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NOTE ON A STABILITY THEOREM

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ABSTRACT. In this note the stability theorem of Albert and Baker concerning the n-th difference equation is proved by using invariant means.

In this note we give a short proof for the following stability theorem:

THEOREM. *Let G be an additive Abelian semigroup with $0, f: G \to \mathbb{C}$ a complex valued function for which $(x, y) \to \Delta_y^n f(x)$ is bounded. Then $f - P$ is bounded for some $P: G \to \mathbb{C}$ satisfying $\Delta_{y_1,\ldots,y_n}^n P(x) = 0$.*

Here $\mathbb{C}$ denotes the set of complex numbers and we have used the following notations: if $f: G \to \mathbb{C}$ then for all x, y in G let

$$\Delta_y f(x) = f(x+y) - f(x)$$

and for all $n = 1, 2, 3, \ldots, x, y_1, \ldots, y_{n+1}$ in G let

$$\Delta_{y_1,\ldots,y_{n+1}}^{n+1} f(x) = \Delta_{y_{n+1}}(\Delta_{y_1,\ldots,y_n}^n f)(x).$$

Although the above result is known (a special case of a theorem of [1]), our idea is new because our method is based on the use of invariant means.

Proof. First we remark that by the results of [2] $\Delta_{y_1,\ldots,y_n}^n f$ is a linear combination of some translates of functions of the type $\Delta_y^n f$ and hence the boundedness of $(x, y) \to \Delta_y^n f(x)$ implies the same property of

$$(x, y_1, \ldots, y_n) \to \Delta_{y_1,\ldots,y_n}^n f(x).$$

Here we also need the notion of invariant mean on G. Let $B(G)$ denote the set of all bounded complex valued functions on G. It is well known [3] that there exists a functional $M: B(G) \to \mathbb{C}$ with the properties: $M(f+g) = M(f) + M(g)$, $M(\lambda f) = \lambda M(f)$, $M(1) = 1$ and $M(f_y) = M(f)$ for all f, g in $B(G)$, λ in $\mathbb{C}$ and y in G (here f_y denotes the function defined by $f_y(x) = f(x+y)$ for all x and y in G). Such functionals are called invariant means. Let M denote one of them and we write M_x if M is applied with respect to the variable x. It is

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obvious, that we have

$$M_x[\Delta^n_{y_1,\dots,y_n} f(x)] = 0.$$

Indeed,

$$M_x[\Delta^n_{y_1,\dots,y_n} f(x)] = M_x[\Delta^{n-1}_{y_1,\dots,y_{n-1}} f(x+y_n) - \Delta^{n-1}_{y_1,\dots,y_{n-1}} f(x)]$$
$$= M_x[\Delta^n_{y_1,\dots,y_{n-1},y_n+x} f(0) - \Delta^n_{y_1,\dots,y_{n-1},x} f(0)] = 0.$$

From this fact we infer by induction:

$$M_{y_{n+1}}, \dots, M_{y_{n+k}}[\Delta^{n+k}_{y_1,\dots,y_{n+k}} f(x)] = (-1)^k \Delta^n_{y_1,\dots,y_n} f(x).$$

Now, without loss of the generality we may assume that $f(0) = 0$. Let, for x in G,

$$f_0(x) = (-1)^{n+1} M_{y_1}, \dots, M_{y_{n-1}}[\Delta^n_{y_1,\dots,y_{n-1},x} f(0)],$$

which is obviously bounded. On the other hand, for all $u_1, \dots, u_n$ in G we have:

$$\Delta^n_{u_1,\dots,u_n}(f - f_0)(x) = \Delta^n_{u_1,\dots,u_n} f(x)$$
$$+ (-1)^n \Delta^n_{u_1,\dots,u_n} M_{y_1}, \dots, M_{y_{n-1}}[\Delta^{n-1}_{y_1,\dots,y_{n-1}} f(x) - \Delta^{n-1}_{y_1,\dots,y_{n-1}} f(0)]$$
$$= \Delta^n_{u_1,\dots,u_n} f(x) + (-1)^n M_{y_1}, \dots, M_{y_{n-1}}[\Delta^n_{u_1,\dots,u_n} \Delta^{n-1}_{y_1,\dots,y_{n-1}} f(x)]$$
$$= \Delta^n_{u_1,\dots,u_n} f(x) + (-1)^n (-1)^{n-1} \Delta^n_{u_1,\dots,u_n} f(x) = 0,$$

hence the theorem is proved.

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