THE ELLIPTIC INTEGRALS OF THE THIRD KIND

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This paper develops a case for adopting as the standard elliptic integrals of the third kind the function $\Pi\mathrm{s}(u, a)$ defined by

$$\Pi\mathrm{s}(u, a) = \int_0^u \frac{\mathrm{qs}\, a\, \mathrm{qs}'\, a\, du}{\mathrm{qs}^2\, u - \mathrm{qs}^2\, a}$$

and the three functions $\Pi\mathrm{s}\,(u, a + K_c)$, $\Pi\mathrm{s}(u, a + K_n)$, $\Pi\mathrm{s}(u, a + K_d)$ where K_c, K_n, K_d are the three quarter-periods of the Jacobian system. The function $\Pi\mathrm{s}(u, a)$ is the same function whether $\mathrm{qs}\, u$ is $\mathrm{cs}\, u$, $\mathrm{ns}\, u$, or $\mathrm{ds}\, u$.

The origin of the paper was a wish to understand how it has come about that the integrals commonly accepted as standard are not related symmetrically to the theta functions in terms of which they are expressed. The explanation of this irregularity is in three parts:

(1) The first of Jacobi's formulae for evaluating an elliptic integral is a deduction from the identity

$$\frac{\Theta^2 0\, \Theta(u + a)\, \Theta(u - a)}{\Theta^2 a\, \Theta^2 u} = 1 - c\, \mathrm{sn}^2\, a\, \mathrm{sn}^2\, u. \tag{0.1}$$

(2) To cover the range of real integrals with real variables it is necessary to use in addition to $\Theta(u + a)\, \Theta(u - a)$ the three products

$$\Theta_1(u + a)\, \Theta_1(u - a),\ \mathrm{H}(u + a)\, \mathrm{H}(u - a),\ \mathrm{H}_1(u + a)\, \mathrm{H}_1(u - a).$$

(3) If the only elliptic functions recognized are $\mathrm{sn}\, u$, $\mathrm{cn}\, u$, $\mathrm{dn}\, u$, the only denominator which can be associated with the products in (2) is $\Theta^2 a\, \Theta^2 u$.

The third part of this answer is the mischief-maker leading to a set of integrals with no community of structure.

1. The notation is the systematic notation used in my *Jacobian Elliptic Functions* **(8)**, including that for bipolar functions suggested in the preface (p. iv) to the second edition (1951). Except that he prefers ω_p to K_p, it is adopted by Lenz in his paper **(7)** written as a tribute to Faber. Glaisher's function $\mathrm{pq}\, u$ is the function with simple zeros congruent with K_p and simple poles congruent with K_q and with 1 for its leading coefficient at the origin.

The bipolar function $\mathrm{bpq}\, u$ has simple poles congruent with K_p and K_q and simple zeros congruent with the other two of the four points K_s, K_c, K_n, K_d; since these other points are the zeros of the derivative $\mathrm{pq}'\, u$, the bipolar function is a multiple of the logarithmic derivative $\mathrm{pq}'\, u/\mathrm{pq}\, u$ and we obtain

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a definite function by again requiring the leading coefficient at the origin to be 1. Then

$$\text{bps}\, u = -\text{ps}'\, u/\text{ps}\, u = \text{sp}'\, u/\text{sp}\, u \tag{1.1}$$

and if the origin is neither pole nor zero

$$\text{bpq}\, u = \text{sp}^2 K_q\, \text{pq}'\, u/\text{pq}\, u. \tag{1.2}$$

Explicitly, $\text{bpq}\, u = \text{rp}\, u\, \text{tq}\, u = \text{tp}\, u\, \text{rq}\, u$, but more often than not the arbitrary coupling of a zero with a pole is an irrelevant nuisance. Since $\text{ps}\, u\, \text{ps}(u + K_p)$ is independent of u, (1.1) implies

$$\text{bps}(u + K_p) = -\text{bps}\, u. \tag{1.3}$$

The theta functions I use also have 1 for leading coefficient at the origin. For $\text{H}u/\text{H}'0$, $\text{H}_1 u/\text{H}_1 0$, $\Theta u/\Theta 0$, $\Theta_1 u/\Theta_1 0$ I write $\vartheta_s u$, $\vartheta_c u$, $\vartheta_n u$, $\vartheta_d u$, relieving the memory by associating each of the functions with its lattice of zeros. The quotient $\vartheta_p u/\vartheta_q u$ is the elliptic function pq u.

The quarter-period relations between the theta functions are

$$\vartheta_c u = A\vartheta_s(u + K_c), \vartheta_n u = Be^{\lambda u}\vartheta_s(u + K_n) \tag{1.4, 1.5}$$

$$\vartheta_d u = C\vartheta_n(u + K_c) = De^{\lambda u}\vartheta_s(u + K_d) \tag{1.6}$$

where A, B, C, D, λ are constants whose values are not needed in this paper. From these relations it follows that the function zp u defined according to Lenz's notation **(7)** by

$$\text{zp}\, u = \vartheta_p'\, u/\vartheta_p u \tag{1.7}$$

satisfies the quarter-period relations

$$\text{zc}\, u = \text{zs}(u + K_c), \qquad \text{zd}\, u = \text{zn}(u + K_c), \tag{1.8}$$

$$\text{zn}\, u = \text{zs}(u + K_n) + \lambda, \qquad \text{zd}\, u = \text{zs}(u + K_d) + \lambda. \tag{1.9}$$

Since $\vartheta_n u$ is a multiple of Θu, the logarithmic derivative zn u is identical with the function Zu defined by Jacobi.

2. In terms of the function $\vartheta_n u$, Jacobi's identity (0.1) becomes

$$\frac{\vartheta_n(a + u)\vartheta_n(a - u)}{\vartheta_n^2 a\,\vartheta_n^2 u} = 1 - c\,\text{sn}^2 a\,\text{sn}^2 u \equiv \Delta_n \tag{2.1}$$

and if we alter the numerators in turn, but not the denominator, we have

$$\frac{\vartheta_d(a + u)\vartheta_d(a - u)}{\vartheta_n^2 a\,\vartheta_n^2 u} = c\,\text{cn}^2 a\,\text{cn}^2 u + c' \equiv \Delta_d, \tag{2.2}$$

$$\frac{\vartheta_s(a + u)\vartheta_s(a - u)}{\vartheta_n^2 a\,\vartheta_n^2 u} = \text{sn}^2 a - \text{sn}^2 u \equiv \Delta_s, \tag{2.3}$$

$$\frac{\vartheta_c(a + u)\vartheta_c(a - u)}{\vartheta_n^2 a\,\vartheta_n^2 u} = c^{-1}\,\text{dn}^2 a\,\text{dn}^2 u - c^{-1} c' \equiv \Delta_c. \tag{2.4}$$

It was all but inevitable that before the discovery by Glaisher in 1882 of the complete group of twelve Jacobian functions the integrands to be associated with Jacobi's integrand

(2.5) $$I_n \equiv -\tfrac{1}{2}\,\partial \log \Delta_n/\partial a = c\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2 u/\Delta_n$$

should be

(2.6) $$I_d \equiv -\tfrac{1}{2}\,\partial \log \Delta_d/\partial a = c\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{cn}^2 u/\Delta_d$$

(2.7) $$I_s \equiv -\tfrac{1}{2}\,\partial \log \Delta_s/\partial a = -\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a/\Delta_s$$

(2.8) $$I_c \equiv -\tfrac{1}{2}\,\partial \log \Delta_c/\partial a = \mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{dn}^2 u/\Delta_c$$

but a revision in the light of Glaisher's discovery is long overdue.

3. If

(3.1) $$\Lambda_p \equiv \Lambda_p(u, a) = \tfrac{1}{2}\log\frac{\vartheta_p(a-u)}{\vartheta_p(a+u)}$$

then

(3.2) $$\frac{\partial \log \Delta_p}{\partial a} = -2\frac{\partial \Lambda_p}{\partial u} - 2\,\mathrm{zn}\,a$$

and therefore

(3.3) $$\int_0^u I_p\,du = \Lambda_p(u, a) + u\,\mathrm{zn}\,a.$$

This is Jacobi's argument. The relation between the integrals is clear if we replace (3.3) by

(3.4) $$\int_0^u (I_p - \mathrm{zn}\,a)du \equiv \Lambda_p(u, a)$$

but zn a is not an elliptic function of a, and we can only regard the integrals in (3.3) as forming not one set of peculiar interest but one of the four sets of the more general form $\Lambda_p(u, a) + u\,\mathrm{zq}\,a$.

So much was evident a century ago, and Enneper (**2**, §34) recorded the integrands corresponding to the sixteen combinations. The calculation is simple. Since $\mathrm{zq}\,a - \mathrm{zn}\,a = \mathrm{qn}'a/\mathrm{qn}\,a$

(3.5) $$\Lambda_p(u, a) + u\,\mathrm{zq}\,a = \int_0^u \left(I_p + \frac{\mathrm{qn}'a}{\mathrm{qn}\,a}\right) du = -\tfrac{1}{2}\int_0^u \frac{\partial}{\partial a}\left(\log\frac{\Delta_p}{\mathrm{qn}^2 a}\right) du.$$

For given p, and q other than n, the denominator Δ_p can be put into the form $U_{pq}\,\mathrm{qn}^2 a + V_{pq}$, where U_{pq}, V_{pq} do not involve a, and then

(3.6) $$\frac{\partial}{\partial a}\left(\log\frac{\Delta_p}{qn^2 a}\right) = -\frac{2V_{pq}\,\mathrm{qn}'a}{\Delta_p\,\mathrm{qn}\,a}.$$

Hence, for q other than n,

(3.7) $$\Lambda_p(u, a) + u\,\mathrm{zq}\,a = \frac{\mathrm{qn}'a}{\mathrm{qn}\,a}\int_0^u \frac{V_{pq}\,du}{\Delta_p}$$

and the integrands which yield the sixteen integrands are given in terms of sn u, cn u, dn u compactly and explicitly in Table I.

