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A MINIMUM DEGREE CONDITION FOR FRACTIONAL ID-[a, b]-FACTOR-CRITICAL GRAPHS

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Abstract

Let G be a graph of order n, and let a and b be two integers with $1 \leq a \leq b$. Let $h : E(G) \to [0, 1]$ be a function. If $a \leq \sum_{e \ni x} h(e) \leq b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is fractional independent-set-deletable $[a, b]$-factor-critical (in short, fractional ID-$[a, b]$-factor-critical) if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G. In this paper, it is proved that if $n \geq ((a + 2b)(a + b - 2) + 1)/b$ and $\delta(G) \geq ((a + b)n)/(a + 2b)$, then G is fractional ID-$[a, b]$-factor-critical. This result is best possible in some sense, and it is an extension of Chang, Liu and Zhu's previous result.

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1. Introduction

For motivation and background to this work, see [15]. Readers are referred to [1] for undefined terms and concepts. The graphs considered in this paper will be finite undirected graphs which have neither loops nor multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For each $x \in V(G)$, we use $d_G(x)$ to denote the degree of x in G, and $N_G(x)$ to denote the neighborhood of x in G. We write $N_G[x]$ for $N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. If $G[S]$ has no edges, then we call S independent. The minimum degree of G is denoted by $\delta(G)$. If G_1 and G_2 are disjoint graphs, the join and union are denoted by $G_1 \vee G_2$ and $G_1 \cup G_2$, respectively.

Let a and b be two positive integers with $1 \leq a \leq b$. Then a spanning subgraph F of G is called an $[a, b]$-factor if $a \leq d_F(x) \leq b$ for each $x \in V(G)$. If $a = b = k$, then

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an $[a, b]$-factor is called a k-factor. If $k = 1$, then we say that a 1-factor is a perfect matching. A graph G is factor-critical [7] if $G - v$ has a perfect matching for each $v \in V(G)$. In [9], the concept of the factor-critical graph was generalised to the ID-factor-critical graph. We say that G is independent-set-deletable factor-critical (in short, ID-factor-critical) if for every independent set I of G which has the same parity with $|V(G)|$, $G - I$ has a perfect matching. It is clear that every ID-factor-critical graph with odd vertices is factor-critical.

Let $h : E(G) \to [0, 1]$ be a function. If $a \leq \sum_{e \ni x} h(e) \leq b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. If $a = b = k$, then a fractional $[a, b]$-factor is called a fractional k-factor. A fractional 1-factor is also called a fractional perfect matching. A graph G is fractional ID-k-factor-critical [2] if $G - I$ has a fractional k-factor for every independent set I of G. In this paper, the concept of the fractional ID-k-factor-critical graph was generalised to the fractional ID-$[a, b]$-factor-critical graph, that is, a graph G is fractional independent-set-deletable $[a, b]$-factor-critical (in short, fractional ID-$[a, b]$-factor-critical) if $G - I$ has a fractional $[a, b]$-factor for every independent set I of G.

Many authors have investigated $[a, b]$-factors [3, 8, 10, 12, 13] and fractional factors [5, 6, 11, 14]. Chang *et al.* [2] obtained a minimum degree condition for a graph to be a fractional ID-k-factor-critical graph.

THEOREM 1.1 [2]. *Let k be a positive integer and G be a graph of order n with $n \geq 6k - 8$. If $\delta(G) \geq 2n/3$, then G is fractional* ID-*k-factor-critical.*

In this paper, we study fractional ID-$[a, b]$-factor-critical graphs, and obtain a minimum degree condition for a graph to be a fractional ID-$[a, b]$-factor-critical graph. Our main result is the following theorem, which is an extension of Theorem 1.1.

THEOREM 1.2. *Let G be a graph of order n, and let a and b be two integers with $1 \leq a \leq b$. If $n \geq ((a + 2b)(a + b - 2) + 1)/b$ and $\delta(G) \geq ((a + b)n)/(a + 2b)$, then G is fractional* ID-*$[a, b]$-factor-critical.*

2. The proof of Theorem 1.2

In order to prove Theorem 1.2, we rely heavily on the following lemma.

LEMMA 2.1 [4]. *Let G be a graph. Then G has a fractional $[a, b]$-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq 0,$$

where

$$T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a\} \quad \text{and} \quad d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x).$$

PROOF OF THEOREM 1.2. By Theorem 1.1, the result obviously holds for $a + b = 2$ (that is, $a = b = 1$). In the following, we assume that $a + b \geq 3$. Let X be an independent set of G and $H = G - X$. Clearly, $|V(H)| = n - |X|$, $n - |X| \geq \delta(G)$ and $\delta(H) \geq \delta(G) - |X|$.

