

THE MATHEMATICAL WORK OF JOHN NAPIER
(1550-1617)

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John Napier, Baron of Merchiston near Edinburgh, lived during one of the most troubled periods in the history of Scotland. He attended St Andrews University for a short time and matriculated at the age of 13, leaving no subsequent record. But a letter to his father, written by his uncle Adam Bothwell, reformed Bishop of Orkney, in December 1560, reports as follows:

"I pray you Sir, to send your son John to the Schools either to France or Flanders; for he can learn no good at home, nor gain any profit in this most perilous world."

He took an active part in the Reform Movement and in 1593 he produced a bitter polemic against the Papacy and Rome which was called *The Whole Revelation of St John*. This was an instant success and was translated into German, French and Dutch by continental reformers. Napier's reputation as a theologian was considerable throughout reformed Europe, and he would have regarded this as his chief claim to scholarship.

Throughout the middle ages Latin was the medium of communication amongst scholars, and translations into vernaculars were the exception until the 17th and 18th centuries. Napier has suffered badly through this change, for up till 1889 only one of his four works had been translated from Latin into English.

Received 16 August 1982. Thesis submitted to University of Auckland, March 1981. Degree approved April 1982. Supervisors: Mr Garry J. Tee Professor H.A. Montgomery (deceased), Dr W.F. Richardson and Dr K.E. Pledger.

Name of Work	Latin Text	English Translation
Descriptio Logarithmorum	1614	1616 Edward Wright
Rabdologiae	1617	1978 W.F. Hawkins
Constructio Logarithmorum	1619	1889 W.R. Macdonald
De Arte Logistica	1839	1978 W.F. Hawkins

The Napier Tercentenary Celebrations took place in 1914, but the onset of the 1914-1918 war was an effective barrier to international relations and remained so for several years. I concluded that Napier's work could not properly be evaluated unless all of his works were available in English. My translations of *De Arte Logistica* and *Rabdologiae* followed.

Napier's four mathematical works

1. *A description of the Wonderful Table of Logarithmes*, Edinburgh 1614 (Latin text), 1616 (English text).

Napier said that if this book was successful and was welcomed by scholars he would publish another book giving full instructions on how to make the Tables. But he need not have worried. The new techniques were welcomed by the mathematicians of Europe, and Keppler admitted that the work of checking the calculations of Tycho Brahe's astronomical records would have been impossible without them. The East India Company hired Edward Wright of Cambridge University to make an English translation "to meet the need of those who know no Latin." This was approved by John Napier, and published in 1616.

The 1616 English translation of DESCRIPTIO contained 20 pages of introductions, including the dedication to Charles, Prince of Wales, later Charles I; the Author's Preface to the Reader, A Preface by Professor Henry Briggs, and two introductory poems in the custom of the period. Book I contains 5 chapters of explanation on the use of sines, cosines and tangents, and the identification of their relevant logarithms.

The second book contains 59 pages providing a complete course of Plane and Spherical Trigonometry in 6 chapters. What a surprise! In a small book of less than 200 pages, introducing logarithms to the world, Napier also introduced the most advanced treatise on Trigonometry, and associated

it with logarithms as the simplest means of obtaining solutions of problems. I wish to deal with Trigonometry as a separate topic together with 20 pages from "The Construction of the Wonderful Table of Logarithms."

Napier's system of logarithms is based upon 6 definitions; 6 Propositions and 27 short sections covering a total of 13 pages. Then Chapter IV deals with finding sines, tangents and secants in the tables by the use of a number of examples. Chapter V extends the scope by the use of problems and the rules of proportion.

"But in the booke following we shall treat of their proper and particular use in that noble kinde of Geometrie which is called Trigonometrie."

This is the treatise in 59 pages to which I have already referred.

2. *The Construction of the Wonderful Table of Logarithmes*, Edinburgh 1619 (Latin text), 1889 (English text).

Most copies of the CONSTRUCTIO were sold bound with the copies of the DESCRIPTIO of 1614. This was the case with the pirated edition sold in France 1619-1620. The CONSTRUCTIO was a direct reply to Napier's promise of 1614, that he would explain the method of making logarithms, if his work pleased his readers. Napier's son Robert with the assistance of Henry Briggs carried the task through successfully. Here is an extract from the Preface to the work:

"You have then, kind reader, in this little book, most amply unfolded the theory of the construction of logarithms, here called by him artificial numbers, for he had this treatise written out beside him several years before the word logarithm was invented; in which their nature, characteristics, and various relations to their natural numbers, are clearly demonstrated."