TABLE I

DERIVATIVE OF $\frac{1}{2}\log\frac{\vartheta_p(a-u)}{\vartheta_p(a+u)}+u\frac{\vartheta_q'a}{\vartheta_q a}$ WITH RESPECT TO u

$p \backslash q$	s	c	n	d	
s	$-\mathrm{sn}^2 u$	$\mathrm{cn}^2 u$	-1	$c^{-1}\mathrm{dn}^2 u$	$\div\ \mathrm{sn}^2 a - \mathrm{sn}^2 u$
c	$\mathrm{cn}^2 u$	$-c'\mathrm{sn}^2 u$	$\mathrm{dn}^2 u$	$-c^{-1}c'$	$\div\ c^{-1}\mathrm{dn}^2 a\,\mathrm{dn}^2 u - c^{-1}c'$
n	1	$\mathrm{dn}^2 u$	$c\,\mathrm{sn}^2 u$	$\mathrm{cn}^2 u$	$\div\ 1 - c\,\mathrm{sn}^2 a\,\mathrm{sn}^2 u$
d	$\mathrm{dn}^2 u$	c'	$c\,\mathrm{cn}^2 u$	$c'\mathrm{sn}^2 u$	$\div\ c\,\mathrm{cn}^2 a\,\mathrm{cn}^2 u + c'$
	$\times\mathrm{sn}'a/\mathrm{sn}a$	$\times\mathrm{cn}'a/\mathrm{cn}a$	$\times\mathrm{sn}a\mathrm{cn}a\mathrm{dn}a$	$\times\mathrm{dn}'a/\mathrm{dn}a$	

As functions of u, the integrands in this table are multiples of the sixteen fractions each of which has one of the four numerators 1, $\mathrm{sn}^2 u$, $\mathrm{cn}^2 u$, $\mathrm{dn}^2 u$ and one of the four denominators Δ_s, Δ_c, Δ_n, Δ_d. In this sense the set is complete, but the structure, so clear from the integrals, is utterly obscure when only the integrands are displayed.

4. In using (3.7) we have completed our table of integrands from its third column, but since zq u − zr u = qr$'u$/qr u, we could as easily complete a row from any one of its members, and we now ask if a different choice of standard integrals and a free use of Glaisher's notation will clarify the pattern of the integrands.

The clue is in the effect of quarter-period addition on the theta functions. A quarter-period addition to a is a quarter-period addition to the arguments of the two theta functions in Λ_p and to the arguments of the two theta functions in zq a, and if p and q are the same, only one transformation is involved. Let us then define a set of four integrals by writing

(4.1) $$\Pi\mathrm{p}(u, a) = \Lambda_p(u, a) + u\,\mathrm{zp}\,a$$

and complete the set of integrals by means of the identity

(4.2) $$\Lambda_p(u, a) + u\,\mathrm{zq}\,a = \Pi\mathrm{p}(u, a) + u\,\mathrm{qp}'a/\mathrm{qp}\,a,$$

From (1.4) and (1.6) applied to (3.1)

$$\Lambda_c(u, a) = \Lambda_s(u, a + K_c), \qquad \Lambda_d(u, a) = \Lambda_n(u, a + K_c)$$

and therefore from (1.8)

(4.3, 4.4) $$\Pi\mathrm{c}(u, a) = \Pi\mathrm{s}(u, a + K_c), \qquad \Pi\mathrm{d}(u, a) = \Pi\mathrm{n}(u, a + K_c).$$

Also from (1.5)

$$\frac{\vartheta_n(a-u)}{\vartheta_n(a+u)} = e^{-2\lambda u}\frac{\vartheta_s(a+K_n-u)}{\vartheta_s(a+K_n+u)}, \frac{\vartheta_n'a}{\vartheta_n a} = \lambda + \frac{\vartheta_s'(a+K_n)}{\vartheta_s(a+K_n)},$$

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and therefore

$$\Lambda_n(u, a) = -\lambda u + \Lambda_s(u, a + K_n), \text{ zn } a = \lambda + \text{zs}(a + K_n),$$

implying

$$\Pi\text{n}(u, a) = \Pi\text{s}(u, a + K_n). \tag{4.5}$$

As functions of a, the integrands with which we are dealing are periodic in $2K_c$ and $2K_n$; hence

$$\Pi\text{s}(u, a + K_c + K_n) = \Pi\text{s}(u, a + K_d),$$

and from (4.4) and (4.5)

$$\Pi\text{d}(u, a) = \Pi\text{s}(u, a + K_d). \tag{4.6}$$

Thus for $p = c, n, d$,

$$\Pi\text{p}(u, a) = \Pi\text{s}(u, a + K_p). \tag{4.7}$$

In my book, $\Pi\text{p}(u, a)$ is defined by this formula, and not directly in terms of the theta function $\vartheta_p u$.

The structure of the set of integrals

$$\Pi\text{s}(u, a),\ \Pi\text{c}(u, a),\ \Pi\text{n}(u, a),\ \Pi\text{d}(u, a)$$

is symmetrical, for if $\Pi\text{p}(u, a)$ is any one of the four functions, then

$$\Pi\text{p}(u, a),\ \Pi\text{p}(u, a + K_c),\ \Pi\text{p}(u, a + K_n),\ \Pi_p(u, a + K_d)$$

are the same four functions looked at, so to speak, from K_p. To put the matter differently, the symmetrical relation

$$\Pi\text{q}(u, a + K_p) = \Pi\text{p}(u, a + K_q) \tag{4.8}$$

shows that no one of the functions dominates the set. Briot and Bouquet **(1,** p. 447**)** complete the set from $\Pi\text{n}(u, a)$ and associate each function $\Pi\text{n}(u, a + K_q)$ with one theta function and each difference $\Pi\text{n}(u, a + K_q) - \Pi\text{n}(u, a)$ with one elliptic function, but their notation does not achieve the economy of typical formulae.

5. To use an integral we must be able to recognize the integrand. We denote the integrand corresponding to $\Pi\text{p}(u, a)$ by J_p or if necessary by $J_p(u, a)$. In terms of theta functions

$$J_p = \partial\Lambda_p/\partial u + \text{zp } a,$$

but what we have to consider is the explicit expression of J_p as an elliptic function. The four integrands satisfy the same quarter-period relations as the functions from which they are derived or, in other words, satisfy the typical relation

$$J_q(u, a + K_p) = J_p(u, a + K_q) \tag{5.1}$$

derived from (4.8).

In our table in §3, the functions J_s, J_c, J_n, J_d occupy the principal diagonal where they appear as follows:

(5.21, 5.22) $$J_s = \frac{\mathrm{sn}'a\,\mathrm{sn}^2u}{\mathrm{sn}\,a(\mathrm{sn}^2a - \mathrm{sn}^2u)},\quad J_c = -\frac{c'\,\mathrm{cn}'a\,\mathrm{sn}^2u}{\mathrm{cn}\,a(c^{-1}\mathrm{dn}^2a\,\mathrm{dn}^2u - c^{-1}c')},$$

(5.23, 5.24) $$J_n = \frac{c\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2u}{1 - c\,\mathrm{sn}^2a\,\mathrm{sn}^2u},\quad J_d = \frac{c'\,\mathrm{dn}'a\,\mathrm{sn}^2u}{\mathrm{dn}\,a(c\,\mathrm{cn}^2a\,\mathrm{cn}^2u + c')}.$$

We may suggest that it is because only the original Jacobian functions sn u, cn u, dn u are used that the symmetry of the quartette cannot be seen, but since each function might be expressed in terms of any one of the twelve functions pq u, we are not likely to find satisfactory transformations by a process of trial and error.

We take a hint from the Weierstrassian theory, in which the fundamental integrand of the third kind is $\wp'a/(\wp u - \wp a)$, and we have

$$\int_0^u \frac{\wp'a\,du}{\wp u - \wp a} = \log\frac{\sigma(a-u)}{\sigma(a+u)} + \frac{2u\,\sigma'a}{\sigma a}.$$

If the Weierstrassian functions have the same lattice as the Jacobian functions, $\vartheta_s u$ and σu are integral functions with the same zeros, and the relation between them is

$$\sigma u = e^{\mu u^2}\vartheta_s u,$$

where μ is a constant. Hence

$$\log\frac{\sigma(a-u)}{\sigma(a+u)} = -4\,\mu a u + \log\frac{\vartheta_s(a-u)}{\vartheta_s(a+u)},\quad \frac{\sigma'a}{\sigma a} = 2\,\mu a + \frac{\vartheta_s'a}{\vartheta_s a},$$

and therefore

$$\int_0^u \frac{\wp'a\,du}{\wp u - \wp a} = 2\{\Lambda_s(u,a) + u\,\mathrm{zs}\,a\},$$

that is,

$$J_s = \frac{\frac{1}{2}\wp'a}{\wp u - \wp a}.$$

Since $\wp u$ differs from qs^2u by a constant, whether q is c, n, or d, we have

(5.3) $$J_s = \frac{\mathrm{qs}\,a\,\mathrm{qs}'a}{\mathrm{qs}^2u - \mathrm{qs}^2a}$$

and therefore

(5.4) $$J_p(u,a) = \frac{\mathrm{qs}(a+K_p)\,\mathrm{qs}'(a+K_p)}{\mathrm{qs}^2u - \mathrm{qs}^2(a+K_p)},$$

a general formula which includes (5.3).

