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A RESULT OF PALEY AND WIENER ON DAMEK–RICCI SPACES

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Abstract

A classical result due to Paley and Wiener characterizes the existence of a nonzero function in $L^2(\mathbb{R})$, supported on a half-line, in terms of the decay of its Fourier transform. In this paper, we prove an analogue of this result for Damek–Ricci spaces.

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1. Introduction

For $f \in L^1(\mathbb{R}^d)$ we define the Fourier transform $\widehat{f}$ by the standard formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^d,$$

where dx denotes the d-dimensional Lebesgue measure and $x \cdot \xi$ denotes the Euclidean inner product of the vectors x and ξ. It is a well-known fact in harmonic analysis that if the Fourier transform of an integrable function defined on the real line decays very rapidly at infinity then the function cannot vanish on a 'large set' unless it vanishes identically. A manifestation of this fact is the following: if f is a compactly supported integrable function defined on $\mathbb{R}$ and its Fourier transform satisfies the estimate

$$|\widehat{f}(\xi)| \leq Ce^{-|\xi|}, \quad \text{for all } \xi \in \mathbb{R},$$

then f extends as a holomorphic function to a strip in the complex plane containing the real line and hence f is identically zero. This initial observation motivates one to endeavour to find a more optimal decay of the Fourier transform for such a conclusion.

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For instance we may ask: Does there exist a nonzero integrable, compactly supported function f on $\mathbb{R}$ with its Fourier transform satisfying the estimate

$$|\widehat{f}(\xi)| \le Ce^{-(|\xi|/\log|\xi|)}, \qquad \text{for large } |\xi|?$$

The answer to the above question is in the negative and follows from a classical result due to Paley and Wiener ([16, page 16, Theorem XII]; [17, Theorem II]). There is a whole body of literature [12–14, 16] devoted to the study of the trade-off between the nature of the set on which a function vanishes and the allowable decay of its Fourier transform. The result due to Paley and Wiener gives the following characterization of the existence of a nonzero function supported on a half-line whose Fourier transform satisfies such an estimate.

THEOREM 1.1. *Let θ be a nonnegative locally integrable even function on $\mathbb{R}$. There exists a nonzero $f \in L^2(\mathbb{R})$ with* $\operatorname{supp} f \subset (-\infty, x_0]$ *for some $x_0 \in \mathbb{R}$ such that*

$$|\widehat{f}(\xi)| \le Ce^{-\theta(\xi)}, \quad \textit{for almost every } \xi \in \mathbb{R},$$

if and only if

$$\int_{\mathbb{R}} \frac{\theta(\xi)}{1+\xi^2}\, d\xi < \infty.$$

Though the result of Paley and Wiener is not available in the exact form given above it can be easily deduced from the following version proved by Paley and Wiener [4, Theorem 1.1].

THEOREM 1.2 [17, Theorem II]. *Let ϕ be a nonnegative, nonzero function in $L^2(\mathbb{R})$. There exists $f \in L^2(\mathbb{R})$ vanishing for $x \ge x_0$ for some $x_0 \in \mathbb{R}$ such that $|\widehat{f}| = \phi$ if and only if*

$$\int_{\mathbb{R}} \frac{|\log \phi(\xi)|}{1+\xi^2}\, d\xi < \infty.$$

Paley and Wiener proved Theorem 1.2 using complex analytic techniques via a holomorphic extension of the Fourier transform in the upper half-plane. This complex analytic technique motivated us to prove an analogue of Theorem 1.1 for compactly supported smooth functions on the Euclidean motion group and connected, noncompact, semisimple Lie groups with finite center [4, 5].

Since for a noncompact, semisimple Lie group G there is no natural way of defining functions supported on a half-space, in [4, 5] we have worked under the assumption that $f \in C_c^\infty(G)$. However, the situation changes if we talk about functions defined on $X = G/K$, where K is a maximal compact subgroup of G. To explain this point let us consider the Iwasawa decomposition $G = NAK$ where $A = \exp \mathfrak{a}$ with $\dim \mathfrak{a} = 1$. Then functions on G/K can be viewed as functions on NA. For fixed $\tau \in \mathbb{R}$, the horosphere

$$H_\tau = \{na_\tau \mid n \in N\},$$

is an analogue of a hyperplane in $\mathbb{R}^d$. We then consider functions f supported on the set

$$E_\tau = \{na_t \mid n \in N, t \ge \tau\}.$$

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This can be thought of as a natural generalization of functions on $\mathbb{R}^d$ which are supported on $\{x \in \mathbb{R}^d \mid x \cdot \eta \geq t_0\}$, for fixed $\eta \in S^{d-1}$, $t_0 \in \mathbb{R}$. We observe that for $d = 1$ and $\eta = 1$ the support of the function becomes $[t_0, \infty)$. Clearly the boundary of E_τ is the horocycle H_τ. For example, for the non-Euclidean disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the Riemannian metric $ds^2 = (1 - x^2 - y^2)^{-2}(dx^2 + dy^2)$ the circles in $\mathbb{D}$ tangential to the boundary $B = \{z \in \mathbb{C} : |z| = 1\}$ are the horocycles. In this case, E_τ is the closure of the set inside the circle tangential to the boundary at the point one and passing through the point $\tanh\tau$. We note that the sets of type H_τ do not exhaust all horocycles in G/K. In disk picture they are all those horocycles tangential to the boundary only at the point one. If a function f is supported on E_τ then for its usual Fourier transform $\widetilde{f}(\lambda, b)$ (as defined in [10]) one cannot hope to gain analytic extension in the variable λ if b is not the normal to the horocycle. On the other hand, the proof of Theorem 2.1 suggests that to have an analogue of Theorem 1.1 it is desirable that the Fourier transform is written using 'Cartesian coordinates' instead of $\widetilde{f}(\lambda, b)$ which corresponds to 'polar coordinates'. This is the point of view we are going to adopt in proving an analogue of Theorem 1.1. This technique was already employed in [2, 3, 9] to prove results of similar nature.