"It seemed also desirable to add to the theory an Appendix as to the construction of another and better kind of logarithms (mentioned by the Author in the Preface to his *Rabdologiae*) in which the logarithm of unity is zero."

Robert paid excellent tribute to the help he had received from Henry Briggs on the death of his father:

"Now however, the burden of the whole business would appear to rest on the shoulders of the most learned Briggs ...".

The method of constructing logarithms was designed by Napier in 60

sections. Sections 1 to 6 deal with geometric and arithmetic progressions and the use of the decimal point:

"In numbers distinguished thus by a period in their midst, what-even is written after the period is a fraction, the denominator of which is unity, with as many cyphers after it as there are figures after the period."

Sections 7 to 12 deal with the addition, multiplication, subtraction and division of limits. Napier was always wary of slipshod numbering and he preferred to obtain his own answers, if approximate, between upper and lower limits.

The first table consists of 100 proportionals derived from the G.P.

$$T_{r+1} = 10^7(1-10^{-7})^r \quad \text{for } r = 0, 1, 2, \dots, 100 .$$

The second table consists of 50 proportionals derived from the G.P.

$$T_{r+1} = 10^7(1-10^{-5})^r \quad \text{for } r = 0, 1, 2, \dots, 50 .$$

The third or radical table

This contains 69 columns, and in each column 21 numbers are placed in the proportion between the first and last numbers of the second table.

These numbers are obtained from the G.P. shown here: $T_{r+1} = 10^7(1-5.10^{-4})^r$ where $r = 0, 1, 2, \dots, 20$.

The 69 leading numbers commence with 10^7 and proceed in the ratio of 100 to 99, subtracting from each one of them its hundredth part.

$$T_{r+1} = 10^7(1-10^{-2})^r \quad \text{where } r = 0, 1, 2, \dots, 68 .$$

Sections 28-33 explain how Napier uses Table 1 to find his first logarithm, $\log_N 9\,999\,999 = 1.000\,000\,05$. He then takes the last number in the second table and in Section 42 he shows that its logarithm is $5\,000.024\,000$. He then gives all the necessary rules for the selection of logarithms to fit the radical table outlined above.

Tables 1 and 2 provide accurate proportionals which can be fitted into the 69 columns of the third table. The third, or radical table, is the one from which the logarithms must be selected. The complete radical table can be expressed quite easily in modern notation.

$$\frac{T}{r+1.c+1} = 10^7 (1-5 \cdot 10^{-4})^r (1-10^{-2})^c$$

$$r = 0, 1, 2, \dots, 20, \quad c = 0, 1, 2, \dots, 68.$$

It must be emphasised that the third or radical table is *Not* a table of logarithms, but is the medium by which logarithms may be evaluated. Sections 28, 39, 41, 43, 45 indicate the methods to be used in evaluating logarithms accurately.

Section 60 refers to the construction of common logarithms. These were discussed by Napier and Briggs during the visit to Merchiston in 1615 and 1616. On the latter occasion it is very likely that Briggs produced the 16 page leaflet of common logarithms (now in the British Museum) - 717C.11.1 dated 1616, and marked for Henry Briggs in ink. Edmond Gunter of Gresham College published the first complete volume of common logarithms in London 1620.

The Explosion of Logarithms 1614-1640

	Title	Author	Date	Base
1	Descriptio Logarithmorum Canonis	John Napier	1614	e^{-1} -
2	Description of the wonderful Canon of Logarithms	John Napier Edward Wright	1616	e^{-1} -
3	Logarithmorum Chilias Prima	John Napier Henry Briggs	1617	- 10
4	Cursus Mathematici	Benj. Ursinus	1618	e^{-1} -
5	Kuntsliche Rechens-tablein	Franz Keszlern	1618	e^{-1} -
6	The Canon of Triangles	Edmond Gunter	1620	- 10
7	New Logarithms	John Speidell	1620	e -
8	Descriptio/Constructio (The Lyons Edition)	Anonymous	1620	e^{-1} -
9	Arithmetica Logarithmica	Henry Briggs	1624	- 10
10	Trigonometry with the Great Canon of Logarithms	Benj. Ursinus	1624	e^{-1} -
11	Chiliades Logarithmorum	Johan Kepler	1624	e^{-1} -
	Supplement to Chiliades	Johan Kepler	1625	e^{-1} -
12	Traite des Logarithmes	Denis Henrion	1625	- 10
13	Arithmetique Logarithmique	Edm. Wingate	1625	- 10