To verify that the formulae (5.21—5.24) extracted from the table in §3 can be deduced from (5.4) is an exercise in algebra. First, $\mathrm{qs}'a = -\mathrm{sq}'a/\mathrm{sq}^2a$, gives

(5.5) $$J_s = \frac{\mathrm{sq}'a\,\mathrm{sq}^2u}{\mathrm{sq}\,a(\mathrm{sq}^2a - \mathrm{sq}^2u)};$$

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this formula includes (5.21), and shows that in spite of appearances the integrand given by (5.21) does not stand in any special relation to K_n.

Next, since $\mathrm{ps}\,a(\mathrm{ps}\,a + K_p) = \mathrm{ps}'K_p$, identification of q with p in (5.4) gives

$$J_p = \frac{\mathrm{ps}'^2 K_p\,\mathrm{sp}\,a\,\mathrm{sp}'a}{\mathrm{ps}^2 u - \mathrm{ps}'^2 K_p\,\mathrm{sp}^2 a},$$

that is,

$$J_p = \frac{\mathrm{ps}'^2 K_p\,\mathrm{sp}\,a\,\mathrm{sp}'a\,\mathrm{sp}^2 u}{1 - \mathrm{ps}'^2 K_p\,\mathrm{sp}^2 a\,\mathrm{sp}^2 u}; \tag{5.61}$$

this formula includes (5.23), identifying Jacobi's integrand with $J_s(u, a + K_n)$. In other words, $\Pi\mathrm{n}(u, a)$ is Jacobi's function $\Pi(u, a)$ seen as one member of a set of which the other two members $\Pi\mathrm{c}(u, a)$ $\Pi\mathrm{d}(u, a)$ have their integrands given by

$$J_c = \frac{c'\,\mathrm{sc}\,a\,\mathrm{sc}'a\,\mathrm{sc}^2 u}{1 - c'\,\mathrm{sc}^2 a\,\mathrm{sc}^2 u}, \qquad J_d = -\frac{cc'\,\mathrm{sd}\,a\,\mathrm{sd}'a\,\mathrm{sd}^2 u}{1 + cc'\,\mathrm{sd}^2 a\,\mathrm{sd}^2 u}. \tag{5.62, 5.63}$$

Lastly, to recover (5.22) and (5.24) from (5.4), we suppose q to be distinct from p and r to be the third member of the set c, n, d; then

$$\mathrm{qs}(a + K_p) = \mathrm{qs}K_p\,\mathrm{rp}\,a, \qquad \mathrm{qr}(a + K_p) = \mathrm{qr}K_p\mathrm{rq}\,a. \tag{5.71, 5.72}$$

Since $\mathrm{sr}^2 u\{\mathrm{qs}^2 u - \mathrm{qs}^2(a + K_p)\}$ is a linear function of $\mathrm{qr}^2 u$ which is zero only if $\mathrm{qr}^2 u = \mathrm{qr}^2(a + K_p)$, it follows that $\mathrm{qs}^2 u - \mathrm{qs}^2(a + K_p)$ is a multiple of $\mathrm{rs}^2 u(\mathrm{qr}^2 a\,\mathrm{qr}^2 u - \mathrm{qr}^2 K_p)$, and is therefore the product of

$$\mathrm{rs}^2 u(\mathrm{pq}^2 K_r\,\mathrm{qr}^2 a\,\mathrm{qr}^2 u + \mathrm{pr}^2 K_q)$$

by a factor independent of u. Determining the factor by putting $u = K_q$ and using (5.71), we have

$$\mathrm{qs}^2 u - \mathrm{qs}^2(a + K_p) = \mathrm{rp}^2 a\,\mathrm{rs}^2 u(\mathrm{pq}^2 K_r \mathrm{qr}^2 a\,\mathrm{qr}^2 u + \mathrm{pr}^2 K_q).$$

Using (5.71) again and replacing $\mathrm{qs}^2 K_p\,\mathrm{rp}'a/\mathrm{rp}\,a$ by $\mathrm{ps}^2 K_q\,\mathrm{pr}'a/\mathrm{pr}\,a$, we have

$$J_p = \frac{\mathrm{ps}^2 K_q\,\mathrm{pr}'a\,\mathrm{sr}^2 u}{\mathrm{pr}\,a(\mathrm{pq}^2 K_r\,\mathrm{qr}^2 a\,\mathrm{qr}^2 u + \mathrm{pr}^2 K_q)}. \tag{5.81}$$

This is the formula of which (5.22) and (5.24) are two cases; a third case is another formula for Jacobi's integrand:

$$J_n = \frac{c\,\mathrm{nc}'a\,\mathrm{sc}^2 u}{\mathrm{nc}\,a(c'^{-1}\mathrm{dc}^2 a\,\mathrm{dc}^2 u - cc'^{-1})}.$$

In fact there are six cases of (5.81), but the interchange of q and r is almost trivial. The direct transformation of (5.82) into (5.23) takes the form

$$\frac{c\,\mathrm{nc}'a\,\mathrm{sc}^2 u}{\mathrm{nc}\,a(c'^{-1}\,\mathrm{dc}^2 a\,\mathrm{dc}^2 u - cc'^{-1})} = -\frac{cc'\,\mathrm{cn}\,a\,\mathrm{cn}'a\,\mathrm{sn}^2 u}{\mathrm{dn}^2 a\,\mathrm{dn}^2 u - c\,\mathrm{cn}^2 a\,\mathrm{cn}^2 u}$$
$$= \frac{cc'\,\mathrm{sn}\,a\,\mathrm{sn}'a\,\mathrm{sn}^2 u}{(1 - c\,\mathrm{sn}^2 a)(1 - c\,\mathrm{sn}^2 u) - c(1 - \mathrm{sn}^2 a)(1 - \mathrm{sn}^2 u)}.$$

6. The relation between $\Pi\mathrm{p}(u,a)$ and $\Pi\mathrm{q}(u,a)$ can be expressed as a relation between functions instead of as a relation between arguments, for (4.1) gives

$$\Pi\mathrm{p}(u,a) - \Pi\mathrm{q}(u,a) = \tfrac{1}{2}\log\frac{\mathrm{pq}(a-u)}{\mathrm{pq}(a+u)} + u\,.\,\frac{\mathrm{pq}'a}{\mathrm{pq}\,a}. \tag{6.1}$$

In other words, an alternative definition of $\Pi\mathrm{p}(u,a)$ in terms of $\Pi\mathrm{s}(u,a)$ is

$$\Pi\mathrm{p}(u,a) = \Pi\mathrm{s}(u,a) + \tfrac{1}{2}\log\frac{\mathrm{ps}(a-u)}{\mathrm{ps}(a+u)} + u\,.\,\frac{\mathrm{ps}'a}{\mathrm{ps}\,a}. \tag{6.2}$$

The additional logarithmic ambiguity is only apparent if it is understood that the logarithm is zero when $u = 0$ and varies continuously as u describes the path of integration implicit in $\Pi\mathrm{s}(u,a)$.

It is interesting to establish (6.2) in terms of integrands. With differences of notation, the algebra is essentially Legendre's (**5**, §46; **6**, §49). With the use of the bipolar function, the addition theorem for ps u can be written

$$\mathrm{ps}(u+v) = \mathrm{ps}\,u\,\mathrm{ps}\,v(\mathrm{bps}\,u - \mathrm{bps}\,v)/(\mathrm{ps}^2u - \mathrm{ps}^2v).$$

Hence

$$\frac{\mathrm{ps}(a-u)}{\mathrm{ps}(a+u)} = \frac{\mathrm{bps}\,u + \mathrm{bps}\,a}{\mathrm{bps}\,u - \mathrm{bps}\,a}, \tag{6.3}$$

and the result to be proved is, that if $a_p = a + K_p$, then

$$\frac{\mathrm{ps}\,a_p\,\mathrm{ps}'a_p}{\mathrm{ps}^2u - \mathrm{ps}^2a_p} = \frac{\mathrm{ps}\,a\,\mathrm{ps}'a}{\mathrm{ps}^2u - \mathrm{ps}^2a} - \frac{\mathrm{bps}\,a\,\mathrm{bps}'u}{\mathrm{bps}^2u - \mathrm{bps}^2a} + \frac{\mathrm{ps}'a}{\mathrm{ps}\,a};$$

since

$$\mathrm{ps}'a = -\mathrm{ps}\,a\,\mathrm{bps}\,a,\ \mathrm{ps}'a_p = -\mathrm{ps}\,a_p\,\mathrm{bps}\,a_p = \mathrm{ps}\,a_p\,\mathrm{bps}\,a.$$

From (1.3), this is equivalent to

$$-\frac{\mathrm{bps}'u}{\mathrm{bps}^2u - \mathrm{bps}^2a} = 1 + \frac{\mathrm{ps}^2a}{\mathrm{ps}^2u - \mathrm{ps}^2a} + \frac{\mathrm{ps}^2a_p}{\mathrm{ps}^2u - \mathrm{ps}^2a_p}. \tag{6.4}$$