The purpose of this paper is to prove an analogue of Theorem 1.1 in the context of Damek–Ricci spaces. Damek–Ricci spaces are solvable Lie groups endowed with a Riemannian structure that makes them harmonic manifolds. These solvable groups were introduced in [7] and are semidirect products of the multiplicative group $(0, \infty)$ with two-step nilpotent Lie groups N of Heisenberg type. Here the action of $(0, \infty)$ on N is by anisotropic dilations. Damek and Ricci [8] proved that NA endowed with a suitable left-invariant Riemannian metric is a harmonic manifold. It is well known that Damek–Ricci spaces include all Riemannian symmetric spaces G/K of noncompact type with rank one via the Iwasawa decomposition $G = NAK$ (see [6]). However, despite being the most distinguishable prototypes, the rank-one Riemannian symmetric spaces, which sit inside the class of Damek–Ricci spaces as Iwasawa NA groups, form a rather small subclass (see [1]).

The following is the main result of this paper which can be viewed as an analogue of the theorem of Paley and Wiener (Theorem 1.1). We refer the reader to Section 3 for the symbols used in the theorem below.

THEOREM 1.3. *Let $\psi : [0, \infty) \to [0, \infty)$ be a locally integrable function and we set*

$$I = \int_0^\infty \frac{\psi(r)}{1 + r^2}\, dr.$$

(a) *Suppose $f \in \mathcal{S}(NA)$ with* supp $f \subseteq E_\tau$, *for some real number τ and satisfies the estimate*

$$\int_{\mathbb{R}} \frac{|\widetilde{f}(\lambda, n)|^q e^{q\psi(|\lambda|)}}{(1 + |\lambda|)^l} |\mathbf{c}(\lambda)|^{-2}\, d\lambda < \infty, \quad \text{for each } n \in N, \tag{1.1}$$

for some $q \in [1, \infty)$ and some $l \geq 0$, or

$$|\widetilde{f}(\lambda, n)| \leq C_n\, (1 + |\lambda|)^l e^{-\psi(|\lambda|)}, \quad \lambda \in \mathbb{R}, n \in N. \tag{1.2}$$

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If I is infinite then f vanishes identically on NA.

(b) *If I is finite and ψ is nondecreasing then there exists a nontrivial $f \in C_c^\infty(NA)^\#$ satisfying the estimate*

$$|\widetilde{f}(\lambda, n)| \le C_n\, e^{-\psi(|\lambda|)}, \quad \lambda \in \mathbb{R}, n \in N$$

or, for some $q \in [1, \infty)$ and for all $n \in N$

$$\int_{\mathbb{R}} |\widetilde{f}(\lambda, n)|^q\, e^{q\psi(|\lambda|)} |\mathbf{c}(\lambda)|^{-2}\, d\lambda < \infty.$$

This paper is organized as follows: In Section 2 we prove an analogue of Theorem 1.1 on $\mathbb{R}^d$, $d > 1$ and prove necessary complex analytic results. In Section 3 we recall the required preliminaries regarding harmonic NA groups and we prove the main result of this paper (Theorem 1.3).

We will use the following notation and conventions in the paper: $\Im z$ denotes the imaginary part of z, $C_c(X)$ denotes the set of compactly supported continuous functions on X, $C_c^\infty(X)$ denotes the set of compactly supported smooth functions on X, $\mathrm{supp}(f)$ denotes the support of the function f and C denotes a constant whose value may vary. For $x, y \in \mathbb{R}^d$, we will use $\|x\|$ to denote the norm of the vector x and $x \cdot y$ to denote the Euclidean inner product of the vectors x and y. We will use $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ to denote the upper half-plane and $\overline{\mathbb{H}}$ for its closure.

2. Paley–Wiener theorem for $\mathbb{R}^d$

In this section, we will first prove an analogue of Theorem 1.1 for $\mathbb{R}^d$. Next, we will prove a complex analytic lemma which is crucial for the proof of an analogue of Theorem 1.1 for Damek–Ricci spaces. We start by briefly recalling some necessary facts regarding the Radon transform on $\mathbb{R}^d$. We refer the reader to [11] for details.

For $\omega \in S^{d-1}$, the unit sphere in $\mathbb{R}^d$ and $t \in \mathbb{R}$, let

$$H_{\omega,t} = \{x \in \mathbb{R}^d \mid x \cdot \omega = t\}.$$

Then $H_{w,t}$ is a hyperplane in $\mathbb{R}^d$ with normal ω and distance $|t|$ from the origin. It is clear from the above definition that $H_{\omega,t} = H_{-\omega,-t}$. For $f \in C_c(\mathbb{R}^n)$, the Radon transform Rf of the function f is defined by

$$Rf(\omega, t) = \int_{H_{\omega,t}} f(x)\, dm(x),$$

where $dm(x)$ is the $d-1$ dimensional Lebesgue measure of $H_{\omega,t}$. The one-dimensional Fourier transform of Rf and the Fourier transform of f are closely connected by the slice projection theorem:

$$\widehat{f}(\lambda\omega) = \mathcal{F}(Rf(\omega, \cdot))(\lambda), \tag{2.1}$$

where $\mathcal{F}(Rf(\omega, \cdot))$ denotes the one-dimensional Fourier transform of the function $t \mapsto Rf(\omega, t)$. Let $C_c^\infty(\mathbb{R}^d)_0$ denote the set of the compactly supported, smooth, radial

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functions on $\mathbb{R}^d$ and $C_c^\infty(\mathbb{R})_e$ denote the set of compactly supported, smooth, even functions on $\mathbb{R}$. By Theorem 2.10 of [11] it is known that

$$R : C_c^\infty(\mathbb{R}^d)_0 \longrightarrow C_c^\infty(\mathbb{R})_e \tag{2.2}$$

is a bijection.

A weaker analogue of Theorem 1.1 has been proved in [4, Theorem 2.3] assuming that the function is compactly supported. We now state and prove an exact analogue of Theorem 1.1 for $\mathbb{R}^d$.

