \le \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{1}{\pi^2}\zeta(2) = \frac{1}{6},$$

$$|\overline{B_4}(x)| \le \frac{2\cdot 4!}{(2\pi)^4}\sum_{m=1}^{\infty}\frac{1}{m^4} \le \frac{2\cdot 4!}{(2\pi)^4}\Big(1 + \frac{1}{2^4} + \int_3^\infty \frac{dy}{y^4}\Big) = 0.033104.$$

Moreover,

$$\int_0^\infty \frac{dx}{2(s+x)^2} = \frac{1}{2s}$$

and

$$\int_0^\infty \frac{dx}{|s+x|^4} \le \int_0^\infty \frac{dx}{(|s|^2+x^2)^2} = \frac{1}{|s|^3}\int_0^\infty \frac{1}{(1+t^2)^2}\,dt$$
$$= \frac{1}{|s|^3}\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2 x}\,dx = \frac{1}{|s|^3}\int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4|s|^3}.$$

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Since $B_2(0) = \frac{1}{6}$ and $B_4(0) = -\frac{1}{30}$, we obtain $|R_2(s)| \le 1/(12|s|)$ and

$$|R_4(s)| \le \frac{0.033104}{16}\frac{\pi}{|s|^3} = \frac{0.0065}{|s|^3}.$$

This completes the proof. □

For $\delta > 0$, $\chi \ne \chi_0$, we define $\Delta_+ \arg \zeta(s,\chi)$ to be the change in argument of $\zeta(\sigma + iT, \chi)$ as σ varies from $\frac{1}{2}$ to $\frac{1}{2} + \delta$, and we define $\Delta_- \arg \zeta(s,\chi)$ to be the change in argument of $\zeta(\sigma + iT, \chi)$ as σ varies from $\frac{1}{2}$ to $\frac{1}{2} - \delta$.

Lemma 2.2. *Fix $\sigma_1 > 1$. Let $s = \sigma + iT$ and $\delta \le \sigma_1$. For $T \ge 1$,*

$$|\Delta_+ \arg \zeta(s,\chi) + \Delta_- \arg \zeta(s,\chi)| \le \tfrac{1}{2} r_2(\tfrac{8}{5}\sigma_1^2 + \tfrac{128}{81}(\sigma_1^6 + 3\sigma_1^3) + 3\sigma_1\pi + \tfrac{2}{3}).$$

Proof. Since $\Phi(s,\chi) = \overline{\Phi(1-\overline{s},\chi)}$, (2.1) shows that

$$\Delta_+ \arg \Phi(s,\chi) = -\Delta_- \arg \Phi(s,\chi).$$

It is easy to check that $\Delta_\pm \arg A(\mathfrak{f})^s = 0$. Hence,

$$|\Delta_+ \arg \zeta(s,\chi) + \Delta_- \arg \zeta(s,\chi)| = |\Delta_+ \arg \Gamma(s)^{r_2} + \Delta_- \arg \Gamma(s)^{r_2}|.$$

For $T \ge 1$, Lemma 2.1 shows that

$$\begin{aligned}
|\Delta_+ \arg \Gamma(s)^{r_2} + \Delta_- \arg \Gamma(s)^{r_2}| &\le r_2 \mathrm{Arg}\Big(\frac{1}{2} + \delta + iT\Big) + r_2 \mathrm{Arg}\Big(\frac{1}{2} - \delta + iT\Big) \\
&\quad + r_2 T \log \frac{\left|\frac{1}{2} + \delta + iT\right|}{\left|\frac{1}{2} + \delta\right|} + r_2 T \log \frac{\left|\frac{1}{2} - \delta + iT\right|}{\left|\frac{1}{2} + \delta\right|} + \frac{r_2}{3T} \\
&\le \frac{r_2}{2} T \log\Big(1 + \frac{\delta^2 + \delta}{\frac{1}{4} + T^2}\Big) + \frac{r_2}{2} T \log\Big(1 + \frac{\delta^2 - \delta}{\frac{1}{4} + T^2}\Big) \\
&\quad + \frac{3\delta\pi r_2}{2} + \frac{r_2}{3T} \\
&\le \frac{r_2}{2}\Big(\frac{8}{5}\delta^2 + \frac{128}{81}(\delta^6 + 3\delta^3) + 3\delta\pi + \frac{2}{3}\Big).
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Let $[K:\mathbb{Q}] = 2r_2$ and $\eta(t) = (\log(e^3(|\Delta| N\mathfrak{f})^{1/2r_2}(|t|+3)))^{-1}$. Then*