Now $\mathrm{ps}^2u(\mathrm{bps}^2u - \mathrm{bps}^2a)$ is a quadratic function of ps^2u which vanishes if $\mathrm{ps}^2u = \mathrm{ps}^2a$ and therefore also, from (1.3), if $\mathrm{ps}^2u = \mathrm{ps}^2a_p$; also the coefficient of ps^4u in $\mathrm{ps}\,u\,\mathrm{bps}^2u$, that is, in ps'^2u, is 1. Hence

$$\mathrm{ps}^2u(\mathrm{bps}^2u - \mathrm{bps}^2a) = (\mathrm{ps}^2u - \mathrm{ps}^2a)(\mathrm{ps}^2u - \mathrm{ps}^2a_p). \tag{6.5}$$

Multiplying by sp^2u, differentiating, and substituting for $\mathrm{ps}'u$ and $\mathrm{sp}'u$ from (1.1), we have

$$\mathrm{bps}\,u\,\mathrm{bps}'u = -\mathrm{bps}\,u(\mathrm{ps}^2u - \mathrm{ps}^2a\,\mathrm{ps}^2a_p\,\mathrm{sp}^2u),$$

that is,

$$-\mathrm{ps}^2u\,\mathrm{bps}'u = \mathrm{ps}^4u - \mathrm{ps}^2a\,\mathrm{ps}^2a_p. \tag{6.6}$$

From (6.5) and (6.6),

$$-\frac{\mathrm{bps}'u}{\mathrm{bps}^2u - \mathrm{bps}^2a} = \frac{\mathrm{ps}^4u - \mathrm{ps}^2a\,\mathrm{ps}^2a_p}{(\mathrm{ps}^2u - \mathrm{ps}^2a)(\mathrm{ps}^2u - \mathrm{ps}^2a_p)}, \tag{6.7}$$

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and the right-hand side of (6.7), resolved into partial fractions in the variable ps^2u, is the right-hand side of (6.4).

7. Since $\Pi\mathrm{s}(u, a)$ is an odd function of a, (6.2) can be written

$$(7.11)\qquad \Pi\mathrm{s}(u, a) + \Pi\mathrm{s}(u, K_p - a) = \tfrac{1}{2}\log\frac{\mathrm{ps}(a+u)}{\mathrm{ps}(a-u)} - u\,.\,\frac{\mathrm{ps}'a}{\mathrm{ps}\,a}\,;$$

further, since $\Pi\mathrm{s}(u, a)$ as a function of a, has $2K_q$ for a period,

$$\Pi\mathrm{s}(u, K_p - (a + K_q)) = \Pi\mathrm{s}(u, (K_p - a) + K_q),$$

and substituting $a + K_q$ for a in (7.11) we have for $\mathrm{q} \neq \mathrm{p}$,

$$(7.12)\qquad \Pi\mathrm{q}(u, a) + \Pi\mathrm{q}(u, K_p - a) = \tfrac{1}{2}\log\frac{\mathrm{rq}(a+u)}{\mathrm{rq}(a-u)} - u\,.\,\frac{\mathrm{rq}'a}{\mathrm{rq}\,a}\,.$$

The formulae (7.11) and (7.12) may be regarded as halving the area of values of a throughout which $\Pi\mathrm{q}(u, a)$ requires a theta function for its evaluation.

From these formulae we see also that if $2a$ is a quarter-period the integrals of the third kind degenerate. Since the value of $\mathrm{ps}(\tfrac{1}{2}K_p + u)\mathrm{ps}(\tfrac{1}{2}K_p - u)$ is $\mathrm{ps}^2\tfrac{1}{2}K_p$, we have from (7.11)

$$(7.21)\qquad \Pi\mathrm{s}(u, \tfrac{1}{2}K_p) = \tfrac{1}{2}\log\{\mathrm{sp}\,\tfrac{1}{2}K_p\,\mathrm{ps}(u + \tfrac{1}{2}K_p)\} + \tfrac{1}{2}\,u\,\mathrm{bps}\,\tfrac{1}{2}K_p.$$

Also $\mathrm{rq}(\tfrac{1}{2}K_p + u)\mathrm{rq}(\tfrac{1}{2}K_p - u)$ is a constant, since addition of K_p to u interchanges the poles and the zeros of $\mathrm{rq}\,u$; this constant is $\mathrm{rq}^2\,\tfrac{1}{2}K_p$, and we have from (7.12)

$$(7.22)\qquad \Pi\mathrm{q}(u, \tfrac{1}{2}K_p) = \tfrac{1}{2}\log\{\mathrm{qr}\,\tfrac{1}{2}K_p\,\mathrm{rq}(u + \tfrac{1}{2}K_p)\} - \tfrac{1}{2}u(\mathrm{bqs}\tfrac{1}{2}K_p - \mathrm{brs}\tfrac{1}{2}K_p),$$

since $\mathrm{rq}\,a = \mathrm{rs}\,a/\mathrm{qs}\,a$.

For the sake of completeness we must add that the identities

$$\Pi\mathrm{p}(u, K_p - a) = -\Pi\mathrm{s}(u, a),\ \Pi\mathrm{p}(u, a) = -\Pi\mathrm{s}(u, K_p - a)$$

imply

$$(7.31)\qquad \Pi\mathrm{p}(u, a) + \Pi\mathrm{p}(u, K_p - a) = \tfrac{1}{2}\log\frac{\mathrm{sp}(a+u)}{\mathrm{sp}(a-u)} - u\,.\,\frac{\mathrm{sp}'a}{\mathrm{sp}\,a},$$

$$(7.32)\qquad \Pi\mathrm{p}(u, \tfrac{1}{2}K_p) = \tfrac{1}{2}\log\{\mathrm{ps}\,\tfrac{1}{2}K_p\,\mathrm{sp}(u + \tfrac{1}{2}K_p)\} - \tfrac{1}{2}\,u\,\mathrm{bps}\,\tfrac{1}{2}K_p.$$

To us, (7.31) and (7.32) are little more than repetitions of (7.11) and (7.21), but we must remember that since the function we are denoting by $\Pi\mathrm{n}(u, a)$ was known long before $\Pi\mathrm{s}(u, a)$ was introduced, the classical formulae implicit in Jacobi's *theorema de additione argumenti parametri* (**4**, p. 159) are cases of (7.22) and (7.32).

The values of the bipolar functions used in (7.21) and (7.22) are easily found. For any value of u,

$$(7.41)\qquad \mathrm{qs}\,2u + \mathrm{rs}\,2u = \mathrm{bps}\,u,$$

and therefore

$$(7.42,\ 7.43)\qquad \mathrm{bps}\,\tfrac{1}{2}K_p = \mathrm{qs}K_p + \mathrm{rs}K_p,\ \mathrm{bqs}\,\tfrac{1}{2}K_p = \mathrm{rs}K_p.$$

Thus (7.22) becomes

(7.44) $\Pi\mathrm{q}(u, \tfrac{1}{2}K_p) = \tfrac{1}{2}\log\{\mathrm{qr}\tfrac{1}{2}K_p\,\mathrm{rq}(u + \tfrac{1}{2}K_p)\} + \tfrac{1}{2}u(\mathrm{qs}\,K_p - \mathrm{rs}\,K_p).$

We can modify the logarithmic terms in (7.21) and (7.22) and take fuller advantage of (7.41) and (7.42). From (6.3), (7.11) is equivalent to

$$(7.51)\quad \Pi\mathrm{s}(u, a) + \Pi\mathrm{s}(u, K_p - a) = \tfrac{1}{2}\log\frac{\mathrm{bps}\,u - \mathrm{bps}\,a}{\mathrm{bps}\,u + \mathrm{bps}\,a} + u\,\mathrm{bps}\,a,$$

and therefore (7.21) is equivalent to

$$(7.52)\quad \Pi\mathrm{s}(u, \tfrac{1}{2}K_p) = \tfrac{1}{4}\log\frac{\mathrm{bps}\,u - \mathrm{qs}K_p - \mathrm{rs}K_p}{\mathrm{bps}\,u + \mathrm{qs}K_p + \mathrm{rs}K_p} + \tfrac{1}{2}u(\mathrm{qs}K_p + \mathrm{rs}K_p).$$

Instead of (7.12) we have

$$(7.53)\quad \Pi\mathrm{q}(u, a) + \Pi\mathrm{q}(u, K_p - a)$$
$$= \tfrac{1}{2}\log\frac{(\mathrm{bqs}\,u + \mathrm{bqs}\,a)(\mathrm{brs}\,u - \mathrm{brs}\,a)}{(\mathrm{bqs}\,u - \mathrm{bqs}\,a)(\mathrm{brs}\,u + \mathrm{brs}\,a)} - u(\mathrm{bqs}\,a - \mathrm{brs}\,a),$$

leading to

$$(7.54)\quad \Pi\mathrm{q}(u, \tfrac{1}{2}K_p)$$
$$= \tfrac{1}{4}\log\frac{(\mathrm{bqs}\,u + \mathrm{rs}K_p)(\mathrm{brs}\,u - \mathrm{qs}K_p)}{(\mathrm{bqs}\,u - \mathrm{rs}K_p)(\mathrm{brs}\,u + \mathrm{qs}K_p)} + \tfrac{1}{2}u(\mathrm{qs}K_p - \mathrm{rs}K_p).$$

The squares of the constants $\mathrm{qs}K_p$ are given by

$$(7.61)\quad \mathrm{ns}^2K_c = -\mathrm{cs}^2K_n = 1,\ \mathrm{ns}^2K_d = -\mathrm{ds}^2K_n = c,\ \mathrm{ds}^2K_c = -\mathrm{cs}^2K_d = c',$$

and depend only on the Jacobian system, but the constants themselves with the exception of $\mathrm{ns}K_c$ depend on the choice of a basis for the lattice. Defining v, k, k' by

$$(7.62)\quad v = \mathrm{sc}K_n,\ k = \mathrm{ns}(K_c + K_n),\ k' = \mathrm{ds}K_c,$$

we have

$$(7.63)\quad v^2 = -1,\ k^2 = c,\ k'^2 = c',$$

and the six critical constants are given by

$$(7.64)\quad \mathrm{ns}K_c = 1,\ \mathrm{cs}K_n = -v,\ \mathrm{ns}K_d = -k,$$
$$\mathrm{ds}K_n = -vk,\ \mathrm{ds}K_c = k',\ \mathrm{cs}K_d = vk'.$$

The relations

$$\mathrm{ns}K_c/\mathrm{cs}K_n = \mathrm{ds}K_n/\mathrm{ns}K_d = \mathrm{cs}K_d/\mathrm{ds}K_c = v$$

express that rotation in the direction $K_c \to K_n \to K_d$ is positive or negative according as v is $+i$ or $-i$.