	Title	Author	Date	Base	
14	Table of Logarithms and Logarithmic sines	Henry Briggs Adrian Vlacq	1626	-	10
15	First 10 000 Logarithms The Trigonometric Tables Table Logarithmetique	Henry Briggs Edmond Gunter Adrian Vlacq	1626	-	10
16	The Rudolphine Tables	Johan Kepler	1627	e^{-1}	-
17	Arithmetica Logarithmica Arithmetique Logarithmic (French Translation)	Adrian Vlacq Adrian Vlacq	1628	-	10
18	Trigonometrie (Forerunner of Admiralty Handbook of Navigation Part II)	Richd. Norwood	1631	-	10
19	Course in Astronomy	Bonaventure Cavalieri	1632	-	10
20	Trigonometria Britannica	Hy. Gellibrand	1633	-	10
21	Trigonometria Artificialis	Adrian Vlacq	1633	-	10
22	Tabulae Logarithmicae Logarithmic sines etc.	Nathanial Roe Edmond Wingate	1633	-	10

3. *Rabdologiae or Numbering Rods in 2 Books: followed by one Book of Local Arithmetic; and an Appendix on Lightning Calculation.*
Edinburgh, 1617. English Text, 1978.

He gives full information, with illustrations, on Napier's popular system of calculating rods, familiarly known as "Napier's Bones". These were very popular, both in Scotland and in Europe. They performed multiplications by the Arabian Lattice using a system of diagonal addition. In Book 1 he introduces two new 'bones', one marked for square root computations, and the other for cube roots.

In Book 2 he performs geometrical calculations by means of tables of proportionals, the rule of three, polygons and the five regular solids.

"Every four numbers in the Table, which can be found within a quadrangle, stand in proportion to each other."

The book of Local Arithmetic uses a system of binary notation when used with counters on the squares of a chessboard, and multiplication, division and square root extraction can also be performed.

This system has been researched by a Russian group in 1978 with the

idea of improving computer training. An extract from their report says:

"On page 158, discussing Napier's alphabetical notation for integers in binary, and remarking on its convenience, they compare it favourably with '0 1' notation. In 3 cases a discrepancy was detected in the results ... produced in calculations in the modern system of writing the operations. No errors were detected with 'Local Arithmetic'".

A study of binary notation (1978) using traditional methods side by side with Napier's approach.

R.S. Guter and Y.L. Polunov (Moscow University)

The appendix to this book deals with Napier's attempt to produce a new calculator. This consisted of 100 direct strips and 100 transverse strips crossing orthogonally over the direct strips. The principle of the Arabian Lattice is used for the addition of all multiples, but the number of multiples is reduced by perforating the transverse strips systematically in only two places, thus showing 2 digits, the tens digit to the left of the main diagonal, and the units digit to the right. These are then added mentally to give the total.

This device was made in the Engineering Workshops of Auckland University from my translation of the text and the original pictures in the book. The calculator was a foot square and five inches deep. The capacity of the calculator depends upon the number of direct and transverse strips; and with one hundred of each, ten digits by ten digits can be calculated in less than a minute. For division an extra minute would be needed. The machine was first displayed at Waikato University in May 1979 and then printed by the New Zealand Mathematical Society in December 1979. The calculator was demonstrated on New Zealand Television early in 1980, and was exhibited to more than 20 United Kingdom Universities during May–July 1980.

4. *De Arte Logistica (The Baron of Merchiston, his booke of Arithmetic and Algebra)* Edinburgh 1839 (Latin text). 1978 English translation.

"John Napier of Merchiston left his manuscripts to his son Robert, who appears to have made the following pages to be written out fair, from his father's notes, for Mr. Briggs, Professor of Geometry at Oxford."

"The manuscript was transcribed and printed in 1839 by Mark Napier, who gave it the title from its opening words. It has a modern binding of red Morocco lettered *Joannis Napier Fragmenta* ... and as a bookplate the arms of Lord Napier."

Napier Memorial Volume (1915) pages 191–192.

After Napier's resounding success with the *Book of Revelations* in 1593, he turned his attentions to astronomy and its associated problems. In the dedication to *Rabdologiae* he says:

"I have always tried, most noble Sir, according to my strength and to the measure of my ability, to do away with the difficulties and prolixity of calculations, the tedium of which deters most people from studying mathematics."