$$|\zeta(\sigma + it, \chi)| \le e^{r_2}(|t|+3)^{r_2(1-\sigma)}(|\Delta| N\mathfrak{f})^{(1-\sigma)/2}(\log(e^4(|\Delta| N\mathfrak{f})^{1/2r_2}(|t|+3)))^{2r_2}$$

for $-\eta(t) \le \sigma \le 1 + \eta(t)$, where $\chi \ne \chi_0$ is a primitive character modulo $\mathfrak{f}$.

Proof. Let $s = \sigma + it$ and $\chi \ne \chi_0$. Consider

$$g(s,\chi) = \frac{\zeta(s,\chi)}{\zeta(1-s,\overline{\chi})}, \tag{2.2}$$

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where χ is a primitive character modulo $\mathfrak{f}$. From the functional equation for $\zeta(s,\chi)$,

$$g(s,\chi) = W(\chi)A(\mathfrak{f})^{1-2s}\Big(\frac{\Gamma(1-s)}{\Gamma(s)}\Big)^{r_2}. \tag{2.3}$$

We estimate $g(s,\chi)$ on the line $s = -\eta + it$, $0 \le \eta \le \frac{1}{4}$, using the inequality

$$\Big|\frac{\Gamma(1-s)}{\Gamma(s)}\Big| \le 1.4 \max(1, |s|^{1+2\eta}) \tag{2.4}$$

(see [2, page 58]). From (2.3) and (2.4),

$$|g(-\eta + it,\chi)| \le 1.4^{r_2} A(\mathfrak{f})^{1+2\eta}(\max(1, |-\eta + it|^{(1+2\eta)}))^{r_2} \tag{2.5}$$

for $-\infty < t < \infty$. Fix $\varepsilon, \eta, \delta$, where $0 < \eta < \frac{1}{4}$ and $-\eta < \delta < 1 + \eta$. Define $G(s,\chi) = G(s,\varepsilon,\sigma_0,\chi)$ by

$$G(s,\chi) = e^{\varepsilon s}((-1 - 3\eta - 2\delta + s)(-1 - 3\eta - s))^{r_2(s-1-\eta)/2}\zeta(s,\chi). \tag{2.6}$$

The function $G(s,\chi)$ is regular in the strip $-\eta \le \sigma \le 1 + \eta$, $-\infty < t < \infty$, since

$$t(\mathrm{Arg}(-1 - 4\eta - 2\delta + it) + \mathrm{Arg}(-1 - 2\eta - it|)) \ge 0,$$

and so, from (2.2) and (2.5),

$$\begin{aligned}|G(-\eta + it,\chi)| &\le e^{-\varepsilon\eta}(|1 + 4\eta + 2\delta - it|\,|1 + 2\eta + it|)^{(-r_2/2)(1+2\eta)}|\zeta(s,\chi)|\\ &\le 1.4^{r_2} e^{-\varepsilon\eta} A(\mathfrak{f})^{1+2\eta}|\zeta(1 + \eta + it,\overline{\chi})|\\ &\le 1.4^{r_2} e^{-\varepsilon\eta} A(\mathfrak{f})^{1+2\eta}\zeta_K(1 + \eta).\end{aligned} \tag{2.7}$$