THEOREM 2.1. *Let $\psi : [0,\infty) \to [0,\infty)$ be a locally integrable function and*

$$I = \int_1^\infty \frac{\psi(t)}{t^2}\,dt.$$

(a) *Let $f \in L^p(\mathbb{R}^d)$, $p \in [1,2]$, be such that*

$$\operatorname{supp} f \subseteq \{x \in \mathbb{R}^d \mid x \cdot \eta \leq s\},$$

for some $\eta \in S^{d-1}$ and $s \in \mathbb{R}$ and $\widehat{f}$ satisfies the estimate

$$\int_{\mathbb{R}^d} \frac{|\widehat{f}(\xi)|^q\, e^{q\psi(|\xi\cdot\eta|)}}{(1+\|\xi\|)^N}\,d\xi < \infty \tag{2.3}$$

for some $q \in [1,\infty)$ and some $N \geq 0$, or

$$|\widehat{f}(\xi)| \leq C\,(1+\|\xi\|)^N\, e^{-\psi(|\xi\cdot\eta|)}, \tag{2.4}$$

for almost every $\xi \in \mathbb{R}^d$. If the integral I is infinite then f is the zero function.

(b) *If ψ is nondecreasing and I is finite then there exists a nontrivial $f \in C_c(\mathbb{R}^d)$ satisfying the estimate (2.3), for some $q \in [1,\infty)$ and all $\eta \in S^{d-1}$ or (2.4), for $q = \infty$ and all $\eta \in S^{d-1}$.*

PROOF. We shall first prove (a) for the case $p = 2$. We now show that it suffices to prove the case $q = 1, N = 0$. Suppose $f \in L^2(\mathbb{R}^d)$ with

$$\operatorname{supp} f \subseteq \{x \in \mathbb{R}^d \mid x \cdot \eta \leq s\},$$

for some $\eta \in S^{d-1}$ and $s \in \mathbb{R}$ and $\widehat{f}$ satisfies the estimate (2.3) for some $q > 1$ and $N \in \mathbb{N}$. If we choose $\phi \in C_c^\infty(\mathbb{R}^d)$ supported in $B(0,l)$ then it is easy to show that

$$\operatorname{supp}(f * \phi) \subseteq \{x \in \mathbb{R}^d \mid x \cdot \eta < s + l\}.$$

Using Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\widehat{(f*\phi)}(\xi)|\, e^{\psi(|\xi\cdot\eta|)}\,d\xi \\
&\quad\leq \bigg(\int_{\mathbb{R}^d} \frac{|\widehat{f}(\xi)|^q e^{q\psi(|\xi\cdot\eta|)}}{(1+\|\xi\|)^N}\bigg)^{1/q} \|(1+\|\xi\|)^N \widehat{\phi}(\xi)\|_{L^{q'}(\mathbb{R}^d)} \\
&\quad< \infty,
\end{aligned}$$

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as N/q is smaller than N. Here q' satisfies the relation

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Hence, by the case $q = 1, N = 0$ it follows that $f * \phi$ vanishes identically. As $\phi \in C_c^\infty(\mathbb{R}^d)$ we have that $\widehat{\phi}$ is nonzero almost everywhere. This implies that $\widehat{f}$ vanishes almost everywhere and so does f. The same technique can be applied to reduce the case $q = 1$ and $N \in \mathbb{N}$ to the case $q = 1$ and $N = 0$ by using Hölder's inequality. For the case $q = \infty$, we get from (2.4) that

$$\int_{\mathbb{R}^d} |\widehat{f * \phi}(\xi)|\, e^{\psi(|\xi\cdot\eta|)}\, d\xi \le \int_{\mathbb{R}^d} (1 + \|\xi\|)^N\, |\widehat{\phi}(\xi)|\, d\xi < \infty.$$

Hence we suppose that $q = 1$ and $N = 0$. Now, rotating the function f, we can assume without loss of generality that $\eta = e_1 = (1, 0, \ldots, 0)$. Then, by writing $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$ the hypothesis (2.3) becomes

$$\int_{\mathbb{R}^d} |\widehat{f}(\xi)|\, e^{\psi(|\xi_1|)}\, d\xi < \infty. \tag{2.5}$$

For $y \in \mathbb{R}^{d-1}$ we define

$$g_y(x) = \mathcal{F}_{d-1} f(x, y),$$

for almost every $x \in \mathbb{R}$. Here $\mathcal{F}_{d-1} f$ denotes the $(d-1)$-dimensional Fourier transform of the Function $f(x, \cdot)$. It then follows that for almost every $y \in \mathbb{R}^{d-1}$, $g_y \in L^2(\mathbb{R})$ with

$$\operatorname{supp} g_y \subseteq \{x \in \mathbb{R} \mid x \le s\},$$

and by (2.5)

$$\int_{\mathbb{R}} |\widehat{g_y}(t)|\, e^{\psi(|t|)}\, dt < \infty. \tag{2.6}$$

As y varies over a set of full $(d-1)$-dimensional Lebesgue measure, we just need to prove that g_y is the zero function. By Theorem 1.2 it suffices to show that

$$\int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)|)|}{1 + t^2}\, dt = \infty.$$

If

$$\int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)| e^{\psi(|t|)})|}{1 + t^2}\, dt < \infty \tag{2.7}$$

then

$$\begin{aligned} \int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)|)|}{1 + t^2}\, dt &= \int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)| e^{\psi(|t|)}) - \psi(|t|)|}{1 + t^2}\, dt \\ &\ge \int_{\mathbb{R}} \frac{\psi(|t|)}{1 + t^2}\, dt - \int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)| e^{\psi(|t|)})|}{1 + t^2}\, dt. \end{aligned}$$

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As I is infinite, it follows from (2.7) that

$$\int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)|)|}{1+t^2}\, dt$$

is divergent. Hence, by Theorem 1.2 it follows that g_y is the zero function. Now, suppose

$$\int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)|e^{\psi(|t|)})|}{1+t^2}\, dt = \infty. \tag{2.8}$$

