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The Complete (L^p, L^p) Mapping Properties of Some Oscillatory Integrals in Several Dimensions

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Abstract. We prove that the operators $\int_{\mathbb{R}^2_+} e^{ix^a \cdot y^b} \varphi(x, y) f(y)\, dy$ map $L^p(\mathbb{R}^2)$ into itself for $p \in J = \left[\frac{a_l+b_l}{a_l+(\frac{b_l r}{2})}, \frac{a_l+b_l}{a_l(1-\frac{r}{2})}\right]$ if $a_l, b_l \geq 1$ and $\varphi(x, y) = |x - y|^{-r}$, $0 \leq r < 2$, the result is sharp. Generalizations to dimensions $d > 2$ are indicated.

0 Introduction

Our purpose here is to study the (L^p, L^p) mapping properties for a class of oscillatory integral operators in several dimensions. For the sake of simplicity we consider a two dimensional case with the understanding that some of these arguments can be extended to dimension $d > 2$. We use iterative methods to go from dimension 1 to 2 and we can continue this approach up to dimension d. But in general we would need to require condition (0.4) below to hold for more j's.

We consider the integral operators with kernel $k(x, y)$, $x, y \in \mathbb{R}^2$,

$$Kf(x) = \int_{\mathbb{R}^2} k(x, y) f(y)\, dy, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{0.1}$$

and k is of the form

$$k(x, y) = \varphi(x, y) \exp\big(ig(x, y)\big) \tag{0.2}$$

and g is a real-valued function; throughout this paper $g(x, y) = x_1^{a_1} y_1^{b_1} + x_2^{a_2} y_2^{b_2}$, $a_1, a_2, b_1, b_2 \geq 1$.

Such operators were considered in various contexts, for example see [St] and the references given there. The one dimensional case was addressed in [PSS], [PS], [SS], [CP] and [S].

Included in our class of operators is the Fourier transform, *i.e.* $g(x, y) = x \cdot y$ and for this choice of g and φ satisfying

$$|\partial_x^\alpha \partial_y^\beta \varphi(x, y)| \leq C_{\alpha\beta} |x - y|^{-|\alpha|-|\beta|}, \quad \text{for all } \alpha \text{ and } \beta \tag{0.3}$$

the L^2-boundedness of K was obtained in [P1], [PhS1]. A similar program to the one we are doing here was carried out in the convolution case, in [JS].

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In Section 1 we consider the case when $r = 0$, and $a_j \geq b_j \geq 1$. There we obtain the condition that $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. We reduce the two (or higher) dimensional case to the one dimensional situation dealt with previously in [PSS], [PS], [S], [CP].

In Section 2 we consider the singular case when $\varphi(x, y)$ is like $|x - y|^{-2}$ and we extend the one dimensional result in [PS] showing that K is a bounded operator from a suitable version of the Hardy space H^1 into L^1.

In Section 3 we return to the case when $r = 0$ but this time we consider the limit cases of the exponents $b_1 = 1$ or $b_2 = 1$.

In Section 4 we prove there that the operator K where $\varphi(x, y)$ is like $|x - y|^{-2}$ maps L^p into itself for $1 < p < \infty$ and in that case $a, b \geq 1$.

The last section deals with the main result of the paper as explained above.

We use the following convenient notations and conventions $x = (x_1, x_2)$, $n = (n_1, n_2)$ with $x \in \mathbb{R}^2_+$, $n \in \mathbb{Z}^2_+$. We also write $x^a = (x_1^{a_1}, x_2^{a_2})$, $x^a \cdot y^b = x_1^{a_1} y_1^{b_1} + x_2^{a_2} y_2^{b_2}$, similarly for $|x|^a \cdot |y|^b$ where $|x|^a = (|x_1|^{a_1}, |x_2|^{a_2})$ which should not be confused with $|x|^a = (x_1^2 + x_2^2)^{a/2}$ $a \in \mathbb{C}$.

We denote by C, indexed if needed, positive constants depending only on k, and it is understood that even in the same string of formulas, C in different places may stand for different constants.

We assume that φ is in C^3 away from the diagonal $y = x$ and that satisfies

$$|\bigtriangledown^j \varphi(x, y)| \leq C|x - y|^{-j-r}, \quad j = 0, 1, 2, 3, \ 0 \leq r \leq 2. \tag{0.4}$$

The model operator occurs when $k(x, y) = |x - y|^{-r+i\tau} e^{ix^a \cdot y^b}$, $\tau \in \mathbb{R}$, with $a_l, b_l \geq 1$ and $a_1/b_1 = a_2/b_2$. This last condition is not required in Section 2 and not needed in Theorem 4.4 where we prove that K maps L^p into itself. The singular case $r = 2$ and the exceptional choices of the parameters a or b ($b_1 = 1, b_2 = 1, a_1 = 1$, or $a_2 = 1$) will require some strengthening of (0.4) and imposition of additional conditions on K near the diagonal. These conditions will be made explicit when needed.

1 L^p Boundedness for $b \geq 1$ and Preliminary Estimates

Our goal in this section is to extend to two dimensions the following one dimensional result (see Section 3 in [PSS]).

Proposition 1.1 *For $x, y \in [0, \infty) = \mathbf{R}_+$ let $k(x, y) = \varphi(x, y) e^{ix^a y^b}$, where $a, b > 1$ and $\varphi \in C^1(\{x \neq y\})$ satisfies*

$$|\nabla^j \varphi(x, y)| \leq C_j |x - y|^{-j}, \quad j = 0, 1. \tag{1.1}$$

Then the integral operator K with kernel k maps $L^p(\mathbf{R}_+)$ into itself for $p = \frac{a+b}{a}$.

In the case when either $a = 1$ or $b = 1$ the conclusion of the proposition remains valid provided $\varphi \in C^\infty(\{x \neq y\})$ satisfies (1.1) for $j = 0, 1, \ldots$. (the bound on the norm of K is of course determined by the constants C_j). This is the result in [CP]. In particular it applies to the model case where $\varphi(x, y) = \left(s^2 + (x - y)^2\right)^{i\tau}$, $\tau \in \mathbf{R}$. This will be taken up in Section 3.

We state a two dimensional version of the proposition in the following theorem. We note that the condition $\frac{b_1}{a_1} = \frac{b_2}{a_2}$ will be used throughout this section.

Theorem 1.2 *Let $a_l \geq b_l > 1$ for l=1,2, $\frac{b_1}{a_1} = \frac{b_2}{a_2}$ and let φ satisfy (0.4). Then the integral operator with the kernel $k(x,y) = e^{ix^a \cdot y^b}\varphi(x,y)$ is bounded from $L^p(\mathbf{R}^2_+)$ into itself for $p = \frac{a_1+b_1}{a_1}$.*

In order to prove Theorem 1.2. we first observe that we can assume that (the amplitude) $\varphi(x,y) = 0$ near the 'diagonal' $\{x = y\}$, say for $|x-y| \leq 1$. This is accomplished by writing $k(x,y) = \eta(|x-y|)k(x,y) + (1-\eta(|x-y|)k(x,y) = k_1(x,y) + k_2(x,y)$, where $0 \leq \eta \in C_0^\infty(\mathbf{R})$, $\eta(t) = 1$ for $|t| < 1$ and $\eta(t) = 0$ for $|t| > 2$. Then, by a simple application of Schur's lemma, the integral operator with the kernel k_1 is bounded in $L^p(\mathbf{R}^2_+)$ for all p, $1 \leq p \leq \infty$ while the kernel k_2 satisfies the same conditions as k, *i.e.*, (0.4). Hence we may replace k by k_2 and assume that

$$\varphi(x,y) = 0 \quad \text{for } |x-y| \leq 1 \tag{1.2}$$

The next step consists of the introduction of a convenient partition of unity on the quadrant $\mathbf{R}^2_+$: $1 = \sum_{n_1,n_2=0}^\infty \psi_n(x)$, where $\psi_n(x) = \psi_{n_1n_2}(x_1,x_2)$ are defined as follows. With η as above we let $\psi_{00}(x) = \eta(x_1)\eta(x_2)$, $\psi_{01}(x) = \eta(x_1)\cdot \big(1-\eta(x_2)\big)$, $\psi_{10}(x) = \big(1-\eta(x_1)\big)\eta(x_2)$ and for $n_1, n_2 \geq 1$ $\psi_n(x) = \big(1-\eta(x_1)\big)\cdot \big(1-\eta(x_2)\big)\psi(2^{-n_1}x_1)\psi(^{-n_2}x_2)$, where $0 \leq \psi \in C_0^\infty(\mathbf{R})$, $\psi(t) = 0$ for $t \notin (\frac{1}{4}, 1)$ is so chosen that $\sum_{j=1}^\infty \psi(2^{-j}t) = 1$ for $t > 1$. We let $k_{mn}(x,y) := \psi_m(x)k(x,y)\psi_n(y)$, denote by K_{mn} the corresponding integral operators and write

$$K = \sum K_{mn}. \tag{1.3}$$

The operators K_{mn}, or partial sums of them, are now estimated by different means depending on the values of the indices m, n. We first consider the partial sum corresponding to one of the four indices running from 0 to 2 and the remaining running from 0 to ∞. In this case we have four possibilities: The sum is equal to $\lambda(x_l)k(x,y)$ or $k(x,y)\lambda(y_l)$ with either $l = 1$ or $l = 2$ and with $\lambda = [\eta + (1-\eta)]\big(\psi(t) + \psi(t/2)\big)$. Note that $\lambda = 0$ outside the interval $[0,4)$. The needed estimate for the sum is in this case obtained by means of the following proposition which also provides the crucial step for descent from dimension d to dimension $d-1$ when $d > 2$. Note that (as opposed to Theorem 1.2) the limiting values of the phase exponents, $a_l, b_l = 1$ are allowed with the caveat that the result is known to be valid in the lower dimension.