The results of expressing the constants in (7.52) and (7.54) in terms of v, k, k' are valid for all Jacobian systems, but it is for the classical systems in which k and k' are real that they are specially required.

8. In proposing that the typical integrand in the table in §3 should be treated as $J_p(u, a) + \mathrm{qp}'a/\mathrm{qp}\, a$ rather than as $I_p(u, a) + \mathrm{qn}'a/\mathrm{qn}\, a$, we are not altering the composition of the table. The integrands are the same sixteen functions of u and a, and the most to be claimed is that with the whole set of Glaisher's functions at our service we have shown that we can move easily from one entry to another within the table. To Hermite **(3)** are due examples of a process by which the tale of recorded integrals of the third kind can be quadrupled in length. The denominator Δ_n in (2.1) is the denominator in the classical expression for $\mathrm{sn}(a + u)$, and since

$$\vartheta_s(a + u)/\vartheta_n(a + u) = \mathrm{sn}(a + u) = (\mathrm{sn}\, a\, \mathrm{cn}\, u\, \mathrm{dn}\, u + \mathrm{cn}\, a\, \mathrm{dn}\, a\, \mathrm{sn}\, u)/\Delta_n$$

we have

$$\text{(8.1)} \qquad \frac{\vartheta_s(a + u)\vartheta_n(a - u)}{\vartheta_n^2 a \vartheta_n^2 u} = \mathrm{sn}\, a\, \mathrm{cn}\, u\, \mathrm{dn}\, u + \mathrm{cn}\, a\, \mathrm{dn}\, a\, \mathrm{sn}\, u.$$

Jacobi's argument now gives

$$\text{(8.2)} \qquad \int_0^u \frac{\mathrm{cn}\, a\, \mathrm{dn}\, a\, \mathrm{cn}\, u\, \mathrm{dn}\, u - \mathrm{sn}\, a(\mathrm{dn}^2 a + c\, \mathrm{cn}^2 a)\mathrm{sn}\, u}{\mathrm{sn}\, a\, \mathrm{cn}\, u\, \mathrm{dn}\, u + \mathrm{cn}\, a\, \mathrm{dn}\, a\, \mathrm{sn}\, u}\, du$$

$$= \log \frac{\vartheta_s(a + u)}{\vartheta_n(a - u)} - 2u \, . \frac{\vartheta_n' a}{\vartheta_n\, a} \, .$$

This method gives integrands corresponding to the 48 integrals

$$\tfrac{1}{2} \log \frac{\vartheta_p(a - u)}{\vartheta_r(a + u)} + u \, . \frac{\vartheta_q' a}{\vartheta_q\, a}$$

with $p \neq r$, but Hermite himself attached no importance to the extension. His comment, "au fond, ces diverses expressions se ramènent à la quantité . . ." $\Pi(u, a)$, suggests only that he was dissatisfied with the incoherent mass of formulae derived from Jacobi's integrand and its three companions.

More interesting than this extension is Hermite's use of the integrand

$$\mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a/(\mathrm{sn}^2 u - \mathrm{sn}^2 a),$$

which is the integrand denoted above (§§2–3) by I_s, in preference to Jacobi's integrand I_n, or, in other words, his use of the integral $\Lambda_s(u, a) + u\, \mathrm{zn}\, a$ in preference to Jacobi's integral $\Pi(u, a)$ which is $\Lambda_n(u, a) + u\, \mathrm{zn}\, a$. "Cette intégrale présente," he says, "plus de facilité que celle de Jacobi pour établir les théorèmes sur l'addition des arguments" **(3**, p. 841**)**. That is to say, he has found that the advantages of using the function $\Lambda_s(u, a)$ associated with the origin instead of the corresponding function $\Lambda_n(u, a)$ associated with the point K_n outweigh any disadvantages due to the heterogeneity of $\Lambda_s(u, a) + u\, \mathrm{zn}\, a$ as compared with $\Lambda_n(u, a) + u\, \mathrm{zn}\, a$. And this in spite of the fact that for elliptic functions he has only those whose poles are congruent with K_n.

9. The integrands tabulated in §3 are functions to which Jacobi's method of integration is seen in advance to be applicable; we have still to consider the arbitrary integrand $\lambda/(\mathrm{pq}^2 u - \mu)$. Determining a constant a by the condition

$$\mathrm{pq}^2\, a = \mu, \tag{9.11}$$

and inserting a numerator found to be convenient, we deal with the integral

$$\int_0^u \frac{\mathrm{pq}\, a\, \mathrm{pq}'a\, du}{\mathrm{pq}^2 u - \mathrm{pq}^2 a}.$$

If q is s, the integral is already known, for (5.3) is equivalent to

$$\int_0^u \frac{\mathrm{ps}\, a\, \mathrm{ps}'a\, du}{\mathrm{ps}^2 u - \mathrm{ps}^2 a} = \Pi\mathrm{s}(u, a). \tag{9.12}$$

If q is not s, then $\mathrm{pq}^2 u$ is a linear function of $\mathrm{sq}^2\, u$; whether $\mathrm{pq}^2 u$ is $\mathrm{sq}^2 u$ or $1 - \mathrm{qs}^2 K_p \mathrm{sq}^2 u$

$$\frac{\mathrm{pq}\, a\, \mathrm{pq}'a}{\mathrm{pq}^2 u - \mathrm{pq}^2 a} = \frac{\mathrm{sq}\, a\, \mathrm{sq}'a}{\mathrm{sq}^2 u - \mathrm{sq}^2 a},$$

and since

$$\frac{\mathrm{qs}\, a\, \mathrm{qs}'a}{\mathrm{qs}^2 u - \mathrm{qs}^2 a} = \frac{\mathrm{qs}'a}{\mathrm{qs}\, a}\cdot\frac{\mathrm{qs}^2 a}{\mathrm{qs}^2 u - \mathrm{qs}^2 a} = \frac{\mathrm{sq}'a}{\mathrm{sq}\, a}\cdot\frac{\mathrm{sq}^2 u}{\mathrm{sq}^2 u - \mathrm{sq}^2 a}$$

we have

$$\frac{\mathrm{pq}\, a\, \mathrm{pq}'a}{\mathrm{sq}^2 a}\int_0^u \frac{\mathrm{sq}^2 u\, du}{\mathrm{pq}^2 u - \mathrm{pq}^2 a} = \Pi\mathrm{s}(u, a), \tag{9.13}$$

$$\int_0^u \frac{\mathrm{pq}\, a\, \mathrm{pq}'a\, du}{\mathrm{pq}^2 u - \mathrm{pq}^2 a} = \frac{u\, \mathrm{qs}'a}{\mathrm{qs}\, a} + \Pi\mathrm{s}(u, a). \tag{9.14}$$

Although (9.13) is valid whether or not p is s, it is worth while to separate the two cases for the sake of further simplification. If p is s, the formula is

$$\frac{\mathrm{sq}'a}{\mathrm{sq}\, a}\int_0^u \frac{\mathrm{sq}^2 u\, du}{\mathrm{sq}^2 u - \mathrm{sq}^2 a} = \Pi\mathrm{s}(u, a), \tag{9.15}$$

a simple variant of (9.12), and if p is not s it can be written

$$\frac{\mathrm{ps}^2 K_q\, \mathrm{sq}'a}{\mathrm{sq}\, a}\int_0^u \frac{\mathrm{sq}^2 u\, du}{\mathrm{pq}^2 u - \mathrm{pq}^2 a} = \Pi\mathrm{s}(u, a), \tag{9.16}$$

since

$$\mathrm{qs}^2 K_p = -\mathrm{ps}^2 K_q.$$

The earliest of all integrals of the third kind, Legendre's function Π defined by (**5**, p. 17; **6**, p. 17)

$$\Pi = \int_0^\phi \frac{du}{(1 + n \sin^2\phi)\Delta},$$

where $\Delta = \sqrt{(1 - c\sin^2\phi)}$, is the integral

$$\int_0^u \frac{du}{1 + n\,\mathrm{sn}^2 u}.$$

It is usual now to change the sign in the denominator, and we take the integral of this form with sn u replaced by pq u as