For a measurable function F on $\mathbb{R}^d$, we define

$$\log^+ |F(x)| = \max\{\log |F(x)|, 0\}$$
$$\log^- |F(x)| = -\min\{\log |F(x)|, 0\},$$

and hence

$$\big|\log |F(x)|\big| = \log^+ |F(x)| + \log^- |F(x)|.$$

As $\log^+ |F(x)|$ is always smaller than $|F(x)|$ we get from (2.6) that

$$\int_{\mathbb{R}} \frac{\log^+(|\widehat{g_y}(t)|e^{\psi(|t|)})}{1+t^2}\, dt \le \int_{\mathbb{R}} \frac{|\widehat{g_y}(t)|e^{\psi(|t|)}}{1+t^2}\, dt < \infty.$$

From (2.8) we now conclude that

$$\int_{\mathbb{R}} \frac{\log^-(|\widehat{g_y}(t)|e^{\psi(|t|)})}{1+t^2}\, dt = \infty.$$

But

$$\int_{\mathbb{R}} \frac{\log^-(|\widehat{g_y}(t)|e^{\psi(|t|)})}{1+t^2}\, dt = \int_{\{t\in\mathbb{R} \,\mid\, |\widehat{g_y}(t)|e^{\psi(|t|)}\le 1\}} \frac{\log^-(|\widehat{g_y}(t)|e^{\psi(|t|)})}{1+t^2}\, dt$$
$$\le \int_{\mathbb{R}} \frac{\log^- |\widehat{g_y}(t)|}{1+t^2}\, dt,$$

as on the set $\{t \in \mathbb{R} \,\big|\, |\widehat{g_y}(t)|e^{\psi(|t|)} \le 1\}$ we have

$$|\widehat{g_y}(t)| \le |\widehat{g_y}(t)|e^{\psi(|t|)} \le 1,$$

and hence

$$\log^-(|\widehat{g_y}(t)|) \ge \log^-(|\widehat{g_y}(t)|e^{\psi(|t|)}).$$

Therefore the integral

$$\int_{\mathbb{R}} \frac{\log^-(|\widehat{g_y}(t)|)}{1+t^2}\, dt$$

is divergent. Hence, the integral

$$\int_{\mathbb{R}} \frac{|\log(|\widehat{g_y}(t)|)|}{1+t^2}\, dt$$

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is divergent. By Theorem 1.2 it now follows that g_y is the zero function. This completes the proof of part (a) for $p = 2$. Now, if $f \in L^p(\mathbb{R}^d)$, for $p \in [1, 2)$ then for any $\phi \in C_c(\mathbb{R}^d)$ we have $f * \phi \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and hence in $L^2(\mathbb{R}^d)$. Moreover, if we choose ϕ such that supp $\phi \subseteq B(0, r)$, for some r positive, then as before

$$\text{supp } f * \phi \subseteq \{x \in \mathbb{R}^d \mid x \cdot \eta \leq s + r\}.$$

Since $\widehat{f * \phi}$ satisfies the hypothesis (2.3) (or, (2.4)) it follows from the case $p = 2$ proved above that $f * \phi$ is zero. This implies that f is the zero function.

We shall now prove part (b). Since ψ is nondecreasing and I is finite it follows from the converse part of Levinson's theorem [15, Lemma 4] that there exists a nontrivial $g_1 \in C_c(\mathbb{R})$ such that

$$|\widehat{g_1}(\xi)| \leq Ce^{-\psi(|\xi|)}, \quad \text{for all } \xi \in \mathbb{R}.$$

Let $l > 0$ be such that supp $g_1 \subseteq [-l/4, l/4]$. By convolving g_1 with a $\phi \in C_c^\infty(\mathbb{R})$, supp $\phi \subseteq [-l/4, l/4]$ we get $g \in C_c^\infty(\mathbb{R})$ such that supp $g \subseteq [-l/2, l/2]$ and

$$|\widehat{g}(\xi)| \leq Ce^{-\psi(|\xi|)}, \quad \text{for all } \xi \in \mathbb{R}. \tag{2.9}$$

If g turns out to be an even function then the function $R^{-1}(g) = f_0$ (well-defined by 2.2) is a nontrivial function in $C_c^\infty(\mathbb{R}^d)$. By the slice projection theorem (2.1), it satisfies the estimate

$$|\widehat{f_0}(\xi)| \leq Ce^{-\psi(\|\xi\|)} \leq Ce^{-\psi(|\xi \cdot \eta|)}, \quad \text{for all } \xi \in \mathbb{R}^d, \tag{2.10}$$

since ψ is nondecreasing. If g is not even then we consider the translate $\tilde{g}(x) = g(x + l/2)$. Then $\tilde{g} \in C_c^\infty(\mathbb{R})$ with supp $\tilde{g} \subseteq [-l, 0]$ and hence $\tilde{g}$ cannot be an odd function. It follows that $\tilde{g}$ has a nontrivial even part given by

$$\tilde{g}_e(x) = \frac{\tilde{g}(x) + \tilde{g}(-x)}{2}, \quad x \in \mathbb{R},$$

and $\widehat{\tilde{g}_e}$ satisfies the estimate (2.9). We can now consider $f_0 = R^{-1}(\tilde{g}_e)$ and argue as before to show that $\widehat{f_0}$ satisfies the estimate (2.10). This, in particular, proves (b), for the case $q = \infty$.

For $q \in [1, \infty)$ we choose $\phi_1 \in C_c^\infty(\mathbb{R}^d)$ with supp $\phi_1 \subseteq B(0, l/2)$ and consider the function $f = f_0 * \phi_1$. Clearly, support of the function f is contained in $\overline{B(0, l)}$ and by (2.10) it follows that

$$\int_{\mathbb{R}^d} \frac{|\widehat{f}(\xi)|^q \, e^{q\psi(|\xi \cdot \eta|)}}{(1 + \|\xi\|)^N} \, d\xi \leq C \int_{\mathbb{R}^d} |\widehat{\phi_1}(\xi)|^q \, d\xi < \infty.$$

This, in particular, proves (b). □

Now we prove the following complex analytic lemma which will be used in the proof of the main theorem.