In order to complete the proof of Theorem 1.2, we need only to prove that H has a fractional $[a, b]$-factor. By contradiction, we suppose that H has no fractional $[a, b]$-factor. Then, according to Lemma 2.1, there exists some subset $S \subseteq V(H)$ such that

$$\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \leq -1. \tag{2.1}$$

We choose such subsets S and T so that $|T|$ is as small as possible.

Claim 1. We shall show that $d_{H-S}(x) \leq a - 1$ for any $x \in T$.

PROOF. If $d_{H-S}(x) \geq a$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2.1), which contradicts the choice of S and T. The proof of Claim 1 is complete. □

Since $n - |X| \geq \delta(G)$ and $\delta(G) \geq ((a + b)n)/(a + 2b)$,

$$\begin{aligned}
\frac{b(a+b)}{a^2}(\delta(G) - |X|) + \frac{b|X|}{a} - \frac{bn}{a} &= \frac{b(a+b)\delta(G)}{a^2} - \frac{b^2|X|}{a^2} - \frac{bn}{a} \\
&= \frac{b(a+b)\delta(G)}{a^2} + \frac{b^2}{a^2}(n - |X|) - \frac{b^2 n}{a^2} - \frac{bn}{a} \\
&= \frac{b(a+b)\delta(G)}{a^2} + \frac{b^2}{a^2}(n - |X|) - \frac{b(a+b)n}{a^2} \\
&\geq \frac{b(a+b)\delta(G)}{a^2} + \frac{b^2}{a^2}\delta(G) - \frac{b(a+b)n}{a^2} \\
&= \frac{b(a+2b)\delta(G)}{a^2} - \frac{b(a+b)n}{a^2} \\
&\geq \frac{b(a+2b)}{a^2} \cdot \frac{(a+b)n}{a+2b} - \frac{b(a+b)n}{a^2} = 0,
\end{aligned}$$

which implies

$$\delta(G) - |X| \geq \frac{a}{a+b}(n - |X|).$$

Combining this with $\delta(H) \geq \delta(G) - |X|$,

$$\delta(H) \geq \delta(G) - |X| \geq \frac{a}{a+b}(n - |X|). \tag{2.2}$$

Claim 2. We shall show that $|T| \geq b + 1$.

PROOF. According to (2.2),

$$n \geq \frac{(a+2b)(a+b-2)+1}{b} > \frac{(a+2b)(a+b-2)}{b},$$

$$n - |X| \geq \delta(G) \quad \text{and} \quad \delta(G) \geq \frac{(a+b)n}{a+2b},$$

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and

$$\begin{aligned}\delta(H) &\geq \frac{a}{a+b}\cdot\frac{(a+b)n}{a+2b} = \frac{an}{a+2b} > \frac{a}{a+2b}\cdot\frac{(a+2b)(a+b-2)}{b}\\ &= \frac{a(a+b-2)}{b} \geq \frac{a(b-1)}{b} = a - \frac{a}{b} \geq a-1.\end{aligned}$$

In terms of the integrity of $\delta(H)$,

$$\delta(H) \geq a. \tag{2.3}$$

If $|T| \leq b$, then, by (2.1) and (2.3),

$$\begin{aligned}-1 &\geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T|\\ &\geq |T||S| + d_{H-S}(T) - a|T|\\ &= \sum_{x\in T}(|S| + d_{H-S}(x) - a)\\ &\geq \sum_{x\in T}(\delta(H) - a) \geq 0,\end{aligned}$$

which is a contradiction. This completes the proof of Claim 2. □

According to Claim 2, $T \neq \emptyset$. Define

$$h_1 = \min\{d_{H-S}(x) : x \in T\}.$$

Choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T \setminus N_T[x_1] \neq \emptyset$, let

$$h_2 = \min\{d_{H-S}(x) : x \in T \setminus N_T[x_1]\}.$$

Choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$. According to Claim 1, $0 \leq h_1 \leq h_2 \leq a-1$. Obviously, $d_H(x_i) \leq |S| + h_i$ for $i = 1, 2$.

Case 1. $T = N_T[x_1]$.

Using Claim 2 and $T = N_T[x_1]$,

$$a - 1 \geq h_1 = d_{H-S}(x_1) \geq |N_T[x_1]| - 1 = |T| - 1 \geq b \geq a.$$

This is a contradiction.

Case 2. $T \setminus N_T[x_1] \neq \emptyset$.