$$\int_0^u \frac{du}{1 - \lambda\,\mathrm{pq}^2 u}.$$

If we define a by

$$\mathrm{qp}^2 a = \lambda, \tag{9.21}$$

we have

$$\frac{\mathrm{qp}'a/\mathrm{qp}\,a}{1 - \lambda\,\mathrm{pq}^2 u} = \frac{\mathrm{pq}\,a\,\mathrm{pq}'a}{\mathrm{pq}^2 u - \mathrm{pq}^2 a},$$

and we have merely to rewrite (9.12), (9.14), (9.15), and (9.16) as

$$\frac{\mathrm{sp}'a}{\mathrm{sp}\,a}\int_0^u \frac{du}{1 - \mathrm{sp}^2 a\ \mathrm{ps}^2 u} = \Pi\mathrm{s}(u, a), \tag{9.22}$$

$$\frac{\mathrm{qp}'a}{\mathrm{qp}\,a}\int_0^u \frac{du}{1 - \mathrm{qp}^2 a\,\mathrm{pq}^2 u} = \frac{u\,\mathrm{qs}'a}{\mathrm{qs}\,a} + \Pi\mathrm{s}(u, a), \tag{9.23}$$

$$\int_0^u \frac{\mathrm{qs}\,a\,\mathrm{qs}'a\,\mathrm{sq}^2 u\,du}{1 - \mathrm{qs}^2 a\,\mathrm{sq}^2 u} = \Pi\mathrm{s}(u, a), \tag{9.24}$$

$$\frac{\mathrm{ps}^2 K_q\,\mathrm{ps}'a}{\mathrm{ps}\,a}\int_0^u \frac{\mathrm{sq}^2 u\,du}{1 - \mathrm{qp}^2 a\,\mathrm{pq}^2 u} = \Pi\mathrm{s}(u, a). \tag{9.25}$$

There is an alternative substitution. The function pq u has one of the quarter-periods of the Jacobian system for a half-period, and if this quarter-period is K_t, the product $\mathrm{qp}u\ \mathrm{qp}(u + K_t)$ is independent of u, that is, is a constant of the system. If the square of this constant is j_{pq}, to write

$$\lambda = j_{pq}\,\mathrm{pq}^2 a \tag{9.31}$$

is equivalent to writing

$$\mathrm{qp}^2(a + K_t) = \lambda,$$

and this change replaces $\Pi\mathrm{s}(u, a)$ by $\Pi\mathrm{t}(u, a)$. The quarter-period relevant for ps u and sp u is K_p, and if the three quarter-periods of the system are K_p, K_q, K_r, then $\mathrm{sp}(K_p + K_q) = -\mathrm{sp}K_r$ and

$$j_{ps} = \mathrm{sp}^2 K_q\,\mathrm{sp}^2 K_r; \qquad j_{sq} = \mathrm{qs}^2 K_p\,\mathrm{qs}^2 K_r.$$

The quarter-period relevant for pq u is K_r, and

$$j_{pq} = \mathrm{qp}^2 K_r.$$

From (9.22), (9.24), and (9.25) we have

(9.32) $$\frac{\operatorname{ps}'a}{\operatorname{ps} a}\int_0^u \frac{du}{1 - j_{ps}\operatorname{ps}^2 a \operatorname{ps}^2 u} = \Pi\mathrm{p}(u, a),$$

(9.33) $$\int_0^u j_{sq}\frac{\operatorname{sq} a \operatorname{sq}'a \operatorname{sq}^2 u\, du}{1 - j_{sq}\operatorname{sq}^2 a \operatorname{sq}^2 u} = \Pi\mathrm{q}(u, a),$$

(9.34) $$\frac{\operatorname{ps}^2 K_q \operatorname{qr}'a}{\operatorname{qr} a}\int_0^u \frac{\operatorname{sq}^2 u\, du}{1 - j_{pq}\operatorname{pq}^2 a \operatorname{pq}^2 u} = \Pi\mathrm{r}(u, a),$$

and from (9.23)

(9.35) $$\frac{\operatorname{sq}'a}{\operatorname{sq} a}\int_0^u \frac{du}{1 - j_{pq}\operatorname{pq}^2 a \operatorname{pq}^2 u} = \frac{u \operatorname{pr}'a}{\operatorname{pr} a} + \Pi\mathrm{r}(u, a).$$

We can now see the structure of the integrands which compose the leading diagonal of the table in §3. Since the only functions to be used are Jacobi's three functions sn u, cn u, dn u, the denominator has one of the two forms $\operatorname{pn}^2 u - \operatorname{pn}^2 a$, $1 - j_{pn}\operatorname{pn}^2 a \operatorname{pn}^2 u$. The integrand J_s corresponding to $\Pi\mathrm{s}(u, a)$ is the integrand in (9.15) with n for q; to use (9.16) would be merely to substitute $-(\operatorname{cn}^2 u - \operatorname{cn}^2 a)$ or $-(\operatorname{dn}^2 u - \operatorname{dn}^2 a)/c$ for $\operatorname{sn}^2 u - \operatorname{sn}^2 a$. The function $\Pi\mathrm{n}(u, a)$ comes only from (9.33), and since $j_{sn} = \operatorname{ns}^2 K_c \operatorname{ns}^2(K_c + K_n) = c$, we find the integrand J_n as $c \operatorname{sn} a \operatorname{sn}'a \operatorname{sn}^2 u/(1 - c\operatorname{sn}^2 a \operatorname{sn}^2 u)$, precisely as given by Jacobi. The functions $\Pi\mathrm{c}(u, a)$ and $\Pi\mathrm{d}(u, a)$ come from (9.34), the one when pq u is dn u and the other when pq u is cn u, but we must express $\operatorname{qr}'a/\operatorname{qr} a$ as $-\operatorname{rq}'a/\operatorname{rq} a$; the constants required are given by

$$\operatorname{ds}^2 K_n = -c,\ j_{dn} = 1/c';\ \operatorname{cs}^2 K_n = -1,\ j_{cn} = -c/c'$$

and the entries in the table can be verified immediately.

10. As we have said, the substitution $\mu = \operatorname{pq}^2 a$ does not impose any restrictions on μ, and theoretically the two formulae (9.12) and (9.14), together with the expression of $\Pi\mathrm{s}(u, a)$ as $\Lambda_s(u, a) + u \operatorname{zs} a$, reduce any function of the third kind to a combination of functions each of which is a function of a single argument. But if the problem is the evaluation of a real integral by means of real variables, there are complications. A real value of μ does not necessarily give a real value of a, and if u is real and a complex, then functions of $a + u$ are functions which must be dissected before they can be evaluated.

In discussing evaluation, we assume that K_c has a real value K and K_n an imaginary value iK', and we assume also that K and K' are positive; then k and k' are positive, and υ is i. The origin and the points K, $K + iK'$, iK' are the corners of a rectangle which we denote by $SCDN$. In applying general formulae it is important to remember that $K + iK'$ is $-K_d$, since in the formal theory the three quarter-periods satisfy the symmetrical relation $K_c + K_n + K_d = 0$.

The path of integration is a segment of the real axis. For the present we continue to take $u = 0$ for the lower limit; the effect of removing this restriction is considered in our concluding paragraph.

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If one of the twelve functions $\mathrm{pq}^2 a$ is real, all of them are real, and therefore each of the functions $\mathrm{pq}\, a$ and each of the derivatives $\mathrm{pq}' a$ is either real or imaginary. Hence in all that follows each of the functions $\Pi\mathrm{p}(u, a)$ is either real or imaginary. To put in a real form a formula in which $\Pi\mathrm{p}(u, a)$ is in fact imaginary, we write

$$\Pi\mathrm{p}(u, a) = i\Pi'\mathrm{p}(u, a);$$

if one of the two functions $\Pi\mathrm{p}(u, a)$, $\Pi'\mathrm{p}(u, a)$ is imaginary, the other is real. This notation is extremely convenient for our purpose here, but is obviously not susceptible of extension for general use.

11. The three functions $\mathrm{cs}^2 u$, $\mathrm{ds}^2 u$, $\mathrm{ns}^2 u$ are real on the perimeter $SCDNS$, and decrease steadily from $+\infty$ to $-\infty$ as u describes the contour; $\mathrm{cs}^2 u$ changes sign at C, $\mathrm{ds}^2 u$ at D, and $\mathrm{ns}^2 u$ at N. Hence $\mathrm{ps}\, a\ \mathrm{ps}'\, a$, which is $-\mathrm{cs}\, a\ \mathrm{ds}\, a\ \mathrm{ns}\, a$, is real if a is on SC or DN, imaginary if a is on CD or NS. It follows that $\Pi\mathrm{s}(u, a)$ is real if a is on SC or DN, and $\Pi'\mathrm{s}(u,a)$ is real if a is on CD or NS. We identify the side to which a belongs by reference to the value of one of the functions $\mathrm{pq}^2 a$; most simply $\mathrm{ds}^2 a$ decreases from $+\infty$ through c' to 0 along SCD and from 0 through $-c$ to $-\infty$ along DNS.