Proposition 1.3 *With k and p as in Theorem 1.2 and $a_l \geq b_l \geq 1$ suppose that (the amplitude) φ is independent of one of the variables $x_1, \ldots, y_2$. Then the corresponding integral operator is bounded from $L^p(\mathbf{R}^2_+)$ into itself.*

Proof We consider the cases when φ is independent of either x_2 or of y_2. In the first case we write

$$Kf(x) = \int_0^\infty e^{ix_2^{a_2}{y_2}^{b_2}} h(x_1,y_2)\,dy_2,$$

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where $h(x_1, y_2) = \int_0^\infty e^{ix_1^{a_1} y_1^{b_1}} \varphi(x_1, y) f(y)\, dy_1$. By Proposition 1.1 (with amplitude $= 1$) we get $\int |Kf(x)|^p\, dx_2 \leq C \int |h(x_1, y)|^p\, dy_2$. With y_2 considered as a parameter, $h(x_1, y_2)$ is the image of $f(\cdot, y_2)$ by the transform as in Proposition 1.1 with the amplitude $\varphi(x_1, y_1, y_2)$ with the L^p norm bounded uniformly in y_2. Hence $\int \bigl(\int |h(x_1, y_2)|^p\, dx_1 \bigr)\, dy_2 \leq C \|f\|_p{}^p$. The argument is similar when φ is independent of y_2 (but with the order of amplitudes used in Proposition 1.1 reversed):

$$\int \Bigl| \int e^{ix_1^{a_1} y_1^{b_1}} \varphi(x, y_1) \Bigl(\int e^{ix_2^{a_2} y_2^{b_2}} f(y)\, dy_2 \Bigr)\, dy_1 \Bigr|^p\, dx_1$$
$$\leq C \int \Bigl| \int e^{ix_2^{a_2} y_2^{b_2}} f(y)\, dy_2 \Bigr|^p\, dy_1,$$

and the argument is completed as in the first case. ■

Proposition 1.4 *Let $k(x, y)$ and p be as in Theorem 1.2. and let λ be a bounded function vanishing outside of the interval $[0, 4)$ (or for that matter outside of a bounded set). Then the four operators with kernels $\lambda(x_l)k(x, y)$, $k(x, y)\lambda(y_l)$, $l = 1, 2$ map $L^p(\mathbf{R}_+^2)$ into itself.*

Proof Consider the case when the kernel in question is of the form $\lambda(x_1)k(x, y)$. We have by the second order Taylor formula in the variable x_1:

$$\lambda(x_1)k(x, y) = \lambda(x_1)e^{ix^a \cdot y^b} \varphi(x, y) = \lambda(x_1)e^{ix^a \cdot y^b} [\varphi(0, x_2, y) + x_1 \partial_{x_1} \varphi(0, x_2, y)$$
$$+ \frac{1}{2} x_1^2 \partial_{x_1}^2 \varphi(0, x_2, y) + v(x, y)],$$

where by (0.4) and (1.2) $|v(x, y)| \leq C \min\{1, |x - y|^{-3}\}$. Now, the operators corresponding to the first three terms of the sum are operators of the kind described in Proposition 1.4 followed by a bounded multiplier $\lambda(x_1)$ (in L^p). The operator corresponding to the fourth term is bounded in L^p for $1 \leq p \leq \infty$ by Schur's lemma. The operator corresponding to $k(x, y)\lambda(y_1)$ is dealt in the same manner, using Taylor's formula in the variable y_1. This time, the operators (with φ independent of y_1) are preceded by the multiplier $\lambda(y_1)$. The remaining two cases follow by symmetry. ■

We now proceed to estimate the remainder of the sum (1.3). It consists of terms with all indices beginning with 2. It is useful to notice, that in this sum the factor $1 - \eta$ appearing in the partition of unity is one, hence the terms are of the form $\psi(2^{-m_1} x_1) \cdots \psi(2^{-n_2} y_2)$.

We estimate individually each term, the main tool being the following estimate (Proposition 1.1. in [PSS] and Proposition 3 in[PS]):

Proposition 1.5 *For $a \geq 1$ there is a constant C ($C = C(a)$) such that for every $0 < t_1 < t_2$, for every θ, $0 \leq \theta \leq 1$, for every $\xi \in \mathbf{R}$ and for every $\psi \in C^1(t_1, t_2)$,*

$$\Bigl| \int_{t_1}^{t_2} e^{it^a \xi} \psi(t)\, dt \Bigr| \leq C |\xi|^{\frac{\theta - 1}{a} - \theta} t_1^{(1-a)\theta} (\|\psi\|_\infty + \|\psi'\|_1).$$

We remark that for $\theta = 0$ the estimate remains valid also for $t_1 = 0$ and that for $\theta = 1$ the estimate is a version of the Van der Corput lemma.

Lemma 1.6 *With the notations as in (1.3) and with* $\min\{m_1, \dots, n_2\} > 2$ *we have the estimates*

$$\|K_{mn}\|_{1,1} \le C 2^{m_1+m_2}, \tag{1.4}$$

and

$$\|K_{mn}\|_{2,2} \le C \min\{m_1, m_2\}^{\frac{1}{2}} 2^{-\frac{1}{2}\lambda_{mn}}, \tag{1.5}$$

where $\lambda_{mn} = \sum_{l=1}^{2}\{n_l[(\frac{1-\theta_l}{a_l} + \theta_l)(b_l - 1) - (1-\theta_l)(1-\frac{1}{a_l})] + m_l\theta_l(a_l - 1)\}$, *with* $0 \le \theta_1, \theta_2 < 1$ *and* C *depending only on the kernel* k *and the function* ψ *appearing in the definition of the partition of unity.*

Proof (1.4) follows readily from the uniform boundedness of the kernels k_{mn} and from the standard estimate $\|K_{mn}\|_{1,1} \le \sup_y \int |k_{mn}(x,y)|\,dx + \sup_x \int |k_{mn}(x,y)|\,dy$.

To prove (1.5) we write $\|K_{mn}\|_{2,2} = \|K_{mn}^{\star}K_{mn}\|_{2,2}^{\frac{1}{2}}$, observe that $T_{mn} = K_{mn}^{\star}K_{mn}$ is an integral operator with the kernel $t_{mn}(x,y) = \int \overline{k_{mn}(z,x)}k_{mn}(z,y)\,dz$ and estimate $|t_{mn}(x,y)|$. We have

$$|t_{mn}(x,y)| \le C \min\{m_1, m_2\} 2^{(-\lambda_{mn} - \sum_{l=1}^{2} n_l(1-\theta_l)(1-\frac{1}{a_l}))}\beta_n(x,y), \tag{1.6}$$

where

$$\beta_n(x,y) = \psi_n(x)\psi_n(y)\prod_{l=1}^{2}|x_l - y_l|^{-(\theta_l + \frac{1-\theta_l}{a_l})}. \tag{1.7}$$

(1.6) is obtained by writing t_{mn} in the form

$$t_{mn}(x,y) = \psi_n(x)\psi_n(y)\int_{2^m}^{2^{m+2}} \overline{\varphi(z,x)}\varphi(z,y)\partial^2_{z_1z_2}\int_{2^m}^{z} e^{-it^a\cdot(x^b - y^b)}\psi_m(t)^2\,dt\,dz,$$

and integrating by parts. Proposition 1.5 yields the estimate

$$|t_{mn}(x,y)| \le C\beta_n(x,y)2^{(-\lambda_{mn} - \sum_{l=1}^{2} n_l(1-\theta_l)(1-\frac{1}{a_l}))}\int |\partial^2_{z_1z_2}[\overline{\varphi(z,x)}\varphi(z,y)]|\,dz.$$

The last integrand is estimated by the sum $C(|z-x|^{-2} + |z-y|^{-2})$ and is 0 if either $|z-x| \le 1$ or $|z-y| \le 1$. This readily yields (1.6). A variant of this estimate is given below in (1.5i), (1.5ii).

We consider next the integral operator B_n with the kernel $\beta_n(x,y)$. By a simple homogeneity argument $\|B_n\|_{2,2} = \prod_{l=1}^{2} 2^{n_l(1-(\theta_l+\frac{1-\theta_l}{a_l}))}\|B_0\|_{2,2}$, where B_0 is the operator with kernel β_0 considered in L^2 on the square $(\frac{1}{2}, 2) \times (\frac{1}{2}, 2)$. By Schur's Lemma

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$\|B_0\|_{2,2} < \infty$ provided $\theta_l + \frac{1-\theta_l}{a_l} < 1$, a condition which is satisfied as long as $\theta_l < 1$. Combining the above estimate with (1.6) we readily get (1.5). ∎

We are now ready to complete the proof of Theorem 1.2.

Proof With $p = 1 + \frac{b_1}{a_1} = 1 + \frac{b_2}{a_2}$ we write $\frac{1}{p} = \frac{t}{2} + (1-t)$ and interpolate between (1.4) and (1.5) to get

$$\|K_{mn}\|_{p,p} \leq C \min\{m_1, m_2\}^{\frac{t}{2}} 2^{-\lambda_{mn}\frac{t}{2} + (m_1+m_2)(1-t)}.$$

Substituting in $t = \frac{2b_1}{a_1+b_1} = \frac{2b_2}{a_2+b_2}$ and going back to the definition of λ_{mn} we can write

$$\begin{aligned} -\frac{t}{2}\lambda_{mn} &+ (1-t)(m_1+m_2) \\ &= -\sum_{l=1}^{2}\Bigg\{ n_l \frac{b_l}{a_l+b_l}\Bigg[\left(\frac{1-\theta}{a_l} + \theta_l\right)(b_l - 1) - (1-\theta_l)\left(1 - \frac{1}{a_l}\right)\Bigg] \\ &\qquad + \left[\frac{b_l}{a_l+b_l}\theta_l(a_l - 1) - \frac{(a_l - b_l)}{a_l + b_l}\right] m_l \Bigg\}. \end{aligned}$$

For $\theta_l = 1$ the coefficients of n_l and of m_l are respectively $\frac{b_l}{a_l+b_l}(b_l - 1)$ and $\frac{1}{a_l+b_l}(b_l - 1)a_l$ and are both positive (here we use essentially the hypothesis that $b_l > 1$). It follows that these coefficients remain positive also for $\theta_l < 1$ but sufficiently near to 1. Choosing such a θ in the estimates for the norms of K_{mn} we can conclude that $\sum_{m,n} \|K_{mn}\|_{p,p} < \infty$ and complete the proof. ∎

We also notice from Lemma 1.6 that in case $a \geq b > 1$ or $b \geq a > 1$ we get

$$\|K_{11}\|_{2,2} = \sum_{m,n \geq 1} \|K_{mn}\|_{2,2} \leq C. \tag{1.8}$$

We shall delay the proof of the following proposition to Section 4,

Proposition 1.7 *Let $a_1 = b_1 = 1$ or $a_2 = b_2 = 1$ and $p = 1 + \frac{a_1}{b_1} = 1 + \frac{a_2}{b_2} = 2$. Assume that $\varphi(x, y)$ satisfies (0.3). Then*

$$\|K_{11} f\|_2 \leq C \|f\|_2.$$

We notice that for $b_1 = 1$, $a_1 \geq 1$ and $a_2 \geq b_2 > 1$ the above estimates allow us to conclude that $\sum_{(m_1 \leq m_2 + n_2; |m_1 - n_1| \leq 5)} \|K_{mn}\|_{p,p} < \infty$, a fact which will become useful in Section 3.

For future reference we record two versions of (1.5) valid in the cases when the indices m, n satisfy additional conditions.

If for $l = 1$ or for $l = 2$, $|n_l - m_l| \geq 5$, then

$$\|K_{mn}\|_{2,2}^2 \leq C \min\{1, 2^{m_1+m_2-2\max\{m_l,n_l\}}\} 2^{-\lambda_{mn}}, \tag{1.5i}$$

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and if $|m_l - n_l| \geq 5$ for $l = 1$ and 2, then

$$\|K_{mn}\|_{2,2}^2 \leq C2^{m_1+m_2-2(\max\{m,n\})}2^{-\lambda_{mn}}. \tag{1.5ii}$$

These estimates follow readily if we notice that $|m_l - n_l| \geq 5$ implies that $|z_l - x_l|$ and $|z_l - y_l|$ occurring in the integral estimating t_{mn} are both $\geq \frac{1}{2}2^{\max\{m_l,n_l\}}$.