Note that $|N_T[x_1]| \leq d_{H-S}(x_1) + 1 = h_1 + 1$ and $a - h_2 \geq 1$. Let $|V(H)| = p$. Then we obtain $p - |S| - |T| \geq 0$. Thus,

$$\begin{aligned}(a-h_2)(p-|S|-|T|) - 1 &\geq \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T|\\ &\geq b|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - a|T|\\ &= b|S| - (a-h_2)|T| - (h_2-h_1)|N_T[x_1]|\\ &\geq b|S| - (a-h_2)|T| - (h_2-h_1)(h_1+1),\end{aligned}$$

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that is,

$$(a+b-h_2)|S| \le (a-h_2)p + (h_2-h_1)(h_1+1) - 1. \tag{2.4}$$

From (2.2) and $n - |X| = p$, we have $\delta(H) \ge ap/(a+b)$. Combining this with $|S| \ge \delta(H) - h_1$,

$$|S| \ge \frac{ap}{a+b} - h_1. \tag{2.5}$$

According to (2.4) and (2.5),

$$(a+b-h_2)\Big(\frac{ap}{a+b} - h_1\Big) \le (a-h_2)p + (h_2-h_1)(h_1+1) - 1,$$

which implies

$$(bp-a-b)h_2 \le (a+b)^2 h_1 - (a+b)(h_1+1)h_1 - (a+b). \tag{2.6}$$

In terms of

$$p = n - |X| \ge \delta(G), \quad \delta(G) \ge \frac{(a+b)n}{a+2b}, \quad n \ge \frac{(a+2b)(a+b-2)+1}{b}$$

and $a+b \ge 3$, we get

$$\begin{aligned} bp \ge b\delta(G) \ge \frac{b(a+b)n}{a+2b} &\ge \frac{(a+b)(a+2b)(a+b-2)+(a+b)}{a+2b} \\ &> (a+b)(a+b-2) \ge a+b. \end{aligned}$$

Combining this with (2.6) and $h_1 \le h_2$,

$$(bp-a-b)h_1 \le (bp-a-b)h_2 \le (a+b)^2 h_1 - (a+b)(h_1+1)h_1 - (a+b),$$

that is,

$$(a+b)h_1^2 + (bp-(a+b)^2)h_1 + (a+b) \le 0. \tag{2.7}$$

Let $f(h_1) = (a+b)h_1^2 + (bp-(a+b)^2)h_1 + (a+b)$. If $h_1 = 0$, then, by (2.7), we have $2 \le a+b \le 0$, which is a contradiction. In the following, we may assume that $h_1 \ge 1$. Since $bp > (a+b)(a+b-2)$,

$$f'(h_1) = 2(a+b)h_1 + bp - (a+b)^2 > 2(a+b) + (a+b)(a+b-2) - (a+b)^2 = 0.$$

Thus, by (2.7),

$$\begin{aligned} 0 \ge f(h_1) \ge f(1) &= (a+b) + (bp-(a+b)^2) + (a+b) \\ &> 2(a+b) + (a+b)(a+b-2) - (a+b)^2 = 0, \end{aligned}$$

which is a contradiction.

In each of the above cases we obtained contradictions. Hence, H has a fractional $[a, b]$-factor, that is, G is fractional ID-$[a, b]$-factor-critical.

This completes the proof of Theorem 1.2. □

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3. Remark

In this section, we show that the condition $\delta(G) \geq ((a+b)n)/(a+2b)$ in Theorem 1.2 is sharp. To see this, we construct a graph $G = (at)K_1 \vee (bt)K_1 \vee (bt+1)K_1$, where t is a sufficiently large positive integer. Obviously, $|V(G)| = n = (a+2b)t + 1$ and

$$\frac{(a+b)n}{a+2b} > \delta(G) = (a+b)t = (a+b)\cdot\frac{n-1}{a+2b} = \frac{(a+b)n}{a+2b} - \frac{a+b}{a+2b} > \frac{(a+b)n}{a+2b} - 1.$$

In the following, let $X = (bt)K_1$. Clearly, X is an independent set of G. Put $H = G - X = (at)K_1 \vee (bt+1)K_1$, $S = (at)K_1$ and $T = (bt+1)K_1$. Then

$$\begin{aligned}\delta_H(S,T) &= b|S| + d_{H-S}(T) - a|T| \\ &= abt - a(bt+1) = -a < 0.\end{aligned}$$

According to Lemma 2.1, H has no fractional $[a,b]$-factor. Hence, G is not fractional ID-$[a,b]$-factor-critical. In the sense above, the bound of $\delta(G)$ in Theorem 1.2 is sharp.

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