He became interested in "the Doctrine of Triangles", for Arabian scholars of the 12th century had learnt how to convert the product of two sines into the sum of 2 different sines. This technique reached Europe through Arabic translations in Greek and Latin, and it was used by Tycho Brahe to simplify the work of his calculators. He absorbed the technique and used it in his studies of astronomy and trigonometry.

Napier's book of arithmetic and algebra

There are 2 books of arithmetic, and 2 books of algebra, the latter unfinished. There is also a Book III; *The Logistics of Geometry*, which I have called *Chapter A* and placed before Chapter I of the book of algebra. This seems quite logical, since the whole chapter is devoted to a system of index notation invented by John Napier. Chapter I of Napier's algebra was entitled "Definitions, description and notations" and included the current method of expressing roots down to the ninth. The system which Napier invented was superior to all others, was easy to remember, and could be used for the 200 hundredth root or higher.

Book I Chapter III, *Operations on powers and roots*, deals with index notation, using Roman numerals for the indices (or logarithms) and Arabic numerals for the natural numbers. Napier would probably have written 2(V) for 32, and 2(VIII) for 256. Napier used the term 'artificial numbers' for his logarithms and Arabic numerals for the natural numbers.

Book I Chapter V, *Compound calculations*. In this chapter Napier claims to have established a 'single general method of working' in 4 rules which provide a solution to all problems of proportion.

Book II Chapter V, *Short methods of multiplication and division*. Napier produces several neat solutions. But he cannot refrain from exclaiming:

"Best of all perhaps is the use of my 'bones' referred to in Rabdologiae. If these are to hand the whole multiplication or division can be accomplished."

Book II Chapters VII-IX, *Extraction of roots, using the binomial triangle*. Extraction of roots was frequently completed approximately by the use of tables. Napier, Recorde and others all used tables as a first approach and this method was satisfactory in most cases. Using the binomial triangle Napier could reach a high degree of approximation.

He evaluates

$$\sqrt[2]{55\ 225} = 235 ,$$

$$\sqrt[4]{3\ 049\ 800\ 625} = 235 ,$$

$$\sqrt[3]{12\ 977\ 875} = 235 .$$

All three of these roots are exact and were obtained by means of the binomial supplement.

Book II Chapter X, *Contracted multiplication*. The method of contracted multiplication described by John Napier in this chapter is a rare piece of excellence.

In 1623, When Johan Kepler received a copy of this method, he was struck with its excellence, and sent it at once to the Landgrave, urging him to distribute copies to the calculators who were working with Kepler on the Rudolphine Tables.

Charles Naux, in *Histoire des Logarithmes*, Volume 1, p. 23, referred to one of Kepler's letters in which the method of contracted multiplication was used 'for the first time'. [Johannes Kepler, "Opera Cmmia", Vol. VII, Prooemium Editoris, p. 306.]

It seems that Napier was himself the inventor of this method of contracted multiplication. The answer obtained by this method never exceeds the real answer by more than one unit.

The algebra of John Napier

At the Napier Tercentenary Professor J.E. Steggal of the University of St Andrews, read a paper on "De Arte Logistica". His comment on the first 5 chapters of the first book of algebra covered less than half a page! He

was puzzled by the use of 'simple numbers' like OQC and $OQCBaQQC$, and their roots.

Let us return to Book I Arithmetic, Chapter 3: 'Operations on powers and roots'. The following table is demonstrated.

index	.	I	II	III	IV	V	VI	VII
power	1	2	4	8	16	32	64	128
index	0	R	Q	C	QQ	B	QC	BQ

The first line of indices is the suggestion of Napier. The bottom line of indices is the system commonly used in Europe during the 16th century and index 0 is equated with power 1, that is, $x^0 = 1$. Consequently the above 'simple numbers' when expressed in modern terms are:

$$OQC = x^0 x^6 \quad \text{and} \quad OQCBaQQC = x^0 x^{30} a^{12}.$$

The symbol 0 is correctly used in each of the 28 appearances in Chapters III and IV. It can also be regarded as a placeholder, as Professor Stegall nearly suggests.

John Napier's algebra book II

Napier's rhetoric is seen to its best advantage in this book, Chapters 1 and 2 (The addition and subtraction of uninomia), and in Chapter 12 (roots of Plurinomia). Simple formulae for $(a^{1/n} \pm b^{1/n})$ and $\sqrt[n]{a \pm b}$ are given directly in limpid Latin.