On the other hand,

$$|G(1 + \eta + it,\chi)| \le e^{\varepsilon(1+\eta)}|\zeta(1 + \eta + it,\chi)| \le e^{\varepsilon(1+\eta)}\zeta_K(1 + \eta). \tag{2.8}$$

We use the estimate $|\zeta(s,\chi)| \le A_1 e^{A_2|t|}$, valid in the strip $-\eta \le \sigma \le 1 + \eta$, where A_1, A_2 depend on K, χ and $\mathfrak{f}$. This yields

$$G(\delta + it,\chi) = O(e^{A_3|t|}) \tag{2.9}$$

for $-\eta \le \sigma \le 1 + \eta$, $-\infty < t < \infty$. From (2.7)–(2.9) and the well-known theorem of Phragmen–Lindelöf,

$$|G(\delta + it,\chi)| \le \zeta_K(1 + \eta) \max(1.4^{r_2} e^{-\varepsilon\eta} A(\mathfrak{f})^{1+2\eta}, e^{\varepsilon(1+\eta)}).$$

From (2.6),

$$|\zeta(\delta + it,\chi)| \le |1 + 3\eta + \delta + it|^{r_2(1+\eta-\delta)} e^{\varepsilon(1+\eta-\delta)}\zeta_K(1 + \eta),$$

where

$$\varepsilon = \log A(\mathfrak{f}) + \frac{r_2}{1 + 2\eta} \log 1.4.$$

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Hence,

$$|\zeta(\delta + it, \chi)| \le (|t| + 3)^{r_2(1+\eta-\delta)}(|\Delta| N\mathfrak{f})^{(1+\eta-\delta)/2}\zeta_K(1 + \eta).$$

Using

$$\zeta_K(1+\eta) \le (\zeta(1+\eta))^{2r_2} \le \Big(1 + \int_2^\infty \frac{dt}{t^{1+\eta}}\Big)^{2r_2} = \Big(1 + \frac{1}{\eta}\Big)^{2r_2}$$

and putting $\eta = (\log(e^3(|\Delta|N\mathfrak{f})^{1/2r_2}(|t| + 3)))^{-1} \le \frac{1}{4}$ yields

$$|\zeta(\delta + it, \chi)| \le e^{r_2}(|t| + 3)^{r_2(1-\delta)}(|\Delta| N\mathfrak{f})^{(1-\delta)/2}(\log(e^4(|\Delta| N\mathfrak{f})^{1/2r_2}(|t| + 3)))^{2r_2}.$$

This completes the proof. □

Lemma 2.4. *For $\sigma > 1$,*

$$|\zeta(\sigma + it, \chi)| > \frac{1}{\zeta_K(\sigma)}.$$

Proof. Let $s = \sigma + it$. Then

$$\frac{1}{|\zeta(s,\chi)|} = \Big|\prod_{\mathfrak{p}}\Big(1 - \frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^s}\Big)\Big| \le \prod_{\mathfrak{p}}\Big(1 + \frac{1}{(N\mathfrak{p})^\sigma}\Big) < \prod_{\mathfrak{p}}\Big(1 - \frac{1}{(N\mathfrak{p})^\sigma}\Big)^{-1} = \zeta_K(\sigma).$$

This completes the proof. □

3. The proof of the main theorem

Proof of Theorem 1.1. Let $\sigma_1 > 1$, $\sigma_1 = \sigma_1(T)$ and $\mathcal{R}$ be a positively oriented rectangle with vertices at $\sigma_1 \pm iT$, $1 - \sigma_1 \pm iT$, where $\pm T$ does not coincide with the ordinate of a zero of $\zeta(s, \chi)$. Let C be the part of the contour $\mathcal{R}$ with $\sigma \ge \frac{1}{2}$. By Cauchy's theorem,

$$2\pi N_\chi(T) = \Delta_{\mathcal{R}} \arg \Phi(s, \chi) = \Im \int_{\mathcal{R}} \frac{\Phi'}{\Phi}(s, \chi)\, ds. \tag{3.1}$$