2 The H_{ab} Into L^1 Mapping Problem

We now turn our attention to the operator K as defined in (0.1) in the singular case, *i.e.*,

$$Kf(x) = \text{p.v.} \int_{\mathbf{R}^2} \varphi(x,y) e^{i|x|^a \cdot |y|^b} f(y)\, dy, \tag{2.1}$$

with the amplitude φ satisfying

$$|\partial_x^\alpha \partial_y^\beta \varphi(x,y)| \leq C \frac{|x-y|^{-|\alpha|-|\beta|}}{|x-y|^2} \begin{cases} (i) & \text{for all } \alpha \text{ and } \beta, \\ (ii) & \text{for } |\alpha|, |\beta| \leq 2. \end{cases} \tag{2.2}$$

We shall use (2.2)(i) for $b_1 = 1$ or $b_2 = 1$ in case $a \geq b \geq 1$ and for $a_1 = 1$ or $a_2 = 1$ in case $b \geq a \geq 1$. And we use (2.2)(ii) in all the other cases. We shall state this explicitly in the relevant results.

We need of course to assume that the principal value is defined for some non zero functions f, in fact we make the standard assumption that the operator $R_0 f(x) = \text{p.v.} \int_{|x-y|\leq 1} \varphi(x,y) f(y)\, dy$ is densely defined and extends to a bounded operator in L^2.

We consider K as an operator in a version of the Hardy space H^1 adapted to the phase of the kernel, *i.e.*, to the exponents a, b. We denote this space by H_{ab} and define it by using atomic decompositions.

Definition 2.1 An H_{ab} atom is a measurable, real or complex valued function θ defined on $\mathbf{R}^2$ such that for some $\tilde{x} \in \mathbf{R}^2$ and $\delta > 0$:

(a) $\operatorname{supp} \theta \subset \mathbf{D}(\tilde{x}, \delta) = \{y; |\tilde{x} - y| < \delta\}$
(b) $\|\theta\|_\infty \leq |\mathbf{D}|^{-1}$ and (the vanishing moment condition)
(c) $\int e^{i|\tilde{x}|^a \cdot |y|^b} \theta(y)\, dy = 0$.

where $|\mathbf{D}| = \text{area}\big(\mathbf{D}(\tilde{x}, \delta)\big)$.

An atom is normalized if $\tilde{x} = 0$ and $\delta = 1$.

An equivalent and useful version of the above definition is obtained by replacing the disk $\mathbf{D}$ by the square with the center at $\tilde{x}$ and the side 2δ.

A function f belongs to H_{ab} if for some sequences of complex numbers (λ_j) and of atoms (θ_j),

$$f = \sum_j \lambda_j \theta_j.$$

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The norm of f is defined by $\|f\|_{H_{ab}} = \inf\{\sum |\lambda_j|; f = \sum_j \lambda_j \theta_j\}$.

Similar modified versions of the Hardy space were discussed in [P] and [PS]—we refer to these papers for further comments and references.

One of the main results of the paper is the following theorem:

Theorem 2.2 *Let $a_l, b_l \geq 1$ and let K and R_0 be as explained above. Suppose that R_0 extends to a bounded operator from L^2 into itself and assume the hypothesis of Theorem 2.3 (below). Then the operator K extends (by continuity) to a bounded operator from H_{ab} into L^1 with the norm determined by the bounds in (2.2).*

In the proof we will use the following result which is of independent interest. The proof will be delayed to Section 4.

Theorem 2.3 *Let $a_l, b_l \geq 1$ for l=1,2, and K as defined in (0.1). If $\varphi(x, y)$ satisfies (2.2)(ii) in case $|x - y| \leq 1$ for $a_l, b_l \geq 1$, $l = 1, 2$, or in case $|x - y| \geq 1$ and $a_l, b_l > 1$, $l = 1, 2$, and $\varphi(x, y)$ satisfies (2.2)(i) in all the other cases. Suppose that R_0 extends by continuity to a bounded operator of L^2 into itself. Then K extends (by continuity) to a bounded linear operator of L^2 into itself.*

Before getting into details, we outline the main idea of the proof of the theorem. We need to show that for some constant C depending only on the operator K we have $\|K\theta\|_{L^1} \leq C$ for all atoms θ. For a conveniently chosen disk or a square $\mathbf{D}$, *e.g.*, the double of the one appearing in Definition 2.1, we write $K\theta = \chi_{\mathbf{D}} K\theta + (1 - \chi_{\mathbf{D}})K\theta$. By Theorem 2.3 it follows that, $\|\chi_{\mathbf{D}} K\theta\|_{L^1} \leq |\mathbf{D}|^{1/2} \|K\|_{2,2} \|\theta\|_{L^2} \leq C\|K\|_{2,2}$. It remains then to estimate the part of $K\theta$ on the exterior of D. As will be shown later, by suitable translation and homothety argument as well as the assumptions on the amplitude φ, this is reduced to establishing the following proposition.

Proposition 2.4 *With a and b as above, there is a constant C such that*

$$\int_{|x|>2} \Big| \int_{\mathbf{R}^2} e^{i(\lambda(|x+h|^a - |h|^a)) \cdot |y+h|^b} \theta(y)\, dy \Big| \frac{dx}{|x|^2} \leq C, \tag{2.3}$$

for all $h \in \mathbf{R}^2$, $\lambda \in \mathbf{R}^2$, $\lambda_1, \lambda_2 \neq 0$ and for all normalized atoms θ.

Recall that according to our conventions, $\lambda(|x + h|^a - |h|^a) \cdot |y + h|^b = \sum_{l=1}^2 [\lambda_l(|x_l + h_l|^{a_l} - |h_l|^{a_l})|y_l + h_l|^{b_l}$.

In deriving the above estimate it will be convenient to use the norm $|x|_\infty = \max\{|x_1|, |x_2|\}$, $x = (x_1, x_2)$ instead of the usual Euclidean norm $|x|$. This usage will also fulfill our intention of making quite evident the reduction of d-dimensional version of the problem, to the one in dimension $d - 1$.

The proof of the proposition is based on the observation that for an operator S in L^2 we have $\|S\|_{2,2} = \|S^\star S\|_{2,2}^{1/2}$. The norm of the (integral) operator $S^\star S$ is then estimated by means of Proposition 1.5.

We begin with the integral operator $S = S_{I_1, I_2}$ with the kernel

$$s(x, y) = \chi_{I_1 \times I_2}(x) e^{(i\lambda(|x+h|^a - |h|^a)) \cdot |y+h|^b} \chi_{[-1,1]\times[-1,1]}(y),$$

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where $I_1, I_2 \subset \mathbf{R}$ are intervals.

For $l = 1, 2$ we let $l' = 2$ if $l = 1$ and $l' = 1$ if $l = 2$. We have the following

Lemma *There exists a constant depending only on the exponents a and b such that*

$$\|S\|_{2,2} \leq C|\lambda_l|^{-1/(4a_l b_l)}|I_l|^{1/2-1/(4b_l)}|I_{l'}|^{1/2},$$

for all h and λ, $\lambda_1, \lambda_2 \neq 0$, *and*

$$\|S\|_{2,2} \leq C(|\lambda_l|\,|h_l|^{b_l-1})^{-1/(4a_l)}|I_l|^{1/4}|I_{l'}|^{1/2},$$

for $|h_l| \geq 2$.

Proof The operator $S^\star S$ is an integral operator with the kernel $\kappa(x, y) = \kappa_1(x_1, y_1)\kappa_2(x_2, y_2)$ where

$$\kappa_l(t, s) = \chi_{[-1,1]}(t)\chi_{[-1,1]}(s) \int_{I_l} e^{-i\lambda_l(|z+h_l|^{a_l}-|h_l|^{a_l})(|t+h_l|^{b_l}-|s+h_l|^{b_l})}\, dz.$$

We have the trivial estimate (i) $|\kappa_l(t, s)| \leq \chi_{[-1,1]}(t)\chi_{[-1,1]}(s)|I_l|$ and from Proposition 1.5, (ii) $|\kappa_l(t, s)| \leq C\chi_{[-1,1]}(t)\chi_{[-1,1]}(s)\big(|\lambda_l|\big|\,|t+h_l|^{b_l} - |s+h_l|^{b_l}\big|\big)^{-1/a_l}$. For $\beta \geq 1$ we have the inequalities: and $\big|\,|t+h_l|^{\beta} - \big|s+h_l|^{\beta}\big| \geq \beta|h_l|^{\beta-1}|t-s|$, when $|h_l| \geq 2$ and $t, s \in [-1, 1]$. For an arbitrary h_l we consider the bound for $|\kappa_l|$ obtained by taking the convex combination $(\mathrm{i})^{1-1/(2b_l)}(\mathrm{ii})^{1/(2b_l)}$ while for $|h_l| \geq 2$ we take the convex combination $(\mathrm{i})^{1/2}(\mathrm{ii})^{1/2}$. Integration of the resulting estimates with respect to s (or t) completes the proof of the lemma. ■

We next consider the operator S_j with the kernel as in the lemma, except that the factor $\chi_{I_1\times I_2}(x)$ is replaced by $\chi_{[2^j,2^{j+1}]}(|x|_\infty)$: $S_j = S_{[-2^{j+1},2^{j+1}],[-2^{j+1},-2^j]} + S_{[-2^{j+1},2^{j+1}],[2^j,2^{j+1}]} + S_{[-2^{j+1},-2^j],[-2^j,2^j]} + S_{[2^j,2^{j+1}],[-2^j,2^j]}$.

Applying the lemma to each of the four summands we get the following proposition.

Proposition 2.5 *With the notations introduced above there exists a constant C depending only on the exponents a, b and such that*

$$\|S_j\|_{2,2} \leq C2^j \min\{(2^{ja_l}|\lambda_l|)^{-1/(4a_l b_l)}; l = 1, 2\},$$

for all $\lambda_1, \lambda_2 \neq 0$, $h \in \mathbf{R}^2$ *and*

$$\|S_j\|_{2,2} \leq C2^j \min\{(2^{ja_l}|\lambda_l|\,|h_l|^{b_l-1})^{-1/(4a_l)}; l = 1, 2\},$$

for all $\lambda_1, \lambda_2 \neq 0$ *and all h such that* $|h_1|, |h_2| \geq 2$.

With the help of Proposition 2.5 we now proceed to the proof of Proposition 2.4. This is accomplished by reduction to the one dimensional case, where the result is contained in Proposition 8 in [PS] which we take for granted. Similarly the inequality in d-dimensions is reduced to the one in dimension $d-1$.