In Chapter 7 (algebraic division) he gives six examples of square root, the first two of which are easily handled; Examples 3 and 4 contain two unknowns, whilst Examples 5 and 6 are still more difficult. The method he uses for long division was the same as that used by Phillipi Calandri in the first Italian textbook of arithmetic printed in 1491. But it must be said that Calandri's method was a century ahead of his successors.

He gives two examples of algebraic cube root; the first example is an exact cube root with no remainder, whilst the second example evaluates both root and remainder. He uses the expansion of $(a+b)^4$ and $(a+b)^5$ to illustrate the extraction of fourth and fifth roots, but the method is

really that of the binomial supplements.

Napier uses the vinculum below the line equivalent to the use of brackets in modern terms, and he also says:

"An example of the square root of $(\sqrt{q48} + \sqrt{q28})$ is to be extracted. Before this binomium, place the radical sign \sqrt{q} with this period (.). This then makes $\sqrt{q} \cdot \sqrt{q48} + \sqrt{q28}$ pronounced in this way. 'The square root of the universal radical of the square root of 48, added to the square root of 28'."

Modern brackets of this kind () were not used by Napier.

Chapter VI, *The arrangement of algebraic expressions*, caused me considerable time and trouble. The text appears to be in some disorder, and I cannot understand the purpose which motivates the author. It may be that we are near the end of Robert Napier's search for material and the pieces have been badly put together.

Chapter VIII, *Extraction of roots from composite numbers*. This appears to be one of Napier's early attempts to deal with root extraction. Professor Steggal says:

"to me, at least, the passages are not perfectly clear, and there are certain misprints in the text. The general idea seems to be the attainment of an approximate root, not only algebraically but arithmetically - a result only possible when some relation exists between the symbols."

Chapter X, *Equations - preparation and solution*. This is a systematic analysis of the solution of equations containing up to 4 universal quadratic radicals. Napier asks what more can be done to simplify equations

"Over and above the propositions given in this chapter. For instance, the multiplication of simple irrationals, for the most part, produces more solutions than there should be."

He gives as an example: " $12 - \sqrt{x} = x$, when multiplied as shown, produced the equation: $x^2 - 25x + 144 = 0$. This has 2 valid solutions, namely 16 and 9." He then points out that the original equation was only solved by $x = 9$, and not $x = 16$.

This is an important point upon which to end a remarkable book which never saw the light of day until too late to be a useful agent. Robert Napier finished off the text of his father's work with the simple sentence:

There is no more of his algebra orderly set down.

5. *Napier's Trigonometria Logarithmica.*

Napier's contributions to trigonometry have been largely ignored, and I am pleased to quote the famous French Historian of Mathematics, J.E. Montucla (1725-1799).

"Napier's celebrity derives principally from the theory of logarithms. Nevertheless, I do not think that I should ignore several other inventions, which are less brilliant, and therefore have less universal appeal ..."

"Amongst these inventions, I call your attention, above all to a rule for the resolution of right spherical triangles, which, in the judgment of those skilled in the subject, is most convenient and ingenious. In fact, those who use spherical trigonometry, know that there are sixteen cases of this kind; and of these, there are ten or a dozen whose solutions are not lightly come by, Authors who have written on the subject, have been obliged to draw up tables for these special cases, which can be consulted when necessary. And only in this way have they been able to ease the strain on the memory."

"Napier however, reduces these cases into a single rule in two parts, which is very suitable by reason of its elegance to impress itself deeply on the memory ..."

"And I could hardly conceal my surprise to find scarcely a trace of them in French, or for that matter in continental trigonometries, which have been produced since Napier's day."
[*Histoire des Mathematiques*, Volume II, pages 24-25. Translated by W.F. Hawkins.]

The contents of *Trigonometria Logarithmica* (1614) are as follows.

1. Of right lined triangles
2. Of oblique angled right lined triangles
John Napier's Analogy of the tangents
3. Spherical triangles - definitions
4. Of single quadrantals - Napier's Rules of circular parts
First and Second examples
Third Example
Fourth Example
5. Of mixed quadrantals - 2 sides, 1 angle
First example - 2 sides and included angle
Second example - 2 angles and 1 side
Third example - 2 sides and 1 angle
Fourth example - 2 sides/subtended angle
Fifth example - 2 angles/1 subt. by side
Sixth example - 2 angles one side
6. Of pure triangles not quadrantal
Three sides given - Napier's Analogies
Three angles given. Principle of Duality

Extracted from Napier's *Descriptio Logarithmorum*, 1614 (pages 30–89) and *Constructio Logarithmorum*, 1619 (pages 64–81) which contains at least 30 worked examples and includes material invented by Napier (Analogies and circular parts). These are the first problems solved by the use of logarithms.