From the functional equation,

$$\frac{\Phi'}{\Phi}(s,\chi) = -\frac{\Phi'}{\Phi}(1 - s, \overline{\chi}).$$

Since $\overline{\Phi(s, \overline{\chi})} = \Phi(\overline{s}, \chi)$,

$$\Im \int_{1-\sigma_1+iT}^{1-\sigma_1-iT} \frac{\Phi'}{\Phi}(s,\chi)\, ds = \int_T^{-T} \Re \frac{\Phi'}{\Phi}(1 - \sigma_1 + it, \chi)\, dt = -\int_T^{-T} \Re \frac{\Phi'}{\Phi}(\sigma_1 - it, \overline{\chi})\, dt$$
$$= -\int_T^{-T} \Re \frac{\Phi'}{\Phi}(\sigma_1 + it, \chi)\, dt = \Im \int_{\sigma_1-iT}^{\sigma_1+iT} \frac{\Phi'}{\Phi}(s,\chi)\, ds$$

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and

$$\Im\int_{\sigma_1+iT}^{(1/2)+iT}\frac{\Phi'}{\Phi}(s,\chi)\,ds=\int_{\sigma_1}^{1/2}\Im\frac{\Phi'}{\Phi}(\sigma+iT,\chi)\,d\sigma=-\int_{\sigma_1}^{1/2}\Im\frac{\Phi'}{\Phi}(1-\sigma-iT,\overline{\chi})\,d\sigma$$
$$=\int_{\sigma_1}^{1/2}\Im\frac{\Phi'}{\Phi}(1-\sigma+iT,\chi)\,d\sigma=\Im\int_{(1/2)+iT}^{1-\sigma_1+iT}\frac{\Phi'}{\Phi}(s,\chi)\,ds. \tag{3.2}$$

Similarly,

$$\Im\int_{\sigma_1-iT}^{(1/2)-iT}\frac{\Phi'}{\Phi}(s,\chi)\,ds=\Im\int_{(1/2)-iT}^{1-\sigma_1-iT}\frac{\Phi'}{\Phi}(s,\chi)\,ds. \tag{3.3}$$

Consequently, $\Delta_{\mathcal{R}}\arg\Phi(s,\chi)=2\Delta_C\arg\Phi(s,\chi)$. By (3.1), it is obvious that

$$N_\chi(T)=\frac{1}{\pi}\Delta_C\arg A(\mathfrak{f})^s+\frac{1}{\pi}\Delta_C\arg\Gamma(s)^{r_2}+\frac{1}{\pi}\Delta_C\arg\zeta(s,\chi) \tag{3.4}$$

and a trivial verification shows that

$$\Delta_C\arg A(\mathfrak{f})^s=2T\log A(\mathfrak{f}). \tag{3.5}$$

By Lemma 2.1,

$$\begin{aligned}\Delta_C\arg\Gamma(s)^{r_2}&=r_2\Big(\Im\log\Gamma\Big(\frac{1}{2}+iT\Big)-\Im\log\Gamma\Big(\frac{1}{2}-iT\Big)\Big)\\&=2Tr_2\log\frac{T}{e}+Tr_2\log\Big(1+\frac{1}{4T^2}\Big)+\frac{2Tr_2}{3(1+4T^2)}+\frac{4Tr_2}{45(1+4T^2)^2}\\&\quad+r_2\Im R_4\Big(\frac{1}{2}+iT\Big)-r_2\Im R_4\Big(\frac{1}{2}-iT\Big)\\&\le 2Tr_2\log\frac{T}{e}+\frac{r_2}{4T}-\frac{r_2}{32T^3}+\frac{r_2}{192T^5}+\frac{2r_2}{15}+\frac{4r_2}{1125}+\frac{2C_2}{(\frac{1}{4}+T^2)^{3/2}}\\&\le 2Tr_2\log\frac{T}{e}+0.370150r_2\end{aligned}$$