Denote by S the integral operator appearing in the proposition: we are proving that $\int_{|x|_\infty>2} |x|_\infty^{-2}|S\theta(x)|\,dx \le C$.

We notice first that for an atom θ independent of one of the variables, say $\theta(y_1, y_2) \equiv \theta_1(y_1)$ the inequality follows readily from Proposition 8 in [PS] and the inequality $\int_{|x|_\infty>2} |x|_\infty^{-2}\,dx_l \le 4\min\{1, |x_{l'}|^{-1}\}$. Hence, replacing θ by $\theta(y) - \int \theta\,dy_l$ we may assume, when needed, that $\int \theta(y)\,dy_l = 0$ for $l = 1$ or for $l = 2$.

Following [PS] we now consider the quantities $(\tilde{h}_l^{b_l-1}|\lambda_l|)^{-1/a_l}$, where $\tilde{h}_l = \max\{1, |h_l|/2\}$. We may assume by symmetry that $(\tilde{h}_1^{b_1-1}|\lambda_1|)^{-1/a_1} \le (\tilde{h}_2^{b_2-1}|\lambda_2|)^{-1/a_2}$. Divide the region of integration $\{|x|_\infty > 2\} =$

$$\{2 < |x|_\infty \le (\tilde{h}_1^{b_1-1}|\lambda_1|)^{-1/a_1}\} \cup \{(\tilde{h}_1^{b_1-1}|\lambda_1|)^{-1/a_1} \le |x|_\infty < \infty\}.$$

To estimate the integral over the second region we choose the integer $j_0 \ge 1$ such that $2^{j_0} < (\tilde{h}_1^{b_1-1}|\lambda_1|)^{-1/a_1} \le 2^{j_0+1}$. Using the Cauchy-Schwartz inequality and the estimates of Proposition 2.5 we get

$$\int_{|x|_\infty>2^{j_0}} |x|_\infty^{-2}|S\theta(x)|\,dx \le \sum_{j=j_0}^{\infty} 2^{-j}\|S_j\theta\|_2 \le C.$$

If $j_0 = 1$, then the proof is complete. Otherwise, in the remaining integral over the region $\{2 < |x|_\infty \le (\tilde{h}_1^{b_1-1}|\lambda_1|)^{-1/a_1}\}$ we may assume that $\int \theta(y_1, y_2)\,dy_1 \equiv 0$. This allows us to rewrite the integral appearing under the absolute value sign as follows

$$S\theta(x) = \int e^{i\lambda_2(|x_2+h_2|^{a_2}-|h_2|^{a_2})|y_2+h_2|^{b_2}} \cdot \left[\int (e^{i\lambda_1(|x_1+h_1|^{a_1}-|h_1|^{a_1})|y_1+h_1|^{b_1}} - \mu)\theta(y_1, y_2)\,dy_1\right] dy_2,$$

where $\mu = e^{i\lambda_1(|x_1+h_1|^{a_1}-|h_1|^{a_1})|h_1|^{b_1}}$. This reduces the problem to the one dimensional case and the proof may now be completed using the same argument as in the proof of Proposition 8 in [PS].

Proof of Theorem 2.2 As already indicated it suffices to show that $\|K\theta\|_1 \le C$ for every H_{ab} atom, with some C depending only on K. We also indicated that this is equivalent to showing that

$$\int_{|\bar{x}-y|\ge 2\delta} |K\theta(x)|\,dx \le C,$$

for every θ satisfying (2.3).

For such θ define $\tilde{\theta}(y) = \theta(\tilde{x} + \delta y)$; then $\tilde{\theta}$ is a normalized atom. Define also $\tilde{K}(x, y) = \delta^2 K(\tilde{x} + \delta x, \tilde{x} + \delta y)$. Then the amplitude $\tilde{\varphi}(x, y) = \varphi(\tilde{x} + \delta x, \tilde{x} + \delta y)$ satisfies (2.2) with bounds independent of δ.

Under this translation and homothety the inequality in question becomes $\int_{|x|>2} |\tilde{K}\tilde{\theta}(x)|\, dx \le C$, which can be verified by writing

$$\int_{|x|>2} |\tilde{K}\tilde{\theta}(x)|\, dx \le \int_{|x|>2} \Big(\int |\tilde{\varphi}(x, y) - \tilde{\varphi}(x, 0)|\, |\tilde{\theta}(y)|\, dy \Big)\, dx + \int_{|x|>2} |x|^{-2} |S\tilde{\theta}(x)|\, dx,$$

with S appearing in Proposition 2.4, with a suitable choice of λ and h. This readily yields the desired bound. ■

3 The Cases When $b_1 = 1$ or $b_2 = 1$

We address now the statement of Theorem 1.2 in the cases when one of the exponents b_l is one or both of them are equal to 1. In light of [CP] it would be natural to expect that the conclusion should remain valid for the amplitude function in (0.2) which is in C^∞ off the diagonal $x = y$ and satisfies (0.3) for all $j = 0, 1, \dots$ with constants C depending on j. However in this section we consider only the model case when

$$\varphi(x, y) = |x - y|^{i\tau}, \tag{3.1}$$

with some real τ (and we drop the support condition (1.2)). The result we prove is the same as Theorem 1.2 which we repeat.

Theorem 3.1 *Suppose that for $l = 1, 2$, $a_l \ge b_l \ge 1$, $b_1/a_1 = b_2/a_2$ and that $b_l = 1$ for $l = 1$ or for $l = 2$. Let τ be a real number. Then the integral operator K with the kernel $k(x, y) = e^{ix^a \cdot y^b} |x - y|^{i\tau}$ is bounded from $L^p(\mathbf{R}^2_+)$ into itself where $p = 1 + \frac{b_1}{a_1}$.*

Proof The proof of the theorem depends on different arguments in the cases when one or both of the exponents b_l equals 1. In the first case we may assume that $b_1 = 1$ and $b_2 > 1$. The proof is then based on the decomposition (1.3), however estimates of the terms in the sum are more involved than those used in Section 1.

Note that if $b_1 = a_1 = 1$, then in that case p=2, and since here $x_2, y_2 \ge 1/2$, we get our result from Proposition 1.7. Thus, we can suppose that $a_1 > 1$.

We first elaborate on the remark made at the end of Section 1. The interpolation inequality

$$\|K_{mn}\|_{p,p} \le \|K_{mn}\|_{2,2}^t \|K_{mn}\|_{1,1}^{1-t}, \quad t = \frac{2b_l}{a_l + b_l}, \tag{3.2}$$

yields with a suitable choice of $0 < \theta_1, \theta_2 < 1$ sufficiently near to 1 an estimate of the form $\|K_{mn}\|_{p,p} \le C \min\{m_1, m_2\} 2^{\mu_1 m_1 - \mu_2(m_2 + n_2)}$, where $0 < \mu_1 < \mu_2$. This implies

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that

$$\sum_{(m_1 \le m_2+n_2;|m_1-n_1|\le 5)} \|K_{mn}\|_{p,p} < \infty. \tag{3.3}$$

To estimate the remainder of the sum we still use (3.2) but with more elaborate estimates of the $\| \, \|_{1,1}$-norm. These are obtained by writing

$$\varphi = \varphi_j{}^{(1)} + \varphi_j^{(2)}, \quad j = 1, 2, 3, 4, \tag{3.4}$$

with similar notations for the corresponding decompositions of k, K, k_{mn}, K_{mn}, where

$$\varphi_j^{(2)}(x, y) = \varphi\big(\pi_j(x, y)\big)$$

and for $j = 1, 2, 3, 4$ $\pi_j(x, y)$ denote the projections of (x, y) onto the hyperplanes $y_1 = 0$, $y_2 = 0$, $x_1 = 0$ and $x_2 = 0$.

The kernels $k_j^{(1)}$ satisfy (0.4) and it follows that the estimates for $\|K_{jmn}^{(1)}\|_{2,2}$ are of the same form as those for $\|K_{mn}\|_{2,2}$,viz., (1.5),(1.5i) and (1.5ii).

The estimates for $\|K_{jmn}^{(1)}\|_{1,1}$ are as follows:

$$\|K_{jmn}^{(1)}\|_{1,1} \le C2^{m_j+n_j} \quad \text{for } j = 1, 2, \tag{3.5)(a}$$

$$\|K_{jmn}^{(1)}\|_{1,1} \le C \min\{2^{m_1+n_j}, 2^{m_2+n_j}\}, \tag{3.5)(b}$$

$$\text{for } j = 1, 2,\ n_j = \min\{m, n\} \text{and } |m_l - n_l| \ge 5,$$

$$\|K_{jmn}^{(1)}\|_{1,1} \le C2^{2m_{j-2}}, \quad \text{if } m_{j-2} = \min\{m, n\} \text{ for } j = 3, 4. \tag{3.5)(c}$$

The proofs of (3.5)(a,b,c) depend on the inequality

$$|\partial_{x_l}\varphi(x, y)|\big(= |\partial_{y_l}\varphi(x, y)|\big) \le C\frac{|x_l - y_l|}{|x - y|^2} \tag{3.6}$$

and go back to the general fact used already in the estimate (1.4): the L^1-norm of an integral operator with a kernel $\kappa(x, y)$ is bounded by the quantity $\sup_y \int |\kappa(x, y)|\,dx + \sup_x \int |\kappa(x, y)|\,dy$. We have

$$|k_{1mn}(x, y)| \le \psi_m(x)\psi_n(y) \int_0^{y_1} \frac{|x_1 - \xi|}{(x_1 - \xi)^2 + (y_2 - x_2)^2}\,d\xi.$$

Integrating with respect to x_2 and changing the order of integration we get (3.5)(a) for $j = 1$. The estimate for $j = 2$ follows by symmetry.

To get (3.5)(b), again for $j = 1$ we notice that conditions $n_1 = \min\{m, n\}$, $|m_1 - n_1| \ge 5$, imply that $m_1 \ge n_1 + 5$ and that $\frac{1}{2}2^{m_1} \le |x_1 - \xi| \le 52^{m_1}$. Omitting $(x_2 - y_2)^2$ in the denominator in the integrand we obtain the required bound for the integral, $2^{m_2+n_1}$. The case when $j = 2$ follows again by symmetry. (3.5)(c) is obtained similarly.

We proceed now to estimate the sum $\sum_{m,n} K_{mn}$ which we split into 4 parts corresponding to ranges of indices:

1) $n_1 = \min\{m, n\}$,
2) $n_1 > n_2 = \min\{m, n\}$,
3) $n_1, n_2 > m_1 = \min\{m, n\}$,
4) $n_1, n_2, m_1 > m_2 = \min\{m, n\}$.

In case j) we use the decomposition $K_{mn} = K^{(1)}_{jmn} + K^{(2)}_{jmn}$ and estimate each sum separately.

We first address the estimates of the operators $K^{(1)}_j$ and consider the following possibilities.