Lemma 2.2. *Suppose f is a holomorphic function on $\mathbb{H}$ which extends continuously to $\overline{\mathbb{H}}$. Let ψ be a nonnegative even function on $\mathbb{R}$ such that for positive constants τ and C*

$$|f(z)| \leq Ce^{\tau|\Im z|}, \quad \textit{for all } z \in \overline{\mathbb{H}}, \tag{2.11}$$

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and

$$\int_{\mathbb{R}} \frac{|f(x)|\, e^{\psi(x)}}{1+x^2}\, dx < \infty. \tag{2.12}$$

If

$$\int_{\mathbb{R}} \frac{\psi(x)}{1+x^2}\, dx = \infty,$$

then f vanishes identically on $\overline{\mathbb{H}}$.

We shall now state a lemma regarding harmonic majoration of subharmonic functions which is the key point for the proof of the lemma above. A proof of the following lemma can be found in [13, Ch. III, §G2, page 50] and [4].

Lemma 2.3. Suppose v is a subharmonic function on $\mathbb{H}$ which is bounded above. If $\lim_{z\to t} v(z) = v(t)$ exists for almost every $t \in \mathbb{R}$ then

$$v(z) \leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{y v(t)}{y^2 + (x-t)^2}\, dt,$$

for all $z = x + iy \in \mathbb{H}$.

Proof of Lemma 2.2. We consider the holomorphic function

$$g(z) = \frac{1}{C} e^{i\tau z} f(z),$$

on $\mathbb{H}$. Clearly, it extends continuously to $\overline{\mathbb{H}}$. We want to apply the Phragmén-Lindelöf theorem ([19], Theorem 3.4, page 124) to show that, for all $z \in \overline{\mathbb{H}}$,

$$|g(z)| \leq 1. \tag{2.13}$$

Let

$$Q_1 = \{z = x + iy \in \mathbb{C} : x > 0, y > 0\}.$$

It follows from the estimate (2.11) that

$$|g(iy)| = \frac{1}{C} e^{-\tau y} |f(iy)| \leq e^{-\tau y} e^{\tau y} = 1, \quad \text{for all } y > 0.$$

It is immediate from (2.11) that for all $x \in \mathbb{R}$

$$|g(x)| = \frac{1}{C} |f(x)| \leq 1.$$

Therefore, g is bounded by 1 on the positive real and positive imaginary axes. As g satisfies the estimate (2.11) we can apply the Phragmén-Lindelöf theorem to the sector Q_1 to obtain (2.13). A similar argument for the quadrant

$$Q_2 = \{z = x + iy \in \mathbb{C} : x < 0, y > 0\},$$

proves the estimate (2.13) for all $z \in \overline{\mathbb{H}}$. Since g is a holomorphic function $\log |g|$ is subharmonic on $\mathbb{H}$ and

$$\log |g(z)| \leq 0, \quad \text{for all } z \in \overline{\mathbb{H}}.$$

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Moreover,

$$\lim_{z\to x} \log|g(z)| = \log|g(x)|$$

exists for almost every $x \in \mathbb{R}$. Now we use the Lemma 2.3 for the subharmonic function

$$v(z) = \log|g(z)|,$$

to get

$$\log|g(z)| \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\log|g(t)|}{y^2+(x-t)^2}\,dt, \quad \text{for all } z = x+iy \in \mathbb{H}. \tag{2.14}$$

It is easy to see that for fixed $x \in \mathbb{R}$ and y positive there exist positive constants $C_{x,y}$ and $c_{x,y}$ (depending on x and y) such that for all $t \in \mathbb{R}$,

$$\frac{c_{x,y}}{1+t^2} \le \frac{y}{(x-t)^2+y^2} \le \frac{C_{x,y}}{1+t^2}.$$

Using the above inequalities it then follows from the estimate (2.14) and the hypothesis (2.12) that

$$\begin{aligned}
\log|g(x+iy)| &\le \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\log|f(t)|}{y^2+(x-t)^2}\,dt \\
&\le \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\log(|f(t)|e^{\psi(t)})}{y^2+(x-t)^2}\,dt - \frac{1}{\pi}\int_{\mathbb{R}} \frac{y\,\psi(t)}{y^2+(x-t)^2}\,dt \\
&\le \frac{C_{x,y}}{\pi}\int_{\mathbb{R}} \frac{\log^+(|f(t)|e^{\psi(t)})}{1+t^2}\,dt - \frac{c_{x,y}}{\pi}\int_{\mathbb{R}} \frac{\log^-(|f(t)|e^{\psi})}{1+t^2}\,dt - \frac{c_{x,y}}{\pi}\int_{\mathbb{R}} \frac{\psi(t)}{1+t^2}\,dt \\
&= -\infty.
\end{aligned}$$

Hence, for each $x \in \mathbb{R}$ and y positive, it follows that $g(x+iy)$ is zero. Since f is a holomorphic, it follows that f vanishes identically on $\overline{\mathbb{H}}$.

3. Preliminaries on Damek–Ricci spaces

In this section, we will explain the notation and gather relevant results on Damek–Ricci spaces. Most of these results can be found in [1, 6]. Relevant results for the Abel, spherical and Fourier transforms on these spaces can be found in [1–3].