for $T\ge 1$. From this, together with (3.4) and (3.5),

$$\left|N_\chi(T)-\frac{T}{\pi}\log\Big(\Big(\frac{T}{2\pi e}\Big)^{2r_2}|\Delta|N\mathfrak{f}\Big)\right|\le\frac{1}{\pi}\Delta_C\arg\zeta(s,\chi)+C_2r_2, \tag{3.6}$$

where $C_2=0.117823$. We now give an estimate for $|\Delta_C\arg\zeta(s,\chi)|$. To do this, we divide C into C_i, $i=1,2,3$, as follows:

$$\begin{aligned}C_1&:\tfrac{1}{2}-iT\quad\text{to}\quad\sigma_1-iT,\\C_2&:\sigma_1-iT\quad\text{to}\quad\sigma_1+iT,\\C_3&:\sigma_1+iT\quad\text{to}\quad\tfrac{1}{2}-iT,\end{aligned}$$

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where $\sigma_1 = \sigma_1(T)$. We start by estimating the change of argument of $\zeta(s,\chi)$ along C_2:

$$\begin{aligned}|\Delta_{C_2} \arg \zeta_K(s,\chi)| &= \left|\Im \int_{\sigma-iT}^{\sigma+iT} \frac{\zeta'}{\zeta}(s,\chi)\,ds\right| = \left|\Im \int_{\sigma-iT}^{\sigma+iT} \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi(\mathfrak{p}^m)\log N\mathfrak{p}}{(N\mathfrak{p})^{ms}}\,ds\right| \\ &= \left|-\Im \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi(\mathfrak{p}^m)}{m(N\mathfrak{p})^{m(\sigma+iT)}} + \Im \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi(\mathfrak{p}^m)}{m(N\mathfrak{p})^{m(\sigma-iT)}}\right| \\ &= \left|2\sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi(\mathfrak{p}^m)\sin(mT\log N\mathfrak{p})}{m(N\mathfrak{p})^{m\sigma}}\right| \le 2\sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{1}{m(N\mathfrak{p})^{m\sigma}} \\ &\le 4r_2 \sum_{p} \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma}} \\ &= 4r_2 \sum_{p} \log\Big(1-\frac{1}{p^{\sigma}}\Big)^{-1} = 4r_2\log(\zeta(\sigma))\end{aligned}$$

for $\sigma > 1$. Hence,

$$\left|\frac{1}{\pi}\Delta_{C_2} \arg \zeta(s,\chi)\right| \le \frac{2}{\pi}\log \zeta(\sigma_1)^{2r_2} \le \frac{2}{\pi}\log\Big(1+\frac{1}{\sigma_1}\Big)^{2r_2}. \tag{3.7}$$

We next give an estimate for $\Delta_{C_3} \arg \zeta(s,\chi)$. Let N be a positive integer. For any primitive character $\chi \ne \chi_0$ modulo $\mathfrak{f}$, define

$$f(s) = \tfrac{1}{2}(\zeta(s+iT,\chi)^N + \zeta(s-iT,\overline{\chi})^N).$$

Since $\zeta(\overline{s},\chi) = \overline{\zeta(s,\overline{\chi})}$, for σ real,

$$f(\sigma) = \Re\zeta(\sigma+iT,\chi)^N.$$

Let m be the number of zeros of $\Re\zeta(\sigma+iT,\chi)^N$, where $\frac{1}{2} \le \sigma \le \sigma_1$. The interval $[\frac{1}{2},\sigma_1]$ is divided into $m+1$ parts, throughout each of which $\Re\zeta(\sigma+iT,\chi)^N \ge 0$ or $\Re\zeta(\sigma+iT,\chi)^N \le 0$. Hence,

$$|\Delta_{C_3} \arg \zeta(s,\chi)| = \frac{1}{N}|\Delta_{C_3} \arg \zeta(s,\chi)^N| \le \frac{(m+1)\pi}{N}. \tag{3.8}$$