(i) $|m_l - n_l| \leq 5, l = 1, 2$,
(ii) $|m_l - n_l| > 5, l = 1, 2$,
(iii) $|m_1 - n_1| \leq 5, |m_2 - n_2| > 5$,
(iv) $|m_1 - n_1| > 5, |m_2 - n_2| \leq 5$.

We observe that if $n_1 = \min\{m, n\}$ and $|m_1 - n_1| \leq 5$, then $m_2 + n_2 \geq n_1 + n_1 \geq n_1 + m_1 - 5$ and the corresponding sum of the operators K_{mn} is estimated by means of (3.3). Hence for $m_1 = \min\{m, n\}$ and for $n_1 = \min\{m, n\}$ we need only look at the case when $|n_1 - m_1| \geq 5$. In order to avoid tedious repetitions and constantly writing multiple sums explicitly, we keep in mind that all the relevant bounds are of the form $C2^{\eta_{mn}}$, possibly with a power of m as a factor, which can be absorbed into the exponential (as long as the exponent is negative). Note that C does not depend upon θ. Hence it is sufficient to keep track of the values of the exponents η_{mn}.

With this in mind, consider the case when $n_1 = \min\{m, n\}$ and $m_1 - n_1 \geq 5$. We get then, using (1.5i),(3.2) and (3.5)(b), the following value for the exponent η_{mn}:

$$\eta_{mn} = \frac{1}{1 + a_1}[\min\{0, m_2 - m_1\} - \lambda_{mn}] + \frac{a_1 - 1}{a_1 + 1}[n_1 + \min\{m_1, m_2\}].$$

We recall that the term λ_{mn} contains the parameters $0 \leq \theta_1, \theta_2 \leq 1$, which for reasons which were apparent in Section 1 are not allowed to assume the value 1. However the finiteness of the sum $\sum 2^{\eta_{mn}}$ for $\theta_1 = \theta_2 = 1$ does, by a continuity argument, imply the finiteness of the sum for $\theta_1, \theta_2 < 1$ but sufficiently near to 1. This allows us to restrict our attention to the case when $\theta_1 = \theta_2 = 1$. In this case we have

$$\eta_{mn} = \frac{1}{1 + a_1}[\min\{0, m_2 - m_1\} - m_1(a_1 - 1) - n_2(b_2 - 1) - m_2(a_2 - 1)] + \frac{a_1 - 1}{a_1 + 1}[n_1 + \min\{m_1, m_2\}].$$

In looking at exponents rather than exponentials we make the obvious comment that summing over $j \geq j_0, \ldots$ exponential with exponents $-\alpha j$, $\alpha > 0$ results in the exponent $-\alpha j_0$.

We now split the sum corresponding to the above exponents into parts over $\{m_1 < m_2\}$ and over $\{m_1 \geq m_2\}$.

For $m_1 < m_2$, η_{mn} becomes

$$\eta_{mn} = \frac{1}{a_1 + 1}[n_1(a_1 - 1) - n_2(b_2 - 1) - m_2(a_2 - 1)].$$

Summation over m_2, $m_2 > m_1$ and n_2, $n_2 \geq n_1$, *i.e.*

$$\sum_{n_1=0}^{\infty}\sum_{n_2=n_1}^{\infty}\sum_{m_1=n_1}^{\infty}\sum_{m_2=m_1+1}^{\infty}(\cdots)$$

results in the exponent

$$\frac{1}{a_1+1}[-m_1(a_2-1)-n_2(b_2-1)+n_1(a_1-1)],$$

if summed with respect to m_1, $m_1 \geq n_1$, results in the exponent

$$\frac{1}{a_1+1}n_1(a_1-a_2-b_2+1),$$

which results in a finite sum first with $n_2 \geq n_1$ and then with respect to $n_1 \geq 1$, since $a_1 - a_2 - b_2 + 1 < 0$.

We take up now the part of the sum where $m_1 \geq m_2$. This time we have the exponent

$$\frac{1}{a_1+1}[-m_2(a_2-a_1-1)-m_1a_1-n_2(b_2-1)+n_1(a_1-1)].$$

Addition with respect to m_1, $m_1 \geq m_2$ and with respect to n_2, $n_2 \geq n_1$, results in the exponent

$$\frac{1}{a_1+1}[-m_2(a_2-1)-n_1(b_2-1)+n_1(a_1-1)],$$

which summed with respect to m_2, $m_2 \geq n_1$ and then $n_2 \geq n_1$ results in the exponent

$$\frac{1}{a_1+1}[-n_1(a_2+b_2-a_1-1)],$$

yielding again the desired conclusion. This completes the estimate of the sum $\sum \|K^{(1)}_{1mn}\|_{p,p}$ over the range of indices where $n_1 = \min\{m, n\}$ and $m_1 \geq n_1 + 5$.

We next consider the case when $\min\{m, n\} = n_2$ and $|m_1 - n_1| \geq 5$, the case where $|m_1 - n_1| \leq 5$ follows by a similar argument. We estimate the terms $\|K^{(1)}_{2mn}\|_{p,p}$ using (1.5i), (3.2) and (3.5)(b). We distinguish the cases when $m_2 < n_2 + 5$ and when $m_2 \geq n_2 + 5$. In the first case we get an estimate with the exponent

$$\eta_{mn} = \frac{1}{a_1+1}[\min\{0, m_1+m_2-2\max\{m_1,n_1\}\} \\ - m_1(a_1-1)-n_2(b_2-1)-m_2(a_2-1)] + \frac{a_1-1}{a_1+1}(n_2+m_2).$$

for $m_1 \geq n_1$ we get

$$\eta_{mn} = \frac{1}{a_1+1} \times [m_2 - m_1a_1 - n_2(b_2-1) - m_2(a_2-1) + 2n_2(a_1-1) + 5(a_1-1)]$$

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where the last term is left and can be absorbed into the constant outside of the exponential. Summing with respect to m_1, $m_1 \geq n_1$, then with respect to $n_1 \geq n_2$, results with the exponent $n_2(-a_2+a_1-b_2+1)$ which yields the desired conclusion. In the case when $m_1 < n_1$ the term, $\min\{0, m_1+m_2-2\max\{m_1, n_1\}\} = \min\{0, m_1+m_2-2n_1\}$ and then sum here with respect to $m_1, n_2 \leq m_1 \leq n_1$, followed by $n_1 \geq n_2$ and lastly with respect to n_2, $n_2 \geq 0$ to obtain our result.

We next consider the sum, where $m_2 \geq n_2 + 5$. In this case we can use the estimate given by (1.5ii) to get

$$\eta_{mn} = \frac{1}{a_1+1}[m_1 + m_2 - 2\max\{n, m\}$$
$$- m_1(a_1-1) - n_2(b_2-1) - m_2(a_2-1) + (a_1-1)(n_2 + \min\{m_1, m_2\})].$$

We distinguish the cases when $\max\{m, n\} = n_1, m_1, m_2$. In the first case $m_1 + m_2 - 2\max\{m, n\} = m_1+m_2-2n_1$ and summation with respect to n_1, $n_1 \geq \max\{m_1, m_2\}$ results in the exponent

$$\frac{1}{a_1+1}[m_1 + m_2 - 2\max\{m_1, m_2\} - m_1(a_1-1) - n_2(b_2-1) - m_2(a_2-1)$$
$$+ (a_1-1)(n_2 + \min\{m_1, m_2\})].$$

We now split the sum into two parts, $m_1 \leq m_2$ and the rest, in the first part we sum first with respect to m_2, $m_2 \geq m_1$, and then with respect to m_1, $m_1 \geq n_2$ in the second part we do it in reversed order of m_1 and m_2. In either case we arrive at the exponent

$$-n_2\frac{a_2+b_2-a_1-1}{a_1+1}.$$

If $\max\{m, n\} = m_1$, say then the exponent is

$$\eta_{mn} = \frac{1}{a_1+1}[m_2 - m_1a_1 - n_2(b_2-1) - m_2(a_2-1) + (a_1-1)(n_2 + m_2\})].$$

The sum with respect to m_1, $m_1 \geq \max\{n_1, m_2\}$ yields

$$\frac{1}{a_1+1}[-a_1\max\{n_1, m_2\} - n_2b_2 - m_2(a_2-1) + n_2a_1 + m_2(a_1-1)].$$

Depending on whether $n_1 \geq$ or $< m_2$ we sum first with respect to the larger index first and then with respect to the smaller one to conclude the argument in the same manner as above. If $\max\{m, n\} = m_2$ and since here $m_2 \geq n_2 + 5$, we get this time that

$$\eta_{mn} = \frac{1}{a_1+1}[m_1 - m_2 - n_2b_2 - m_2(a_2-1) + n_2a_1]$$

and the argument proceeds as above.

We are left with the cases when $m_l = \min\{m, n\}$ with $l = 1$ or $l = 2$ and, of course, $|m_1 - n_1| \geq 5$. In the first case we estimate $\|K_{3mn}\|_{p,p}$ and in the second $\|K_{4mn}\|_{p,p}$.

We use the estimates (1.5i) and (3.5)(c) and repeating arguments as above we end up in either case with the exponent $-\frac{a_2-a_1+b_2-1}{a_1+1}m_l$, where $m_l = \min\{m, n\}$. ∎

This completes the discussion of the sum $\sum \|K^{(1)}_{jmn}\|_{p,p}$. Notice that in these estimates we could afford some overlaps between sums occurring in different cases. In estimating the complementary sums $\|\sum K^{(2)}_{jmn}\|_{p,p}$ we do not interchange the norm with the sum and thus have to be more careful in having disjoint sums. Thus we look separately at the sums $\sum K^{(2)}_{jmn}$, where $j = 1$ when $\min\{m, n\} = n_1$, $j = 2$ when $n_2 = \min\{m, n\} < n_1$, $j = 3$ when $m_1 = \min\{m, n\} < n_1, n_2$ and $j = 4$ when $m_2 = \min\{m, n\} < n_1, n_2, m_1$.

We now estimate the terms $\sum_{m,n} K^{(2)}_{jmn} f$ in the following lemma. The proof follows closely the argument given in Proposition 1.3.

Lemma *With the hypothesis in Theorem 3.1, we get that*

$$\Big\| \sum_{m,n} K^{(2)}_{jmn} \Big\|_{p,p} \leq C \quad \text{for } j = 1, 2, 3, 4.$$

Proof We shall be brief here. We recall the notation and comments that appear in the proof of Theorem 3.1.

We begin with the operator when $j = 1$, *i.e.* $\sum_{m,n} K^{(2)}_{1mn}$.