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra equipped with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the centre of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement. We say that $\mathfrak{n}$ is an H-type algebra if for every $Z \in \mathfrak{z}$ the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_z X, Y\rangle = \langle [X,Y], Z\rangle, \quad X, Y \in \mathfrak{v}$$

satisfies the condition $J_Z^2 = -|Z|^2 I_{\mathfrak{v}}$, $I_{\mathfrak{v}}$ being the identity operator on $\mathfrak{v}$. A connected and simply connected Lie group N is called an H-type group if its Lie algebra is H-type. Since $\mathfrak{n}$ is nilpotent, the exponential map is a diffeomorphism and hence we can parametrize the elements in $N = \exp\mathfrak{n}$ by (X, Z), for $X \in \mathfrak{v}, Z \in \mathfrak{z}$. It follows from the Campbell–Baker–Hausdorff formula that the group law in N is given by

$$(X,Z)(X',Z') = (X+X', Z+Z'+\tfrac{1}{2}[X,X']), \quad X, X' \in \mathfrak{v};\ Z, Z' \in \mathfrak{z}.$$

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The group $A = R^{+}$ acts on an H-type group N by nonisotropic dilation: $(X, Z) \mapsto (\sqrt{a}X, aZ)$. Let $S = NA$ be the semidirect product of N and A under the above action. Thus the multiplication in S is given by

$$(X, Z, a)(X', Z', a') = \Big(X + \sqrt{a}X', Z + aZ' + \frac{\sqrt{a}}{2}[X, X'], aa'\Big),$$

for $X, X' \in \mathfrak{v}$; $Z, Z' \in \mathfrak{z}$; $a, a' \in \mathbb{R}^{+}$. Then S is a solvable, connected and simply connected Lie group having Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ with Lie bracket

$$[(X, Z, l), (X', Z', l'\rangle)] = (\tfrac{1}{2}lX' - \tfrac{1}{2}l'X, lZ' - lZ + [X, X'], 0).$$

We write $na = (X, Z, a)$ for the element $\exp(X + Z)a, X \in \mathfrak{v}, Z \in \mathfrak{z}, a \in A$. We note that for any $Z \in \mathfrak{z}$ with $|Z| = 1$, $J_Z^2 = -I_{\mathfrak{v}}$; that is, J_Z defines a complex structure on $\mathfrak{v}$ and hence $\mathfrak{v}$ is even-dimensional. We suppose $\dim \mathfrak{v} = m$ and $\dim \mathfrak{z} = k$. Then $Q = (m/2) + k$ is called the homogenous dimension of S. For convenience we will use the symbol ρ for $Q/2$ and d for $m + k + 1 = \dim \mathfrak{s}$. The group S is equipped with the left-invariant Riemannian metric induced by

$$\langle (X, Z, l), (X', Z', l')\rangle = \langle X, X'\rangle + \langle Z, Z'\rangle + ll'$$

on $\mathfrak{s}$. The associated left-invariant Haar measure dx on S is given by

$$dx = a^{-(Q+1)}\, dX\, dZ\, da = a^{-(Q+1)}\, dn\, da,$$

where dX, dZ, da are the Lebesgue measures on $\mathfrak{v}, \mathfrak{z}$ and R^{+} respectively. The geodesic distance of $x = (X, Z, a)$ from the identity e of S is

$$\sigma(x) = d(x, e) = \log \frac{1 + r(x)}{1 - r(x)},$$

where $r(x)$ lies in the interval $(0, 1)$ and is given by

$$1 - r(x)^2 = \frac{4a}{\Big(1 + a + \frac{|X|^2}{4}\Big)^2 + |Z|^2}.$$

The Fourier transform on S requires the notion of the Poisson kernel $\mathcal{P}(x, n)$. The Poisson kernel $\mathcal{P} : S \times N \to \mathbb{R}$ is given by

$$\mathcal{P}(na_t, n_1) = \mathcal{P}_{a_t}(n_1^{-1}n),$$

where

$$\mathcal{P}_{a_t}(n) = \mathcal{P}_{a_t}(V, Z) = Ca_t^{Q}\Big(\Big(a_t + \frac{|V|^2}{4}\Big)^2 + |Z|^2\Big)^{-Q}$$

and $a_t = e^t, t \in \mathbb{R}$; $n = (V, Z) \in N$. For the precise value of C we refer to [2], (2.6). For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel is defined by

$$\mathcal{P}_\lambda(x, n) = \mathcal{P}(x, n)^{\frac{1}{2} - (i\lambda/Q)}.$$

It is known that for every fixed $n_1 \in N$, the function $\mathcal{P}_\lambda(x, n_1)$ is an eigenfunction of the Laplace–Beltrami operator $\mathcal{L}$ with eigenvalue $(\lambda^2 + Q^2/4)$ (see [2]), and $\mathcal{P}_\lambda(x, n_1)$

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is constant on the hypersurface

$$H_{n_1,a_t} = \{n_1\mathfrak{r}(a_t n) \mid n \in N\},$$

where $\mathfrak{r}$ stands for the geodesic inversion (see [18]). In view of this it is natural to define the Fourier transform of a function $f \in C_c^\infty(S)$ by [2, page 406]

$$\widetilde{f}(\lambda, n) = \int_S f(x)\mathcal{P}_\lambda(x, n)\,dx, \quad \lambda \in \mathbb{C},\ n \in N.$$

It is known that for $f \in C_c^\infty(S)$ the following Fourier inversion and the Plancherel formula holds [2]:

(a) For $f \in C_c^\infty(S)$

$$f(x) = C_1 \int_{\mathbb{R}\times N} \widetilde{f}(\lambda, n)\mathcal{P}_{-\lambda}(x, n)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,dn, \quad \text{for all } x \in S, \tag{3.1}$$

where

$$\mathbf{c}(\lambda) = \frac{2^{Q-2i\lambda}\Gamma(2i\lambda)\Gamma(\frac{2m+k+1}{2})}{\Gamma(\frac{Q}{2} + i\lambda)\Gamma(\frac{m+1}{2} + i\lambda)}.$$

(b) The Fourier transform extends from $C_c^\infty(S)$ to an isometry from $L^2(S)$ onto the space $L^2(R^+ \times N, C_2\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,dn)$.

The precise value of the constants C_1, C_2 are given in [2].

Remark. If $f \in L^1(S)$, then for almost every fixed $n \in N$, the map $\lambda \mapsto \widetilde{f}(\lambda, n)$ is continuous on the strip

$$S_1 = \{\lambda \in \mathbb{C} \mid |\Im\lambda| \le \rho\},$$

and analytic in the interior of S_1 [18, Theorem 5.4].

We also have the following estimate of the function $|\mathbf{c}(\lambda)|^{-2}$ (see [18, Lemma 4.8]).