We now give an estimate for m. Let

$$\eta = \eta(T) = (\log(e^3(|\Delta|N\mathfrak{f})^{1/2r_2}(|T|+3)))^{-1} \le \tfrac{1}{4}.$$

We define

$$\sigma_0 = \sigma_0(T) = 1+\eta(T), \quad \sigma_1(T) = \sigma_0 + \tfrac{1}{2}(\tfrac{1}{2}+\eta(T)).$$

We apply [12, Lemma 2] with $F(s) = \zeta(s,\chi)$. Lemma 2.2 shows that

$$|\Delta_+ \arg \zeta(s,\chi) + \Delta_- \arg \zeta(s,\chi)| \le E = E(T,r_2) = 14.200774r_2$$

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for $T \geq 1$. Firstly, we assume that

$$|\Delta_{C_3} \arg \zeta(s,\chi)^N| < 3 + \left\lfloor \frac{NE}{\pi} \right\rfloor.$$

By (3.6) and (3.7),

$$\left| N_\chi(T) - \frac{T}{\pi} \log\left(\left(\frac{T}{2\pi e}\right)^{2r_2} (|\Delta| N\mathfrak{f})\right)\right| \leq \left(\frac{4}{\pi} \log \zeta(\sigma_1) + 0.1178230\right) r_2 + \frac{2E}{\pi^2} + \frac{6}{N\pi}.$$

Secondly, assume that there exists $m \geq 3 + \lfloor NE/\pi \rfloor$ such that

$$m\pi \leq |\Delta_{C_3} \arg \zeta(s,\chi)^N|.$$

By [12, Lemma 2], there are at least m distinct zeros ϱ_j of $\Re\zeta(\sigma + iT,\chi)^N$, where $\varrho_j = x_j + iT$ for $j = 1,\ldots,m$ and $\frac{1}{2} \leq x_m < \cdots < x_1 \leq \sigma_1$, and there are at least $m - 2 - \lfloor NE/\pi \rfloor$ distinct zeros ϱ_j of $\Re\zeta(\sigma + iT,\chi)^N$, where $\varrho'_j = x'_j + iT$ for $j = 1,\ldots,m - 2 - \lfloor NE/\pi \rfloor$ and $1 - \sigma_1 \leq x'_1 < \cdots < x'_{m-2-\lfloor NE/\pi \rfloor} \leq \frac{1}{2}$. Moreover,

$$x_j \geq 1 - x'_j, \quad j = 1,\ldots,m - 2 - \left\lfloor \frac{NE}{\pi} \right\rfloor. \tag{3.9}$$

Fix $R = R(T) = 1 + 2\eta(T)$. (We remark that Trudgian [12] takes $R = r(\frac{1}{2} + \eta)$, where $r > 1$, and optimises R over r.) Let $n(T)$ be the number of zeros of $f(s)$ in the circle

$$|s - \sigma_0| \leq R.$$

To estimate $n(T)$, define k to be the number of zeros $\varrho_j = x_j + iT$ of $f(\sigma)$ such that $\sigma_0 < x_j < \sigma_1$, where $j = 1,\ldots,k$. From (3.9), there are zeros ϱ' of $f(\sigma)$ satisfying $1 - \sigma_1 < x'_j < -\eta$, where $j = 1,\ldots,k$. Let $x_{m-2-\lfloor NE/\pi \rfloor} < x_j < \sigma_0$. From (3.9), there are zeros ϱ' of $f(\sigma)$ such that $x'_j \geq -\eta$. Since $f(s)$ is regular in the circle $|s - \sigma_0| < R$ and $f(\sigma_0) \neq 0$, Jensen's theorem shows that

$$\log \frac{R^{2m-k-2-\lfloor NE/\pi \rfloor}}{M} = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \log |f(\sigma_0 + Re^{i\theta})|\, d\theta - \log |f(\sigma_0)| = J_1 - \log |f(\sigma_0)|,$$

where

$$M = \prod_{j=1}^{k} |\sigma_0 - x_j| \prod_{j=k+1}^{m-2-\lfloor NE/\pi \rfloor} |\sigma_0 - x_j||\sigma_0 - x'_j| \prod_{j=m-1-\lfloor NE/\pi \rfloor}^{m} |\sigma_0 - x_j|$$

and

$$J_1 = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \log |f(\sigma_0 + Re^{i\theta})|\, d\theta.$$