We first deal with the term

$$\sum_{m_1=0}^{\infty} \sum_{n_2=m_1}^{\infty} \sum_{m_2=n_2}^{\infty} \sum_{n_1=0}^{m_1} \cdot \int_{\mathbb{R}^2} \beta(x-y)\psi_m(x)\psi_n(y)e^{ix^a\cdot y^b}\varphi(x, 0, y_2) f(y)\, dy.$$

Where $\beta(x) \in C^\infty(\mathbb{R}^2)$, $\beta(x) = 1$ if $|x| \geq 1$, $= 0$ if $|x| \leq 1/2$ and $0 \leq \beta(x) \leq 1$. Because of Schur's lemma we can drop the $\beta(x - y)$, and we need only estimate

$$I(x) = \sum_{m_1=0}^{\infty} \sum_{n_2=m_1}^{\infty} \sum_{m_2=n_2}^{\infty} \cdot \int_0^\infty \psi_m(x)\psi_{n_2}(y_2)e^{ix_2{}^{a_2}y_2{}^{b_2}}\varphi(x, 0, y_2)H(x_1, y_2)\, dy_2$$

where $H(x_1, y_2) = \int_0^\infty \sum_{n_1=0}^{m_1} \psi_{n_1}(y_1)e^{ix_1{}^{a_1}y_1} f(y)\, dy_1$.

We first notice that

$$\Big(\int_{\mathbb{R}^2} |I(x)|^p\, dx\Big)^{1/p} \leq C \sum_{m_1=0}^{\infty} \sum_{n_2=m_1}^{\infty} \sum_{m_2=n_2}^{\infty} \cdot \Big(\int_{\mathbb{R}^2} \psi_m(x)\Big|\int_0^\infty \psi_{n_2}(y_2)e^{ix_2{}^{a_2}y_2{}^{b_2}}\varphi(x, 0, y_2)H(x_1, y_2)\, dy_2\Big|^p\, dx\Big)^{1/p}.$$

Since $1 < b_2 \le a_2$, as in the proof of Theorem 1.2, there exists, $\lambda_1, \lambda_2 > 0$ depending only on a_2 and b_2 such that

$$\int_0^\infty \psi_{m_2}(x_2)\Big|\int_0^\infty \psi_{n_2}(y_2)e^{ix_2{}^{a_2}y_2{}^{b_2}}\varphi(x,0,y_2)H(x_1,y_2)\,dy_2\Big|^p\,dx_2 \\ \le C\frac{m_2{}^{\alpha p}}{2^{m_2\lambda_1 p}2^{n_2\lambda_2 p}}\int_0^\infty |H(x_1,y_2)|^p\,dy_2$$

where $\alpha = \frac{b_2}{a_2+b_2}$. Note we used here that $x_1 \ge 1/2$, thus $\varphi(x,0,y_2)$ which satisfies (0.4) is bounded and its derivatives are bounded.

We thus get that

$$\|I(x)\|_p \le C\sum_{m_1=0}^{\infty}\sum_{n_2=m_1}^{\infty}\sum_{m_2=n_2}^{\infty}\frac{m_2{}^{\alpha}}{2^{m_2\lambda_1}2^{n_2\lambda_2}}\|f\|_p,$$

since this sums, we get our result. All the other cases follow in a similar fashion.

The next interesting case occurs when $j = 2$. In that case after disposing of $\beta(x-y)$, we consider the expression

$$II_1(x) + II_2(x) = \sum_{n_2=0}^{\infty}\sum_{m_2=n_2}^{\infty}\sum_{m_1=m_2}^{\infty}\int_0^\infty \psi_m(x)\Big(\sum_{n_1=n_2}^{m_1-6} + \sum_{n_1=m_1-5}^{m_1}\Big) \\ \cdot\,\psi_{n_1}(y_1)e^{ix_1{}^{a_1}y_1}\varphi(x,y_1,0)H(x_2,y_1)\,dy_2$$

where $H(x_2,y_1) = \int_0^\infty \psi_{n_2}(y_2)e^{ix_2{}^{a_2}y_2{}^{b_2}}f(y)\,dy_2$. We wish to show that this term maps L^p into itself. By the above argument (as in our p-estimate of $I_1(x)$) we get that the term $II_2(x)$ maps L^p into itself, thus we are left with the term $II_1(x)$.

We begin by showing that the kernel

$$\sum_{m_1=m_2}^{\infty}\int_0^\infty \psi_{m_1}(x_1)\sum_{n_1=n_2}^{m_1-6}\psi_{n_1}(y_1)e^{ix_1{}^{a_1}y_1}\varphi(x,y_1,0)$$

maps L^p into itself.

Arguing as in Section 4 of [PSS] it is enough to consider (a modified version) of the dual operator with $\varphi(x,y) = |x-y|^{i\tau}$, $T_\lambda f(x_1) = \int_0^\infty \sum_{m_1=m_2}^\infty \psi_{m_1}(y_1)\cdot$

$$\cdot\sum_{n_1=n_2}^{m_1-6}\psi_{n_1}(x_1)e^{ix_1y_1{}^{a_1}}\varphi(x,y_1,0)\big(1-\eta(x_1-y_1)\big)\,\eta\Big(\frac{x_1-y_1}{\lambda}\Big)\,f(y_1)\,dy_1$$

and

$$\widehat{T_\lambda f}(\xi) = \int_0^\infty f(y_1)e^{-iy_1(\xi-y_1{}^{a_1})}\tilde{K_\lambda}(\xi,y_1,x_2)\,dy_1.$$

It suffices to check that this kernel $\tilde{K_\lambda}(\xi,y_1,x_2)$ satisfies (4.6) of [PSS]. As noted above since $x_2 \ge 1/2$, $\varphi(x,y_1,0)$ and its derivatives are bounded.

Another one of the key ideas in these arguments is that

$$\psi_{m_1}(y_1)\Big|\sum_{n_1=0}^{m_1-6}\partial_u\psi_{n_1}(u+y_1)\Big| \le C\frac{\psi_{m_1}(y_1)\chi(2^{m_1-6}\le u+y_1\le 3/2\cdot 2^{m_1-6})}{2^{m_1}}$$

and

$$\psi_{m_1}(y_1)\sum_{n_1=0}^{m_1-6}\psi_{n_1}(u+y_1)\ne 0$$

implies that $-C_1 2^{m_1}\le u\le -C_2 2^{m_1}$.

The difficulty here is that $\tilde{K_\lambda}(\xi,y_1,x_2)$ is not a convolution kernel.

Using the notation in Section 4 of [PSS] we have that $S_\lambda f(\xi)=\Omega_\lambda(g)$, with

$$g(y_1)=\begin{cases}\text{(a)}\ \ y_1^{\frac{1}{a_1}-1}f(y_1^{\frac{1}{a_1}}), & 0<y_1<\infty,\\ \text{(b)}\ \ 0 & \text{elsewhere,}\end{cases}$$

and the kernels $\omega_\lambda(\xi,y_1)=\tilde{K_\lambda}(\xi,y_1^{\frac{1}{a_1}},x_2)$. It follows that

$$\begin{cases}\text{(i)}\ \ |\omega_\lambda(\xi,y_1)|\le\frac{C}{|\xi-y_1|}, & \text{and}\\ \text{(ii)}\ \ |D\omega_\lambda(\xi,y_1)|\le\frac{C}{|\xi-y_1|^2}, & \end{cases}$$

and by standard arguments $\|\Omega_\lambda(g)\|_2\le C\|g\|_2$.

Finally by 6.13, p. 221 of [St], we get that $(1/p+1/q=1)$

$$\int_{-\infty}^{\infty}|\Omega_\lambda(g)|^q|x|^{q-2}\,dx\le C\int_{-\infty}^{\infty}|g|^q|x|^{q-2}\,dx$$

and this completes the outline of the proof. ∎

We are left with the case when $b_1=b_2=1$ and $a_1=a_2=a>1$, and the operator is of the form (in dimension 2)

$$Tf(x)=\int_{\mathbb{R}^2_+}e^{ix^a\cdot y}|x-y|^{i\tau}f(y)\,dy,\quad \tau\in\mathbb{R},\ \tau\text{ not zero.}$$

The objective is to show that T maps $L^p(\mathbb{R}^2)$ into itself, for $p=\frac{a+1}{a}$. The case when $\tau=0$ follows by our remark following Proposition 1.4.

As in [PSS] we prove the equivalent statement that the transposed operator

$$f\to\int_{\mathbb{R}^2_+}e^{iy^a\cdot x}|x-y|^{-i\tau}f(y)\,dy$$

maps $L^{a+1}(\mathbb{R}^2_+)$ into itself. This is done using the weighted L^p estimates for Fourier transforms applied to the operator T_λ with kernel

$$e^{iy^a\cdot x}|x-y|^{i\tau}\big(1-\psi(|x-y|)\big)\,\psi\big((|x-y|)/\lambda\big)=e^{iy^a\cdot x}K_\lambda(x-y)$$

with $\lambda > 1$ and $\psi(t)$ being a radial cutoff function $\in C^\infty$, where,

$$\psi(t) = 1 \quad \text{for } t \in [0, 1/2], \text{ and } \psi(t) = 0 \quad \text{for } t \geq 1$$
$$\text{(we eventually let } \lambda \to \infty).$$

As in [PSS] Section 4, the Fourier transform can be written in the form

$$\widehat{(T_\lambda f)}(\xi) = \int_{\mathbb{R}_+{}^2} e^{iy\cdot(y^a - \xi)} \hat{K}_\lambda(\xi - y^a) f(y)\, dy,$$

and the aim is to establish the inequality

$$\iint |\widehat{(T_\lambda f)}(\xi)|^p |\xi|^{2(p-2)}\, d\xi \leq C(\|f\|_p)^p,$$

with $p = a + 1$, and C independent of λ.

This is done by using an argument from [PSS], Section 4, provided we establish the following estimates

$$\text{(3.7)} \qquad \begin{cases} \text{(a)} \ |\hat{K}_\lambda(\xi)| \leq C|\xi|^{-3}, & |\xi| \geq 1, \\ \text{(b)} \ |\hat{K}_\lambda(\xi)| \leq C|\xi|^{-2}, & |\xi| \leq 1, \\ \text{(c)} \ |\bigtriangledown \hat{K}_\lambda(\xi)| \leq C|\xi|^{-3}, & |\xi| \leq 1, \end{cases}$$

with C independent of λ.

The following proof of (3.7) is valid for arbitrary dimension $d > 1$, we give it here just for $d = 2$.

Proof To prove (3.7)(b) we write

$$\hat{K}_\lambda(\xi) = -|\xi|^{-2} \iint \Delta_x e^{-ix\cdot\xi} K_\lambda(x)\, dx = |\xi|^{-2} \iint e^{-ix\cdot\xi} \Delta K_\lambda(x)\, dx,$$

where $\Delta_x = \partial_1{}^2 + \partial_2{}^2$. We take advantage of the radial nature of K_λ to rewrite the above equations in polar coordinates with $x \cdot \xi = |x|\,|\xi| cos(\theta)$,

$$\text{(3.8)} \quad \hat{K}_\lambda(\xi) = |\xi|^{-2} \int_0^\infty \Big(\int_0^{2\pi} e^{-iu|\xi|cos(\theta)}\, d\theta \Big) D_u \Big(u D_u \big(\eta(u)\psi(u/\lambda) u^{i\tau} \big) \Big)\, du$$
$$= |\xi|^{-2} \int_0^\infty J_0(u|\xi|) D_u \Big(u D_u \big(\eta(u)\psi\big(u/\lambda\big) u^{i\tau} \big) \Big)\, du,$$

where J_0 is the Bessel function of order 0 and $\eta(u) = 1 - \psi(u)$.