To locate a on a side of the fundamental rectangle by means of a real variable, we write $a = K_p + b$ or $a = K_p + ib'$, where K_p is one of the two corners available, and we have four pairs of formulae:

$$\mathrm{ds}^2 a > c'$$

$$(11.11)\quad a = b \qquad \Pi\mathrm{s}(u, a) = \Pi\mathrm{s}(u, b) = \Lambda_s(u, b) + u\,\mathrm{zs}\, b$$

$$(11.12)\quad a = K - b \qquad \Pi\mathrm{s}(u, a) = -\Pi\mathrm{c}(u, b) = -\Lambda_c(u, b) - u\,\mathrm{zc}\, b,$$

$$c' > \mathrm{ds}^2 a = 0$$

$$(11.13)\quad a = K + ib' \qquad i\Pi'\mathrm{s}(u, a) = i\Pi'\mathrm{c}(u, ib') = \Lambda_c(u, ib') + u\,\mathrm{zc}\, ib'$$

$$(11.14)\quad a = K + iK' - ib', \qquad i\Pi'\mathrm{s}(u, a) = -i\Pi'\mathrm{d}(u, ib') = -\Lambda_d(u, ib') - u\,\mathrm{zd}\, ib',$$

$$0 > \mathrm{ds}^2 a > -c$$

$$(11.15)\quad a = K + iK' - b \qquad \Pi\mathrm{s}(u, a) = -\Pi\mathrm{d}(u, b) = -\Lambda_d(u, b) - u\,\mathrm{zd}\, b$$

$$(11.16)\quad a = iK' + b \qquad \Pi\mathrm{s}(u, a) = \Pi\mathrm{n}(u, b) = \Lambda_n(u, b) + u\,\mathrm{zn}\, b,$$

$$-c > \mathrm{ds}^2 a$$

$$(11.17)\quad a = iK' - ib' \qquad i\Pi'\mathrm{s}(u, a) = -i\Pi'\mathrm{n}(u, ib') = -\Lambda_n(u, ib') - u\,\mathrm{zn}\, ib'$$

$$(11.18)\quad a = ib' \qquad i\Pi'\mathrm{s}(u, a) = i\Pi'\mathrm{s}(u, ib') = \Lambda_s(u, ib') + u\,\mathrm{zs}\, ib'.$$

For any one value of a there is a choice between two formulae, and we can cover the whole perimeter either using two theta functions with b, b' in the intervals $(0, K)$, $(0, K')$ or using the four theta functions with b, b' in the intervals $(0, \frac{1}{2}K)$, $(0, \frac{1}{2}K')$; in the first case we have a further choice, for we can use $\vartheta_s u$ on CSN and $\vartheta_d u$ on CDN or $\vartheta_c u$ on SCD and $\vartheta_n u$ on SND.

With a on SC or ND the choice between functions is more apparent than real. Writers from Legendre onwards ignore (11.12) and (11.15) without explaining why these alternatives can be ignored. For the final evaluation from (11.11) and (11.12) we have explicitly

$$\Lambda_s(u, b) = \tfrac{1}{2}\log\frac{\vartheta_s(b-u)}{\vartheta_s(b+u)}, \qquad \mathrm{zs}\, b = \frac{\vartheta_s' b}{\vartheta_s\, b},$$

$$\Lambda_c(u, b) = \tfrac{1}{2}\log\frac{\vartheta_c(b-u)}{\vartheta_c(b+u)}, \qquad \mathrm{zc}\, b = \frac{\vartheta_c' b}{\vartheta_c\, b}.$$

Since $\vartheta_s(K-u) = \vartheta_s K\ \vartheta_c u$, tables of $\vartheta_s u$ and $\vartheta_s' u$ have only to be provided with the complementary argument $K-u$ to become tables of $\vartheta_s K\ \vartheta_c u$ and $-\vartheta_s K\ \vartheta_c' u$, and we use the same entries and do the same arithmetic whether we compute $\Lambda_s(u, K-b)$ and $\mathrm{zs}(K-b)$ as

$$\tfrac{1}{2}\log\frac{\vartheta_s(K-b-u)}{\vartheta_s(K-b+u)}, \qquad \frac{\vartheta_s'(K-b)}{\vartheta_s\,(K-b)}$$

or compute $-\Lambda_c(u, b)$ and $-\mathrm{zs}\, b$ as

$$-\tfrac{1}{2}\log\frac{\vartheta_s K\vartheta_c(b+u)}{\vartheta_s K\vartheta_c(b-u)}, \qquad -\frac{\vartheta_s K\vartheta_c' b}{\vartheta_s K\vartheta_c\, b}.$$

The same considerations apply to (11.15) and (11.16): tables of $\vartheta_n u$ and $\vartheta_n' u$ provided with the complementary argument $K-u$ are tables of $\vartheta_n K\ \vartheta_d u$ and $-\vartheta_n K\ \vartheta_d' u$.

With a on SN or CD the process of evaluation is more elaborate and the distinction between the alternatives is not trivial. The theta function in $\Lambda_p(u, ib')$ has the complex arguments $ib' \pm u$ and must be dissected before $\Pi'\mathrm{p}(u, ib')$ can be computed. We take the four functions in turn. The theta functions are defined in terms of v, where $v/\tfrac{1}{2}\pi = u/K$, that is, where $v = \pi u/2K$ and we write also $\beta = \pi b'/2K$. It is to be noticed that $\vartheta_p' ib'$ means $(d\vartheta_p/du)_{u=ib'}$ that is, $(d\vartheta_p/dv)_{u=ib'}\,.\, dv/du$, and that therefore

$$u\,\vartheta_p' ib' = v(d\vartheta_p/dv)_{v=i\beta}.$$

The functions are defined in terms of v and q, where

$$q = e^{-\pi K'/K}, \tag{11.21}$$

but q is a constant of the Jacobian system and variation of q is not contemplated.

The functions $\vartheta_s u$, $\vartheta_c u$ are multiples of

$$\sin v - q^{1.2}\sin 3v + q^{2.3}\sin 5v - q^{3.4}\sin 7v + \ldots$$

$$\cos v + q^{1.2}\cos 3v + q^{2.3}\cos 5v + q^{3.4}\cos 7v + \ldots$$

and therefore $\vartheta_s(ib'+u)$ is a multiple of

$$(\cosh\beta\sin v - q^{1.2}\cosh 3\beta\sin 3v + q^{2.3}\cosh 5\beta\sin 5v - \ldots)$$
$$+\, i\,(\sinh\beta\cos v - q^{1.2}\sinh 3\beta\cos 3v + q^{2.3}\sinh 5\beta\cos 5v - \ldots)$$

and $\vartheta_c(ib' + u)$ is a multiple of

$$(\cosh\beta\cos v + q^{1.2}\cosh 3\beta\cos 3v + q^{2.3}\cosh 5\beta\cos 5v + \ldots)$$
$$- i\,(\sinh\beta\sin v + q^{1.2}\sinh 3\beta\sin 3v + q^{2.3}\sinh 5\beta\sin 5v + \ldots).$$

Hence

(11.22) $\Pi' \mathrm{s}(u, ib') =$

$$\arctan\frac{\cosh\beta\sin v - q^{1.2}\cosh 3\beta\sin 3v + q^{2.3}\cosh 5\beta\sin 5v\ldots}{\sinh\beta\cos v - q^{1.2}\sinh 3\beta\cos 3v + q^{2.3}\sinh 5\beta\cos 5v\ldots}$$
$$- u\,.\,\frac{\cosh\beta - 3q^{1.2}\cosh 3\beta + 5q^{2.3}\cosh 5\beta - \ldots}{\sinh\beta - q^{1.2}\sinh 3\beta + q^{2.3}\sinh 5\beta - \ldots},$$

and

(11.23) $\Pi' \mathrm{c}(u, ib\ ) =$

$$\arctan\frac{\sinh\beta\sin v + q^{1.2}\sinh 3\beta\sin 3v + q^{2.3}\sinh 5\beta\sin 5v + \ldots}{\cosh\beta\cos v + q^{1.2}\cosh 3\beta\cos 3v + q^{2.3}\cosh 5\beta\cos 5v + \ldots}$$
$$- u\,.\,\frac{\sinh\beta + 3q^{1.2}\sinh 3\beta + 5q^{2.3}\sinh 5\beta + \ldots}{\cosh\beta + q^{1.2}\cosh 3\beta + q^{2.3}\cosh 5\beta + \ldots}.$$

Similarly, since $\vartheta_n u$, $\vartheta_d u$ are multiples of

$$1 - 2q\cos 2v + 2q^4\cos 4v - 2q^9\cos 6v + 2q^{16}\cos 8v - \ldots$$
$$1 + 2q\cos 2v + 2q^4\cos 4v + 2q^9\cos 6v + 2q^{16}\cos 8v + \ldots$$

we have

(11.24) $\Pi' \mathrm{n}(u, ib') =$

$$\arctan\frac{2q\sinh 2\beta\sin 2v - 2q^4\sinh 4\beta\sin 4v + 2q^9\sinh 6\beta\sin 6v - \ldots}{1 - 2q\cosh 2\beta\cosh 2v + 2q^4\cosh 4\beta\cos 4v - 2q^9\cosh 6\beta\cos 6v + \ldots}$$
$$+ u\,.\,\frac{4q\sinh 2\beta - 8q^4\sinh 4\beta + 12q^9\sinh 6\beta - \ldots}{1 - 2q\cosh 2\beta + 2q^4\cosh 4\beta - 2q^9\cosh 6\beta + \ldots},$$

(11.25) $\Pi' \mathrm{d}(u, ib') =$

$$-\arctan\frac{2q\sinh 2\beta\sin 2\nu + 2q^4\sinh 4\beta\sin 4\nu + 2q^9\sinh 6\beta\sin 6\nu + \ldots}{1 + 2q\cosh 2\beta\cos 2\nu + 2q^4\cosh 2\beta\cos 4\nu + 2q^9\cosh 6\beta\cos 6\nu + \ldots}$$
$$- u\,.\,\frac{4q\sinh 2\beta + 8q^4\sinh 4\beta + 12q^9\sinh 6\beta + \ldots}{1 + 2q\cosh 2\beta + 2q^4\cosh 4\beta + 2q^9\cosh 6\beta + \ldots}.$$

If b' and u are real, the functions $\Pi' \mathrm{p}(u, ib')$ have real values and (11.22)—(11.25) are formulae from which these values can be calculated. The hyperbolic functions do not retard appreciably the convergence of the several series; if b' is in the range $(0, K')$, both $\sinh n\beta$ and $\cosh n\beta$ are smaller than q^{-n}, and if b' is in $(0, \frac{1}{2}K')$, then $\sinh 2n\beta$ and $\cosh 2n\beta$ are smaller than q^{-n}. The restriction on the path of u implies that the inverse tangents are all in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

The dissection of the theta functions for the evaluation of elliptic integrals is classical; the improvement on current practice lies in avoiding a mixture of functions in any one formula.