For $j = 1,\ldots,k$,

$$\log \frac{R}{|\sigma_0 - x_j|} \geq \log \frac{R}{|\sigma_0 - \sigma_1|} \geq 2 \log 2$$

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and, for $j = k+1, \dots, m-2-\lfloor NE/\pi \rfloor$,

$$\log \frac{R^2}{|\sigma_0 - x_j||\sigma_0 - x'_j|} = \log \frac{4(\frac{1}{2}+\eta)^2}{(1+\eta-x_j)(\eta+x_j)} \geq 2\log 2$$

and, for $j = m-1-\lfloor NE/\pi \rfloor, \dots, m$,

$$\log \frac{R}{|\sigma_0 - x_j|} \geq \log \frac{R}{|\sigma_0 - \frac{1}{2}|} \geq \log 2,$$

whence

$$J_1 - \log|f(\sigma_0)| \geq 2\log 2\Big(m-k-2-\Big\lfloor \frac{NE}{\pi} \Big\rfloor\Big) + 2k\log 2 + \log 2\Big(2+\Big\lfloor \frac{NE}{\pi} \Big\rfloor\Big)$$

and, consequently,

$$m \leq \frac{1}{2\log 2}J_1 - \frac{1}{2\log 2}\log|f(\sigma_0)| + 1 + \frac{NE}{2\pi}. \tag{3.10}$$

To estimate J_1, write

$$J_1 = \frac{1}{2\pi}\Big(\int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}\Big)\log|f(\sigma_0 + R(T)e^{i\theta})|\,d\theta = J_2 + J_3. \tag{3.11}$$

From Lemma 2.3,

$$\begin{aligned}
J_3 &\leq \frac{N}{2\pi}\int_{\pi/2}^{3\pi/2} \log|\zeta(\sigma_0(T) + R(T)\cos\theta + i(R(T)\sin\theta + T), \chi)|\,d\theta \\
&\leq \frac{Nr_2}{2\pi}R(T)\log(T+5)\int_{\pi/2}^{3\pi/2}(-\cos\theta)\,d\theta \\
&\quad - \frac{Nr_2}{2\pi}\eta(T)\int_{\pi/2}^{3\pi/2}\log(|R(T)\sin\theta + T| + 3)\,d\theta \\
&\quad + \frac{Nr_2}{4\pi}\log(|\Delta|N\mathfrak{f})R(T)\int_{\pi/2}^{3\pi/2}(-\cos\theta)\,d\theta - \frac{1}{4}\eta(T)Nr_2\log(|\Delta|N\mathfrak{f}) \\
&\quad + Nr_2\log\log(T+5) + Nr_2\log\log(|\Delta|N\mathfrak{f})^{1/2r_2} + Nr_2\log\log e^4 + \frac{Nr_2}{2} \\
&\leq \frac{1}{\pi}Nr_2R(T)\log(T+5) + \frac{1}{2\pi}R(T)Nr_2\log(|\Delta|N\mathfrak{f}) + Nr_2\log\log(T+5) \\
&\quad + Nr_2\log\log(|\Delta|N\mathfrak{f})^{1/2r_2} + Nr_2\log\log e^4 + \frac{Nr_2}{2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
J_2 &\leq \frac{1}{2}N\log\zeta(1+\eta(T))^{2r_2} \leq Nr_2\log\Big(1 + \frac{1}{\eta(T)}\Big) \\
&\leq Nr_2\log\log(e^4(|\Delta|N\mathfrak{f})^{1/2r_2}(T+3)).
\end{aligned}$$