Lemma 3.1. $|\mathbf{c}(\lambda)|^{-2} \asymp \lambda^2(1 + |\lambda|)^{m+k-2}$, *for all* $\lambda \in \mathbb{R}$, *that is, there exist two positive constants* C_1, C_2 *such that*

$$C_1\lambda^2(1 + |\lambda|)^{m+k-2} \le |\mathbf{c}(\lambda)|^{-2} \le C_2\lambda^2(1 + |\lambda|)^{m+k-2}, \quad \text{for } \lambda \in \mathbb{R}. \tag{3.2}$$

Let $\mathcal{U}$ be the universal enveloping algebra of S and $U, V \in \mathcal{U}$. We will denote by $U^L f$ and by $V^R f$ the corresponding left-invariant and right-invariant vector fields applied to a C^∞ function f on S. The Schwartz space $\mathcal{S}(S)$ is then defined as the space of smooth functions f on S such that

$$\sup_{x\in NA} e^{(Q\sigma(x)/2)}(1 + \sigma(x))^l|(U^L V^R f)(x)| < \infty,$$

for every positive integer l and for every $U, V \in \mathcal{U}$. It can be verified that if f is a function in $\mathcal{S}(S)$ then its Fourier transform $\widetilde{f}(\lambda, n)$ is given by an absolutely convergent integral for $(\lambda, n) \in \mathbb{R} \times N$. Moreover, the inversion formula (3.1) holds for f in $\mathcal{S}(S)$.

We will now specialize to the case of radial functions on S. A function $f : S \to \mathbb{C}$ is said to be radial if, for all $x \in S$, $f(x)$ depends only on the geodesic distance $\sigma(x)$ of x from the identity e of S. A spherical function ϕ on S is a radial eigenfunction of

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the Laplace–Beltrami operator $\mathcal{L}$ normalised so that $\phi(e) = 1$. For $\lambda \in \mathbb{C}$, we denote by ϕ_λ the spherical function with eigenvalue $-(\lambda^2 + Q^2/4)$. Let $f \in C_c^\infty(S)^\#$ denote the subspace of radial functions in $f \in C_c^\infty(S)$. The spherical Fourier transform of a function $f \in C_c^\infty(S)^\#$ is given by

$$\mathcal{F} f(\lambda) = \int_{NA} f(x)\, \phi_\lambda(x)\, dx, \quad \lambda \in \mathbb{C}.$$

Since $\phi_\lambda(x) = \phi_{-\lambda}(x)$, it follows that $\mathcal{F} f$ is an even function on $\mathbb{R}$. If f is radial then, unlike the case of Riemannian symmetric spaces, the Helgason Fourier transform does not boil down to its spherical Fourier transform; indeed, they are related as [18, Equation (2.9)]

$$\widetilde{f}(\lambda, n) = \mathcal{P}_\lambda(e, n) \mathcal{F} f(\lambda). \tag{3.3}$$

We also have the following convolution relation.

Lemma 3.2 [2, Proposition 3.2]. *If $f \in C_c^\infty(S)$ and $\phi \in C_c^\infty(S)^\#$, then*

$$(\widetilde{f * \phi})(\lambda, n) = \widetilde{f}(\lambda, n) \mathcal{F} \phi(\lambda), \textit{ for } \lambda \in \mathbb{R}, n \in N.$$

Remark. It can be easily seen that the same result continues to be true for $f \in \mathcal{S}(S)$ [2].

We also need the notion of Abel transform on S. For a suitable radial function f on S, the Abel transform is defined by

$$\mathcal{A} f(t) = e^{-(Qt/2)} \int_N f(na_t)\, dn, \quad \text{where } a_t = e^t.$$

It is not hard to see that $\mathcal{A} f$ is an even function in t [1].

Lemma 3.3 [1]. *Abel transform satisfies the following properties:*

(a) *If $f \in C_c^\infty(S)$ is radial then*

$$(\widehat{\mathcal{A} f})(\lambda) = \mathcal{F} f(\lambda), \quad \lambda \in \mathbb{R}.$$

(b) *The map $\mathcal{A} : C_c^\infty(S)^\# \to C_c^\infty(\mathbb{R})_e$ is a bijection, where*

$$C_c^\infty(\mathbb{R})_e = \{f \in C_c^\infty(\mathbb{R}) : f \textit{ is even}\}.$$

In this paper our main concern are the functions with support contained in sets of the type

$$E_\tau = \{na \in S \mid a \geq e^\tau\}, \quad \tau \in \mathbb{R}.$$

We note that the boundary of E_τ is a horocycle in S. To prove the main result of this paper we use the following result proved in [3].

Lemma 3.4. *Let f be in $\mathcal{S}(S)$ and τ be a real number. The support of f is contained in the set*

$$E_\tau = \{na \in S : a \geq e^\tau\}$$

if and only if the following conditions hold:

(i) *For each fixed $n \in N$, the function $\lambda \mapsto \widetilde{f}(\lambda, n)$ is holomorphic in $\{\lambda \in \mathbb{C} : \Im \lambda > 0\}$;*

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(ii) *The map* $(\lambda, n) \mapsto \widetilde{f}(\lambda, n)$ *is* C^∞ *on* $(\{\lambda \in \mathbb{C} : \Im\lambda \geq 0\} \times N)$*;*

(iii) *for every positive integer* l *and for* $1 < p \leq \infty$*,*

$$\sup_{\Im\lambda \geq 0} \|\widetilde{f}(\lambda, \cdot)\|_{L^p(N)}(1 + |\lambda|)^l e^{\tau\Im\lambda} < \infty.$$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. For part (a), we first show that it suffices to prove the case $q = 1, l = 0$. Suppose $f \in \mathcal{S}(S)$ is supported on the set E_τ, for some real number τ and satisfies (1.1), for some $q > 1$ and $l \in \mathbb{N}$. We choose $\phi \in C_c^\infty(S)^\#$ supported on a compact subset of the set $\{na \in NA \mid a > e^\eta\}$, for some positive real number η. Then $f * \phi$ is supported on the set $E_{(\tau+\eta)}$. In fact, since f is supported in E_τ

$$f * \phi(n'a') = \int_{E_\tau} f(na)\, \phi((na)^{-1}n'a')a^{-(Q+1)}\, dn\, da.$$