To estimate the right hand side of (3.8), we look separately at terms containing: i) the factor $\eta'(u)$ or $\eta''(u)$ (which vanish outside of $[1/2, 1]$), (ii) the factor $\psi'(u/\lambda)$ or $\psi''(u/\lambda)$ (which vanishes outside of $[\lambda/2, \lambda]$) and (iii) the term $\eta(u)\psi(u/\lambda)u^{i\tau-1}$. We decompose the integral in (3.8) accordingly into the sum of

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three terms $I_1 + I_2 + I_3$. The estimates for I_1 and I_2 are straightforward. I_3 can be written as follows:

$$I_3 = \tau^2 \int_0^\infty J_0(u|\xi|)\eta(u)\psi(u/\lambda)u^{i\tau-1}\,du = \int_{1/2}^1 + \int_1^{\lambda/2} + \int_{\lambda/2}^{\lambda}.$$

The first term and the third term are estimated in a straight-forward manner. The middle term is a little more delicate and it requires a more careful analysis. Notice that the integral

$$\int_\epsilon^N J_0(u)u^{i\tau-1}\,dr$$

is bounded independently of $\epsilon, N > 0$. Here $\epsilon \to 0$, and $N \to \infty$. Indeed

$$\int_\epsilon^N J_0(u)u^{i\tau-1}\,du = \int_\epsilon^1 \big(J_0(u) - J_0(0)\big)\,u^{i\tau}\,du + J_0(0)\int_\epsilon^1 u^{i\tau-1}\,du + \int_1^N J_0(u)u^{i\tau-1}\,du.$$

Hence

$$\Big|\int_1^{\lambda/2} J_0(u|\xi|)u^{i\tau-1}\,du\Big| = \Big|\,|\xi|^{i\tau}\int_{|\xi|}^{(\lambda/2)|\xi|} J_0(u)u^{i\tau-1}\,du\Big|, \tag{3.9}$$

which is bounded uniformly in ξ and λ. This ends the proof of (3.7)(b).

To prove (3.7)(c) we differentiate (3.8):

$$\partial_j\hat{K}_\lambda(\xi) = \partial_j(|\xi|^{-2})\iint e^{-ix\cdot\xi}\,\Delta K_\lambda(x)\,dx + |\xi|^{-2}\iint e^{-ix\cdot\xi}(-ix_j)\Delta K_\lambda(x)\,dx.$$

The first term satisfies (3.7)(c) because of (3.7)(b). While

$$\iint e^{-ix\cdot\xi}(-ix_j)\Delta K_\lambda(x)\,dx = \frac{-i\xi_j}{|\xi|}\int_0^\infty J_1(u|\xi|)D_u\big(uD_uK_\lambda(u)\big)\,du$$

with J_1 being the Bessel function of order 1. This gets our estimate of (3.7)(c) for this term.

The last integral, below (3.9) is estimated in a straightforward manner. This completes the discussion of (3.7)(c).

We turn now to (3.7)(a). We write

$$\hat{K}_\lambda(\xi) = \frac{1}{|\xi|^4}\iint(\Delta^2 e^{-ix\cdot\xi})K_\lambda(x)\,dx = \frac{1}{|\xi|^4}\iint e^{-ix\cdot\xi}\Delta^2 K_\lambda(x)\,dx,$$

$$\Delta^2 = (\partial_1^2 + \partial_2^2)^2.$$

In this case we look separately at terms a) involving derivatives of η, b) involving derivatives of ξ but not η and c) involving derivatives of $|x|^{i\tau}$ only. In each of these cases the integrals are estimated by bringing the absolute value under the integral sign.

The case $d > 2$ is handled similarly: we use higher order integrations by parts and higher order Bessel functions. This completes our proof of Theorem 3.1. ■

4 The L^p Into L^p Mapping Problem

In this section, we prove Theorem 2.3, the $(2, 2)$ result, for the operator K defined in (2.1). We shall also prove Theorem 4.4 which gives the (p, p) result for K. Finally we note that Proposition 1.7 follows from Proposition 4.3.

We notice that $a, b \geq 1$, but because of duality it suffices to prove our results in case $a \geq b \geq 1$. Also note that the operator R_0 is defined in Section 2 (below (2.2)).

We shall begin with the proof of Theorem 2.3. First we notice that our operator $K = K_0 + K_1$, where the kernel of K_0 is given by $\big(1 - \beta(x - y)\big)\,\varphi(x, y)e^{i|x|^a\cdot|y|^b}$ and the kernel of K_1 is given by $\beta(x - y)\varphi(x, y)e^{i|x|^a\cdot|y|^b}$ and $\beta(x)$ is defined in Section 3 (in the proof of the Lemma).

Proposition 4.1 *Suppose R_0 maps $L^2(\mathbb{R}^2)$ into itself and $\varphi(x, y)$ satisfies (2.2)(ii) in case $a, b \geq 1$ for $|x - y| \leq 1$. Then,*

$$\|K_0\|_{2,2} \leq C$$

Proof Since R_0 maps $L^2(\mathbb{R}^2)$ into itself, we employ the argument in Section 2 of [PS] and adapt it to two dimensions. ■

We finish off this argument once we show that K_1 maps $L^2(\mathbb{R}^2)$ into itself.

Proposition 4.2 *Let $\varphi(x, y)$ be supported in one of the sets $\{(x, y) : 0 \leq x_j \leq 1\}, \{(x, y) : 0 \leq y_j \leq 1\}$ for $j = 1$ or 2. If*

$$|\varphi(x, y)| \leq C|x - y|^{-2}, \tag{4.1}$$

then the operator $Tf(x) = \int_{\mathbb{R}^2} |\varphi(x, y)|\beta(x - y)f(y)\,dy$, is a $(2, 2)$ map, i.e.

$$\|Tf\|_2 \leq C\|f\|_2.$$

Proof We assume that $\varphi(x, y)$ is supported in $0 \leq x_1 \leq 1$ (note that $0 \leq y_1 \leq 1$ is the dual case), all the remaining cases are similar and their proofs will be omitted.

In this case, we estimate $I = \int_0^1 (\int_0^\infty |Tf(x)|^2\,dx_2)\,dx_1$. Note $|\varphi(x, y)| = |\varphi|^{\frac{1+\epsilon}{2}}|\varphi|^{\frac{1-\epsilon}{2}}$ for some $0 < \epsilon < 1$, and here we use $\epsilon = 1/4$.

Therefore,

$$|Tf(x)| \leq \Big(\int_{\mathbb{R}^2} |\varphi(x, y)|^{1+\epsilon}\beta(x - y)\,dy\Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbb{R}^2} |\varphi(x, y)|^{1-\epsilon}\beta(x - y)|f(y)|^2\,dy\Big)^{\frac{1}{2}}.$$

By (4.1) we get that ($\epsilon = 1/4$)

$$\int_{\mathbb{R}^2} |\varphi(x,y)|^{1+\epsilon}\beta(x-y)\,dy \le \int_{|x-y|\ge 1} \frac{1}{|x-y|^{2(1+\epsilon)}}\,dy \le 4\pi.$$

Therefore, (with $\epsilon = 1/4$) we get that

$$I \le 4\pi\Big(\int_{\mathbb{R}^2}\Big(\int_0^1\int_0^\infty |\varphi(x,y)|^{1-\epsilon}\beta(x-y)\,dx\Big)|f(y)|^2\,dy\Big).$$

But,

$$\int_0^1\int_0^\infty |\varphi(x,y)|^{1-\epsilon}\beta(x-y)\,dx\,dy$$
$$\le \int_0^1\Big(\int_0^\infty \frac{\beta(x-y)}{|x-y|^{2(1-\epsilon)}}\big(\chi(|x_2-y_2|\le 1/2)+\chi(|x_2-y_2|\ge 1/2)\big)\,dx_2\Big)\,dx_1$$

$= II_1 + II_2$. For II_1: since $|x-y| \ge 1$ and $|x_2-y_2| \le 1/2$, then $(x_1-y_1)^2 \ge 3/4$. Therefore, (for $\epsilon = 1/4$), $II_1 \le \int_0^1\int_0^\infty \frac{\chi(|x_2-y_2|\le 1/2)}{(3/4)^{1-\epsilon}} \le 1/2\cdot(4/3)^{3/4}$. For II_2:

$$II_2 \le \int_0^1\Big(\int_0^\infty \frac{\chi(|x_2-y_2|\ge 1/2)}{(|x_2-y_2|)^{2(1-\epsilon)}}\,dx_2\Big)\,dx_1.$$
$$\le \int_0^1\Big(\int_{|x_2|\ge 1/2}\frac{1}{|x_2|^{2(1-\epsilon)}}\,dx_2\Big)\,dx_1 \le 4\cdot 2^{1/2}.$$

■

We complete the $(2,2)$ estimate for the operator K_1, where $\varphi(x,y)$ is supported in $x, y \ge 1$. We show this result in d-dimensions. Define $f(y^{\frac{1}{b}}) = f(y_1^{\frac{1}{b_1}}, y_2^{\frac{1}{b_2}}, \dots, y_d^{\frac{1}{b_d}})$, and define the monomials as usual, namely, $y^{\frac{1}{b}-\bar{1}} = y_1^{\frac{1}{b_1}-1}\cdot y_2^{\frac{1}{b_2}-1}\cdots y_d^{\frac{1}{b_d}-1}$, and similarly for $x^{\frac{1}{a}-\bar{1}}$.

We recall the symbol class $S^0_{0,0}$ as defined in Chapter VII of [St].