Appendix

The following is a table of contents of the PhD thesis. This table is included in order to indicate the scope of the work.

Volume	IA	De Arte Logistica 1639 Latin (translated into English)	
		Arithmetic Books I and II	78
		Algebra Books I and II	88
Volume	IB	Rabdologiae 1617 Latin (translated into English)	
		Book I Numbering Rods	34
		Book II Ditto using Rule of Three and Tables	38
		Book III Local Arithmetic, binary notation	35
		Appendix Napier's Calculating Machine	27
Volume	IIA	Descriptio Canonis Logarithmorum 1614	
		Wright's English translation 1616.	
		1. Summary with notes	38
		Constructio Canonis Logarithmorum 1619	
		Macdonald's English translation 1889	
		2. Summary with notes	38
		3. Napier's Trigonometry from Descriptio	
		Summary and solutions with notes	58
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		Summary, notes and modern version	61
		Includes table of sines, cosines and tangents at 1	
		minute intervals 0° – 45°	
Volume	IIB	The explosion of Logarithms in Britain, France,	
		Germany, Holland, Italy, etc (1614–1633)	43
		2. The Pioneers of Navigation in Britain	
		(a) Henry Briggs 1561–1631	38
		(b) Edmund Gunter 1581–1626	47
		(c) Edward Wright 1558–1615	19
		(d) Richard Norwood 1590–1675	17
		(e) John Speidell fl. 1600–1634	12
Volume	IIIA	(a) Notes on Napier's Arithmetic	60
		(b) Notes on Napier's Algebra	62
		(c) Commentary on Napier's Trigonometry	32
		Comparison with modern style by W.F.H.	17

Volume IIIB

The History of Trigonometry

- (a) The Legacy of antiquity: Babylonian astronomy; Egyptian astronomy; Thales of Miletus (624-546 BC) Aristarchus of Samos (310-230 BC); Eratosthenes (c. 230BC); Hipparchus (161-126 BC); Menelaus of Alexandria (c. 100 AD); Claudius Ptolemaeus (c. 100-170 AD) and the *Almagest*. 39 pages
- (b) The Legacy of Islam: Al-Khwarizmi (c. 800-847 AD) Al-Farghani (861+); Thabit Ibn Qurra (836-901 AD); Al-Battani (858-929 AD); Abu'l Wefa (940-998 AD); Ibn Yunus (950-1009 AD); Al-Biruni (973-1050 AD); Omar Khayyam (1044-1123 AD); Jabir Ibn Aflah (1100-1150 AD). 57 pages
- (c) Historical Introduction 12th-15th century Fibonacci (1175-1250); Jordanus (fl. 1220-1240); Roger Bacon (1214-1294); Sacrobosco (1200(?) -1256); Richard of Wallingford (1291(?) -1336) the maker of the wonderful clock "ALBION", was aware of the works of Thabit Ibn Qurra, Al-Battani, Abu'l Wefa, Al-Biruni and Jabir, otherwise Gebir, and uses their work to solve his problems. Copies of the text of the *Albion* were used by John of Gmunden, Georg Peurbach and Regiomontanus, successive professors of astronomy/mathematics at Vienna. The new technology was produced by Henry the Navigator by establishing a research centre at Sagres in 1418. Nearly a century later the Spanish establishment near Seville carried on this work and produced the best navigators and technicians in seamanship. 51 pages
- (d) The Foundations of Trigonometry in Europe Peurbach (1423-1461), Regiomontanus (1436-1476) and Walther (1450-1504) 20 pages
- (e) The "Doctrine of triangles" and the computation of very large numbers by means of a technique copied from the Arabs called *PROSTHAPHAERESIS*. By this means the sum of 2 sines can be converted into the product of 2 other sines. 16 pages
- (f) The makers of tables for astronomy and navigation The Alfonsine Tables (1260 c); Copernicus (1473-1543); Rheticus (1514-1576); Erasmus Reinhold (1511-1553) the Prutenic Tables; Valentine Otho (1550-1596) *Opus Palatinum*; Tycho Brahe (1546-1601) *Historia Coelestis*; Bartholomew Pitiscus (1561-1613) "*Trigonometrie, or the Doctrine of Triangles*"; first translated into English by Ralph Handson in 1614; Johan Kepler and the Rudolphine Tables 1627. These are the first tables to use logarithms, both Napier's and Kepler's. 36 pages