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To complete the bound for m, we estimate $-N\log(|f(\sigma_0)|)$. To do this, we write

$$\zeta(1+\eta+iT,\chi)=re^{i\varphi}.$$

Choose a sequence of integers N tending to infinity such that $N\varphi$ tends to 0 modulo 2π (by Dirichlet's approximation theorem). It follows that

$$\lim_{N\to\infty}\frac{f(\sigma_0)}{|\zeta(\sigma_0+iT,\chi)|^N}=1.$$

By Lemma 2.4,

$$\begin{aligned}-\log|f(\sigma_0)|&\le N\log\frac{1}{|\zeta(\sigma_0+iT,\chi)|}\le N\log|\zeta_K(\sigma_0)|\le N\log\zeta(1+\eta(T))^{2r_2}\\&\le 2r_2N\log\log(e^4(|\Delta|N\mathfrak{f})^{1/2r_2}(|T|+3))\end{aligned}\tag{3.12}$$

as $N\to\infty$. From (3.8), (3.10), (3.11) and (3.12),

$$\begin{aligned}\Big|\frac{1}{\pi}\Delta_{C_3}\arg\zeta(s,\chi)\Big|&\le\frac{r_2}{2\pi\log 2}R(T)\log T+\frac{r_2}{2\pi\log 2}R(T)\log\Big(1+\frac{5}{T}\Big)\\&\quad+\frac{r_2}{2\log 2}\log\log(T+5)+\frac{r_2}{4\pi\log 2}R(T)\log(|\Delta|N\mathfrak{f})\\&\quad+\frac{r_2}{2\log 2}\Big(\log\log(|\Delta|N\mathfrak{f})^{1/2r_2}+\log\log e^4+\frac{1}{2}\Big)\\&\quad+\frac{3r_2}{2\log 2}\log\log(e^4(|\Delta|N\mathfrak{f})^{1/2r_2}(T+3))+\frac{E}{2\pi}.\end{aligned}\tag{3.13}$$

By (3.2) and (3.3), the same bound holds with C_1 in place of C_3. From (3.7) and (3.13), with $R(T)=1+2\eta(T)$,

$$\begin{aligned}\Big|\frac{1}{\pi}\Delta_C\arg\zeta(s,\chi)\Big|&\le\frac{2r_2}{2\pi\log 2}\log T+\frac{4r_2}{\log 2}\log\log(T+5)+\frac{r_2}{4\pi\log 2}\log(|\Delta|N\mathfrak{f})\\&\quad+\frac{r_2}{\log 2}\log\log(|\Delta|N\mathfrak{f})^{1/2r_2}+C_3(T,r_2,\Delta,\mathfrak{f}),\end{aligned}$$

where

$$\begin{aligned}C_3(T,r_2,\Delta,\mathfrak{f})&=\frac{2r_2}{\pi\log 2}\Big(1+\frac{3}{\log(T+3)}+\frac{\log(|\Delta|N\mathfrak{f})^{1/2r_2}}{\log(T+3)}\Big)^{-1}\\&\quad+\frac{2r_2}{2\pi\log 2}(1+2\eta(T))\log\Big(1+\frac{5}{T}\Big)+\frac{r_2}{2\pi\log 2}\eta(T)\log(|\Delta|N\mathfrak{f})\\&\quad+\frac{3r_2}{\log 2}\log\Big(\Big(1+\frac{4}{\log((|\Delta|N\mathfrak{f})^{1/2r_2}(T+3))}\Big)\Big(1+\frac{\log((|\Delta|N\mathfrak{f})^{1/2r_2}}{\log(T+3))}\Big)\Big)\\&\quad+\frac{E}{\pi}+\frac{4r_2}{\pi}\log\frac{9}{5}+\frac{r_2}{\log 2}\Big(\log\log e^4+\frac{1}{2}\Big).\end{aligned}$$

From (3.6), Theorem 1.1 follows. □

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