Since A normalizes N, the A-component of $(na)^{-1}n'a'$ is $a^{-1}a'$. If $a' < e^{\tau+\eta}$, then for all $a \in E_\tau$

$$a^{-1}a' < e^\eta,$$

and hence $f * \phi(n'a')$ is zero. By Lemma 3.2, it follows that

$$(\widetilde{f * \phi})(\lambda, n) = \widetilde{f}(\lambda, n)\mathcal{F}\phi(\lambda), \quad \text{for } \lambda \in \mathbb{R}, n \in N.$$

Using Hölder's inequality

$$\begin{aligned}
&\int_{\mathbb{R}} |(\widetilde{f * \phi})(\lambda, n)|e^{\psi(|\lambda|)}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&\leq \bigg(\int_{\mathbb{R}} \frac{|\widetilde{f}(\lambda, n)|^q e^{q\psi(|\lambda|)}}{(1 + |\lambda|)^l}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\bigg)^{1/q} \|(1 + |\cdot|)^l \mathcal{F}\phi(\cdot)\|_{L^{q'}(\mathbb{R},\, |\mathbf{c}(\lambda)|^{-2}d\lambda)} \\
&< \infty,
\end{aligned}$$

as l/q is smaller than l. Here q' satisfies the relation

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Hence, by the case $q = 1, N = 0$ it follows that $f * \phi$ vanishes identically. As $\phi \in C_c^\infty(S)$, it follows from Remark 3 that $\mathcal{F}\phi$ is nonzero almost everywhere. This implies that $\widetilde{f}$ vanishes almost everywhere and so does f. The same technique can be applied to reduce the case $q = 1$ and $l \in \mathbb{N}$ to the case $q = 1$ and $l = 0$ by using Hölder's inequality. For the case $q = \infty$, we get from (1.2) that

$$\int_{\mathbb{R}} |\widetilde{f * \phi}(\lambda, n)|\, e^{\psi(|\lambda|)}|\mathbf{c}(\lambda)|^{-2}\, d\lambda \leq \int_{\mathbb{R}} (1 + |\lambda|)^l\, |\widehat{\phi}(\lambda)||\mathbf{c}(\lambda)|^{-2}\, d\lambda < \infty.$$

So, without loss of generality, we assume that $f \in \mathcal{S}(S)$ is such that $\widetilde{f}$ satisfies the condition

$$\int_{\mathbb{R}} |\widetilde{f}(\lambda, n)|e^{\psi(|\lambda|)}|\mathbf{c}(\lambda)|^{-2}\, d\lambda < \infty. \tag{3.4}$$

Since $f \in \mathcal{S}(S)$ and its support is contained in E_τ, for some real number τ, it follows from Lemma 3.4 that for each $n \in N$, the function F_n defined by

$$F_n(\lambda) = \widetilde{f}(\lambda, n),$$

is holomorphic on the upper half-plane $\mathbb{H}$ which extends continuously to $\overline{\mathbb{H}}$. Moreover, by Lemma 3.4, iii) the function F_n satisfies the estimate

$$|F_n(\lambda)| \leq C_n \frac{e^{|\tau||\Im\lambda|}}{1+|\lambda|^l}, \quad \lambda \in \mathbb{H},$$

for some positive integer l. Now, using the estimate of $|\mathbf{c}(\lambda)|^{-2}$ given in (3.2) and the hypothesis (3.4), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{|F_n(\lambda)|\, e^{\psi(|\lambda|)}}{1+\lambda^2}\, d\lambda \\
&\quad \leq \frac{1}{C_1} \int_{|\lambda|\geq 1} \frac{|F_n(\lambda)| e^{\psi(|\lambda|)}}{(1+\lambda^2)} \frac{|\mathbf{c}(\lambda)|^{-2}}{\lambda^2(1+|\lambda|)^{m+k-2}}\, d\lambda + \int_{|\lambda|<1} \frac{|F_n(\lambda)| e^{\psi(|\lambda|)}}{1+\lambda^2}\, d\lambda \\
&\quad \leq \frac{1}{C_1} \int_{|\lambda|\geq 1} |\widetilde{f}(\lambda, n)|\, e^{\psi(|\lambda|)} |\mathbf{c}(\lambda)|^{-2}\, d\lambda + C \\
&\quad < \infty.
\end{aligned}$$

Therefore, by Lemma 2.2, it follows that F_n vanishes identically on $\mathbb{R}$. Since this is true for every $n \in N$, therefore $\widetilde{f}$ and hence f vanish identically on S. This completes the proof of part (a).

Now we shall prove (b). Since I is finite and ψ is nondecreasing, by Theorem 2.1, (b) there exists a nontrivial even function $f_0 \in C_c^\infty(\mathbb{R})$ satisfying the estimate

$$|\widehat{f_0}(\lambda)| \leq C e^{-\psi(|\lambda|)}, \quad \lambda \in \mathbb{R}. \tag{3.5}$$

By Lemma 3.3, there exists $f \in C_c^\infty(S)$ such that $\mathcal{A}f = f_0$ with $\mathcal{F}f(\lambda) = \widehat{f_0}(\lambda)$, for all $\lambda \in \mathbb{R}$. Therefore, by (3.5) it follows that

$$|\mathcal{F}f(\lambda)| \leq C e^{-\psi(|\lambda|)}, \quad \lambda \in \mathbb{R}.$$

Since $|\mathcal{P}_\lambda(e, n)|$ is independent of λ it follows from the relation (3.3) and the above equation that

$$|\widetilde{f}(\lambda, n)| \leq C_n\, e^{-\psi(|\lambda|)}, \quad \lambda \in \mathbb{R}, n \in N.$$

This in particular proves part (b) for the case $q = \infty$. For $q \in [1, \infty)$ we choose a radial function $\phi \in C_c(S)$ and consider the nontrivial function $f_1 = f * \phi$. It is easy to show that $\widetilde{f_1}$ satisfies the estimate (1.1). This completes the proof. □

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