Define the operator (in d-dimensions)

$$K_{11}f(x) = \psi_1(x)\int_{\mathbb{R}_+{}^d} e^{ix^a\cdot y^b}\varphi(x,y)\psi_1(y)\beta(x-y)f(y)\,dy,$$

$\psi_1(x) = \psi(x_1)\psi(x_2)\cdots\psi(x_d)$, $\psi(t)$ as defined below (1.2). We prove

Proposition 4.3 *Suppose that $a, b \ge 1$. Then the operator K_{11} maps $L^2(\mathbb{R}^d)$ into itself, if $\varphi(x,y)$ satisfies (0.3).*

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Proof We consider L the operator below (in place of K_{11}), namely

$$Lh(x) = \psi_1(x^{\frac{1}{a}}) \int_{\mathbb{R}_+{}^d} e^{ix\cdot y} \varphi(x^{\frac{1}{a}}, y^{\frac{1}{b}}) \psi_1(y^{\frac{1}{b}}) \beta(x^{\frac{1}{a}} - y^{\frac{1}{b}}) h(y)\, dy$$

We notice from (0.3) that

$$\psi_1(x^{\frac{1}{a}}) \varphi(x^{\frac{1}{a}}, y^{\frac{1}{b}}) \psi_1(y^{\frac{1}{b}}) \beta(x^{\frac{1}{a}} - y^{\frac{1}{b}}) \in S^0_{0,0}.$$

Hence by the proposition on p. 282 of [St], we get that

$$\|Lh\|_2 \le C\|h\|_2. \tag{4.2}$$

Since $x_1, y_1, x_2, y_2, \dots, x_d, y_d \ge 1$, $a, b \ge 1$, we get:

$$\int_{\mathbb{R}_+{}^d} x^{\frac{1}{a} - \bar{1}} |Lh(x)|^2\, dx \le \|Lh\|_2^2 \le C\|\psi(y^{\frac{1}{b}}) h(y)\|_2^2 \le C \int_{\mathbb{R}_+{}^d} y^{\bar{1} - \frac{1}{b}} |h(y)|^2\, dy. \tag{4.3}$$

Now set $h(y) = f(y^{\frac{1}{b}}) y^{\frac{1}{b} - \bar{1}}$ into (4.3), and after changing variables, we get our result. ■

We are now in a position to complete the proof that the operator K_1 maps $L^2(\mathbb{R}_+{}^2)$ into itself.

Proof of Theorem 2.3 Note that here it suffices to do the cases where $a \ge b \ge 1$.

In case $b_1 = 1$ or $b_2 = 1$, we use Proposition 4.2 to handle the cases where $\varphi(x, y)$ is supported in one of the sets $0 \le x_j$ or $y_j \le 1$ for some $j = 1$ or 2. In order to settle the remaining cases, namely when $x_1, x_2, y_1, y_2 \ge 1$ we employ Proposition 4.3. Note that we suppose $\varphi(x, y)$ satisfies (2.2)(ii) in case $|x - y| \ge 1$. This completes the proof of the cases where $b_1 = 1$ or $b_2 = 1$.

We are left with the cases where $a \ge b > 1$ Once again we appeal to Proposition 4.2 in cases when $\varphi(x, y)$ is supported in one of the sets $0 \le x_j$ or $y_j \le 1$ for $j = 1$ or 2. In order to see the remaining cases, we notice that from our estimates in (1.5) and (1.8) we get

$$\|K_{11} f\|_2 \le C\|f\|_2.$$

This time we assumed that $\varphi(x, y)$ satisfies (2.2)(ii). This completes our estimates for the operator K_1 and by Proposition 4.1, the proof is complete. ■

We end this section with the (p, p) result for K.

Theorem 4.4 *Let $a, b \ge 1$ and the operator as defined in (2.1). Assume the hypothesis in Theorem 2.3. Then*

$$\|Kf\|_p \le C\|f\|_p \quad \textit{for all } 1 < p < \infty. \tag{4.4}$$

Proof Since Theorem 2.3 holds for K, we get the result for $p = 2$. Next, by Theorems 2.2 and 2.3 and arguing as in 5.2, p. 175 in [St] we conclude that

$$\|Kf\|_p \le C\|f\|_p \quad \text{for all } 1 < p < \infty,$$

and this completes the proof. ∎

5 Main Theorem and Necessary Conditions

Here we prove our (L^p, L^p) mapping theorem for the operator defined in (0.1) where the kernel φ satisfies (0.4).

We begin with the case where $a_l, b_l > 1$.

Proposition 5.1 *Assume* $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ *and* $a_l, b_l > 1$ *for* $l = 1, 2$. *Let* $r \in [0, 2)$ *and let* φ *satisfy (0.4). Then for* $p \in J = \left[\frac{a_l+b_l}{a_l+\frac{(b_l r)}{2}}, \frac{a_l+b_l}{a_l(1-(r/2))}\right]$,

$$\|Kf\|_p \le C\|f\|_p. \tag{5.1}$$

Remark If $|\varphi(x, y)| \ge C_1|x - y|^{-r}$, then (5.1) is both necessary and sufficient in order that $p \in J$.

Our main theorem is stated below,

Theorem 5.2 *Assume* $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ *and* $a_l, b_l \ge 1$ *for* $l = 1, 2$. *Let* $r \in [0, 2)$ *and* $\varphi(x, y) = |x - y|^{-r+i\tau}$, $\tau \in \mathbb{R}$, *then*

$$\|Kf\|_p \le C\|f\|_p, \tag{5.2}$$

if and only if $p \in J$.

Proofs of Proposition 5.1 and Theorem 5.2 Define the analytic family of operators,

$$S_z f(x) = \int_{\mathbb{R}^2} e^{i|x|^a\cdot|y|^b}\varphi(x, y)\beta(x - y)|x - y|^{-z} f(y)\, dy \tag{5.3}$$

where φ is as above.

Suppose that $a_l \ge b_l \ge 1$ (or $a_l \ge b_l > 1$ in case of Proposition 5.1) the cases where $b_l \ge a_l \ge 1$ (or $b_l \ge a_l > 1$) follow then by duality. By Theorem 2.2, we get (5.4),

$$\|S_{2-r+i\tau} f\|_1 \le C(\tau)\|f\|_{H_{ab}}, \quad \text{and} \tag{5.4}$$

$$\|S_{-r+i\tau} f\|_{p_0} \le C(\tau)\|f\|_{p_0}, \quad \text{for } p_0 = \frac{a_l + b_l}{a_l}, \tag{5.5}$$

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where $C(\tau)$ in both estimates grow polynomially in τ. In case $b_1, b_2 > 1$ we get (5.5) by Theorem 1.2, and in case b_1 or $b_2 = 1$, we get (5.5) by Theorem 3.1. By (5.4), (5.5) and analytic interpolation we obtain

$$\|S_0 f\|_p \le C\|f\|_p, \quad \text{for } p = \frac{a+b}{a+(br/2)},$$

that gives us the left endpoint for J.

While using the argument found in (2.5) and below in [PS] we get:

$$\|S_0 f\|_p \le C\|f\|_p \quad \text{where } p = \frac{a+b}{a(1-\frac{r}{2})} \tag{5.6}$$

and that gets us the right endpoint of J. This completes the proof of Proposition 5.1 and the suffiency in Theorem 5.2.

We need to prove the necessity part of Theorem 5.2. This requires the condition

$$C_2|x-y|^{-r} \le |\varphi(x,y)| \le C_1|x-y|^{-r}. \tag{5.7}$$

Let

$$Tf = \int_{\mathbb{R}^2} e^{ix^a \cdot y^b} f(y)\varphi(x,y)\,dy,$$

where $\varphi(x,y)$ satisfies (5.7). We want to show that

$$\|Tf\|_p \le C\|f\|_p, \quad \text{for all } f \in L^p \tag{5.8}$$

implies that $p \in J$. First for $N_1, N_2 \ge 1$ we write (5.8) in the form,

(5.9)

$$\iint_{I_N} s_1^{\frac{1}{a_1}-1} s_2^{\frac{1}{a_2}-1} \Big| \iint_{I_{\frac{1}{N}}} \varphi(s^{1/a}, t^{1/b}) e^{is\cdot t} f(t^{1/b}) \prod_{l=1}^{2} t_l^{\frac{1}{b_l}-1}\,dt \Big|^p\,ds \le C\int |f(y)|^p\,dy,$$

with $I_N = [N_1/2, N_1] \times [N_2/2, N_2]$ and similarly for $I_{\frac{1}{N}}$ and take

$$f(t) = \begin{cases} 1 & \text{if } t_l \in [(2N_l)^{\frac{-1}{b_l}}, N_l^{\frac{-1}{b_l}}], \text{ for } l = 1,2 \\ 0 & \text{elsewhere,} \end{cases}$$

and note that $1/4 \le s_1 t_1, s_2 t_2 \le 1$.

Then we get from (5.9)

$$\frac{N_1^{\frac{1}{a_1}} N_2^{\frac{1}{a_2}}}{(N_1^{\frac{2}{a_1}} + N_2^{\frac{2}{a_2}})^{\frac{pr}{2}}} \le C N_1^{\frac{p-1}{b_1}} N_2^{\frac{p-1}{b_2}}. \tag{5.10}$$

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With $N = N_1^{\frac{1}{b_1}} = N_2^{\frac{1}{b_2}}, N \to \infty$ then (5.8) and (5.10) imply that

$$\frac{b+a}{a+\frac{br}{2}} \le p.$$

Next for $N_1, N_2 \le 1$ in (5.9): we get from (5.7) and (5.8):

$$\frac{N_1^{\frac{1}{a_1}} N_2^{\frac{1}{a_2}}}{N_1^{\frac{p_1-1}{b_1}} N_2^{\frac{p_2-1}{b_2}}} \le C\Big((1/N_1)^{\frac{2}{b_1}} + (1/N_2)^{\frac{2}{b_2}}\Big)^{\frac{pr}{2}},$$

and again as in (5.10) we get

$$N^{\frac{2b}{a}+2+pr} \le CN^{2p},$$

letting $N \to 0$, we conclude

$$p \le \frac{a+b}{a(1-\frac{r}{2})},$$

the proof is complete. ■

References

[CP] L. C. Cheng and Y. Pan, *L^p Estimates For Oscillatory Integral Operators*. Proc. Amer. Math. Soc. (10) **127**(1999), 2995–3002.

[JS] W. B. Jurkat and G. Sampson, *The complete solution to the (L^p, L^q) mapping problem for a class of oscillating kernels*. Indiana Univ. Math. J. **30**(1981), 403–413.

[P1] Y. Pan, *Uniform estimates for oscillatory integral operators*. J. Funct. Anal. **100**(1991), 207–220.

[P2] ______, *Hardy spaces and oscillatory singular integrals*. Rev. Mat. Ibero **7**(1991), 55–64.

[PSS] Y. Pan, G. Sampson and P. Szeptycki, *L^2 and L^p estimates for oscillatory integrals and their extended domains*. Studia Math. (3) **122**(1997), 201–224.

[PS] Y. Pan and G. Sampson, *The complete (L^p, L^p) mapping properties for a class of oscillatory integrals*. J. Fourier Anal. Appl. (1) **4**(1998), 93–103.

[PhS1] D. H. Phong and E. M. Stein, *Hilbert integrals, singular integrals and Radon transforms I*. Acta Math. **157**(1986), 99–157.

[PhS2] ______, *Oscillatory integrals with polynomial phases*. Invent. Math. **110**(1992), 39-62.

[RS] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I*. J. Funct. Anal. **73**(1987), 179–184.

[S] G. Sampson, *L^2 Estimates for oscillatory integrals*. Colloq. Math. (2) **76**(1998), 201–211.

[SS] G. Sampson and P. Szeptycki, *Estimates of oscillatory integrals*. preprint.

[St] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Othogonality and Oscillatory Integrals*. Princeton Univ. Press, Princeton, N.J., 1993.

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