12. Light is thrown on the alternatives in (11.11)—(11.18) by the relation (6.2) between $\Pi s(u, a)$ and $\Pi p(u, a)$:

$$(12.1) \qquad \Pi p(u, a) = \Pi s(u, a) + \tfrac{1}{2} \log \frac{ps(a-u)}{ps(a+u)} + u \cdot \frac{ps'a}{ps\, a}.$$

Denote the midpoints of SC, CD, DN, NS by E, F, G, H, and let $b \equiv b_s$ be a point in SE and $ib' \equiv b^1{}_s$ be a point in SH.

In the half-sides EC, DG, GN there are points b_c, b_d, b_n at distance b from the corners C, D, N, and we have

$$\begin{aligned} b_c &= K_c - b_s, & \Pi s(u, b_c) &= -\Pi c(u, b_s), \\ b_d &= -K_d - b_s, & \Pi s(u, b_d) &= -\Pi d(u, b_s), \\ b_n &= K_n + b_s, & \Pi s(u, b_n) &= \Pi n(u, b_s). \end{aligned}$$

If b_s traverses SE, the four points b_s, b_c, b_n, b_d together traverse the two sides SC, ND, and the evaluation of $\Pi s(u, a)$ is extended from SE to the two sides by means of the elliptic functions ps u:

$$(12.21) \qquad \Pi s(u, b_s) = \Pi s(u, b)$$

$$(12.22) \qquad \Pi s(u, b_c) = -\Pi s(u, b) - \tfrac{1}{2} \log \frac{cs(b-u)}{cs(b+u)} - u \cdot \frac{cs'b}{cs\, b}$$

$$(12.23) \qquad \Pi s(u, b_n) = \Pi s(u, b) + \tfrac{1}{2} \log \frac{ns(b-u)}{ns(b+u)} + u \cdot \frac{ns'b}{ns\, b}$$

$$(12.24) \qquad \Pi s(u, b_d) = -\Pi s(u, b) - \tfrac{1}{2} \log \frac{ds(b-u)}{ds(b+u)} - u \cdot \frac{ds'b}{ds\, b}.$$

Since the operation of evaluating the difference

$$\tfrac{1}{2} \log \frac{ps(b-u)}{ps(b+u)} + u \cdot \frac{ps'b}{ps\, b}$$

from tables of ps u and ps$'$ u is precisely the same as the operation of evaluating $\Pi p(u, a)$ in the form

$$\tfrac{1}{2} \log \frac{\vartheta_p(a-u)}{\vartheta_p(a+u)} + u \frac{\vartheta_p' a}{\vartheta_p a}$$

from tables of $\vartheta_p u$ and $\vartheta_p' u$, no practical advantage is to be expected from these formulae.

It is different when we deal with the half-sides HN, CF, FD. On them we have points b'_n, b'_c, b'_d such that

$$\begin{aligned} b'_n &= K_n - b'_s, & \Pi s(u, b'_n) &= -\Pi n(u, b'_s), \\ b'_c &= K_c + b'_s, & \Pi s(u, b'_c) &= \Pi c(u, b'_s), \\ b'_d &= -K_d - b'_s, & \Pi s(u, b'_d) &= -\Pi d(u, b'_s). \end{aligned}$$

Since b'_s is imaginary, we take (6.2) in the form

$$(12.3)\qquad i\Pi'\mathrm{p}(u, a) = i\Pi'\mathrm{s}(u, a) + \tfrac{1}{2}\log\frac{\mathrm{bps}\, a + \mathrm{bps}\, u}{\mathrm{bps}\, a - \mathrm{bps}\, u} - u\,\mathrm{bps}\, a.$$

Using Jacobi's imaginary transformation we have

$$\mathrm{bcs}(ib'|c) = i\,\mathrm{bns}(b'|c'),\ \mathrm{bds}(ib'|c) = i\,\mathrm{bds}(b'|c'),\ \mathrm{bns}(ib'|c) = i\,\mathrm{bcs}(b'|c')$$

and therefore

$$(12.41)\qquad \Pi'\mathrm{s}(u, b'_s) = \Pi'\mathrm{s}(u, ib'),$$

$$(12.42)\qquad \Pi'\mathrm{s}(u, b'_n) = -\Pi'\mathrm{s}(u, ib') + \arctan\frac{\mathrm{bcs}(b'|c')}{\mathrm{bns}(u|c)} - u\,\mathrm{bcs}(b'|c'),$$

$$(12.43)\qquad \Pi'\mathrm{s}(u, b'_c) = \Pi'\mathrm{s}(u, ib') - \arctan\frac{\mathrm{bns}(b'|c')}{\mathrm{bcs}(u|c')} + u\,\mathrm{bns}(b'|c'),$$

$$(12.44)\qquad \Pi'\mathrm{d}(u, b'_d) = \Pi'\mathrm{s}(u, ib') + \arctan\frac{\mathrm{bds}(b'|c')}{\mathrm{bds}(u|c)} - u\,\mathrm{bds}(b'|c').$$

It is far quicker to evaluate a difference

$$\arctan\frac{\mathrm{bqs}(b'|c')}{\mathrm{bps}(u|c)} - u\,\mathrm{bqs}(b'|c')$$

than to find an isolated value of a function $\Pi'\mathrm{p}(u, ib')$ by means of a dissected q-series, and (12.41)—(12.44), unlike (12.21)—(12.24), can be recommended to computers.

13. To conclude, we have to consider the integral

$$L = \int_{u_1}^{u_2} \frac{du}{1 - \mu\,\mathrm{pq}^2 u}$$

between arbitrary real limits. If the integral can be expressed as the difference between integrals from 0, the evaluation in one of the forms

$$\frac{\mathrm{sp}\, a}{\mathrm{sp}' a}\Pi_{12},\ (u_2 - u_1) + \frac{\mathrm{ps}\, a}{\mathrm{ps}' a}\Pi_{12},\ \frac{u_2 - u_1}{1 - \mu} + \frac{\mathrm{qp}\, a}{\mathrm{qp}' a}\Pi_{12},$$

where $\Pi_{12} = \Pi\mathrm{s}(u_2, a) - \Pi\mathrm{s}(u_1, a)$ introduces no fresh problems. But since the integral has a logarithmic singularity at any point where $\mathrm{pq}^2 u = \mathrm{pq}^2 a$, there is a tacit assumption throughout that there is no such point on the u-path.

If a is not real, this assumption does not come into operation. But if a is real, $\Pi\mathrm{s}(u, a)$ is defined as a real integral only for values of u in $(-a, a)$ and L is expressible by means of Π_{12} only if u and u_2 are in this interval, whereas the condition implicit in the existence of the integral does not restrict u and u_2 separately. The problem is the same as in the integration of $1/x$. If neither $\vartheta_s(a - u)$ nor $\vartheta_s(a + u)$ is zero for any value of u in (u_1, u_2), the two quotients $\vartheta_s(a - u_2)/\vartheta_s(a - u_1)$, and $\vartheta_s(a + u_1)/\vartheta_s(a + u_2)$ are positive and Π_{12}, defined as

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$$\int_{u_1}^{u_2} J_s(u, a)\, du,$$

can be compu ted as

$$\tfrac{1}{2}\log\frac{\vartheta_s(a-u_2)\vartheta_s(a+u_1)}{\vartheta_s(a-u_1)\vartheta_s(a+u_2)} + (u_2-u_1)\frac{\vartheta_s' a}{\vartheta_s\, a}.$$

If there are points $b_1, b_2, \ldots, b_m$ in (u_1, u_2) such that $\text{qp}^2 b_r = \text{qp}^2 a$ the substitution of

$$\tfrac{1}{2}\log\left|\frac{\vartheta_s(a-u_2)\vartheta_s(a+u_1)}{\vartheta_s(a-u_1)\vartheta_s(a+u_2)}\right| + (u_2-u_1)\frac{\vartheta_s' a}{\vartheta_s\, a}$$

for Π_{12}, in the formal evaluation gives the limit of the sum

$$\int_{u_1}^{b_1-\epsilon_1} + \int_{b_1+\epsilon_1}^{b_2-\epsilon_2} + \int_{b_{m-1}+\epsilon_{m-1}}^{b_m-\epsilon_m} + \int_{b_m+\epsilon_m}^{u_2} \frac{du}{1-\mu\,\text{pq}^2 u}$$

when $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ tend independently to zero.

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