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GENERALIZED INDEPENDENT INCREMENTS PROCESSES(*)

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Dedicated to Professor K. Urbanik on his 60th birthday

We study a class of Markov processes which arise in the theory of generalized convolutions and stand for a generalization of processes with independent increments.

1. Notation and preliminaries

Let P be the set of all probability measures (p.m.'s) on the positive half-line $R_+ = [0, \infty)$ with the weak convergence $\xrightarrow{w}$. We write δ_x for the unit mass at point x and write T_x for the map given by

$$T_x\mu(B) = \mu(x^{-1}B)$$

for $x > 0$, $\mu \in P$ and $B \in \mathfrak{B}$, the σ-field of Borel subsets of R_+. We define $T_0\mu = \delta_0$. We denote by Q the class of all sub-probability measures (sub-p.m.'s) on R_+. Let C_b be the Banach space of all real bounded continuous functions on R_+ with supremum norm $\|\cdot\|$ and C_0 its subspace consisting of functions vanishing at infinity.

A commutative and associative P-valued binary operation $\circ$ on P with δ_0 as the unit element is called a *generalized convolution*, if it is continuous in each variable separately and distributive with respect to convex combinations and maps T_x, and if it satisfies the following law of large numbers:

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(LLN) There exists a sequence of positive numbers c_n such that the sequence $T_{c_n}\delta_1^{\circ n}$ is convergent to a limit other than δ_0.

Here $P^{\circ n}$ denotes the nth power of P under the operation $\circ$.

The pair $(P, \circ)$ is called a *generalized convolution algebra*, which was introduced by K. Urbanik in [6] and studied by many researchers (cf. [2], [10], [11], [12], [17-22], [23]).

We assume throughout the paper that the algebra $(P, \circ)$ is regular, i.e. it admits a *characteristic function* $\hat{\mu} \in C_b$ defined by the following properties: the correspondence $\mu \leftrightarrow \hat{\mu}$ is one-to-one, $\hat{\mu}$ is distributive with respect to convex combinations, $\widehat{\mu\circ\nu} = \hat{\mu}\hat{\nu}$, $\widehat{T_x\mu}(t) = \hat{\mu}(xt)$, and the uniform convergence of $\hat{\mu}_n$ to $\hat{\mu}$ on every finite interval is equivalent to $\mu_n \xrightarrow{w} \mu$. The characteristic function $\hat{\mu}$ is represented as

$$\hat{\mu}(t) = \int \Omega(tx)\mu(dx). \tag{1.1}$$

Here and in the sequel the symbol $\int$ denotes the integral over $[0, \infty)$. The function Ω is called a kernel of the characteristic function. The system of characteristic functions is unique in the following sense: If there are two systems of characteristic functions with kernels Ω_1 and Ω_2, respectively, then

$$\Omega_1(t) = \Omega_2(ct) \quad (t \geq 0)$$

for some $c > 0$ (cf. Urbanik [18], Theorem 2.1). Henceforth we fix a system of characteristic functions.

The limiting measure in (LLN), denoted by σ_κ, is called the *characteristic measure* of the algebra in question and (with c_n replaced by their constant multiples if necessary) has the following characteristic function:

$$\hat{\sigma}_\kappa(t) = \exp(-t^\kappa) \tag{1.2}$$

where $t \geq 0$ and κ is a positive constant called the *characteristic exponent* of the generalized convolution $\circ$. The concepts of infinite divisibility and self-decomposability are introduced in the algebra $(P, \circ)$.

In a natural way the operation $\circ$ as well as the characteristic function can be extended to the set Q. Moreover, one can also extend the generalized convolution $\circ$ and the map T_x $(x > 0)$ to the set $\bar{P}$ of all p.m.'s defined on the compactified half-line $\bar{R}_+ = [0, \infty]$. Namely,

$$(a\mu' + (1-a)\delta_\infty)\circ(b\nu' + (1-b)\delta_\infty) = ab(\mu'\circ\nu') + (1-ab)\delta_\infty,$$

$$T_c(a\mu' + (1-a)\delta_\infty) = aT_c\mu' + (1-a)\delta_\infty$$

for $0 \leq a \leq 1, 0 \leq b \leq 1, 0 < c < \infty$ and $\mu', \nu' \in P$. The pair $(\bar{P}, \circ)$ is called the *extended generalized convolution algebra* (cf. Urbanik [21]). The concepts of infinitely divisible measures and self-decomposable measures can be defined in terms of the operation $\circ$ also in the extended algebra $(\bar{P}, \circ)$. Consider $\mu \in \bar{P}$ with $\mu = a\mu' + (1-a)\delta_\infty$ where $\mu' \in P$ and $0 < a \leq 1$. Then μ is infinitely divisible in $(\bar{P}, \circ)$ if and only if μ' is infinitely divisible in $(P, \circ)$. Similarly, μ is self-decomposable in $(\bar{P}, \circ)$ if and only if μ' is self-decomposable $(P, \circ)$.

Now we quote some examples of regular generalized convolutions which will be needed in the subsequent discussion. The examples will be given in terms of the kernel Ω and the characteristic measure $\sigma_\varkappa$ or its density $g_\varkappa$. Except Example 4, which was essentially considered by S. Cambanis, R. Keener and G. Simons in [4], the examples can be found in Urbanik's and Kingman's standard papers [16, 17, 18] [10]. The symmetric unimodal convolution in Example 3 and relation (1.3) are given by N. V. Thu.

EXAMPLE 1. *α-convolutions* $*_\alpha$ $(0 < \alpha < \infty)$: $\Omega(t)$ $\exp(-t^\alpha)$, $\kappa = \alpha$, $\sigma_\varkappa = \delta_1$. For $\alpha = 1$ we get the ordinary convolution i.e. $*_1 = *$

EXAMPLE 2. *Symmetric convolution* $*_{1,1}$: $\Omega(t) = \cos t$, $\kappa = 2$,

$$g_\varkappa(x) = \frac{1}{\sqrt{\pi}} \exp(-4^{-1}x^2).$$

EXAMPLE 3. *Kingman convolutions* $*_{1,\beta}$ $(\beta = 2(s+1)) > 1)$: We have $\kappa = 2$,

$$\Omega(t) = \Lambda_s(t) = \Gamma(s+1)J_s(t) \Big/ \left(\frac{1}{2}t\right)^s,$$

where J_s is the Bessel function and

$$g_\varkappa(x) = 2^{-2s-1}x^{2s+1}\exp(-4^{-1}x^2)/\Gamma(s+1).$$

The limiting case $s = -\frac{1}{2}$ reduces to the symmetric convolution. Moreover, as observed by Bingham [2], every Kingman convolution is subordinate to the symmetric convolution:

The case $\beta = 3$, $s = \frac{1}{2}$ reduces to the following symmetric unimodal convolu-

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tion.

Let W denote the uniform distribution on $[-1,1]$. For two independent random variables X and Y with distributions F and G we denote by FG the distribution of the product XY. By Khintchine-Shepp representation (cf. e.g. [6], Theorem 1.5, p. 10), every symmetric unimodal distribution μ on the real line can be uniquely represented by $\mu = FW$ with $F \in P$. Furthermore, by a routine computation we have the following equation:

$$FW * GW = (F *_{1,3} G) W \quad (F, G \in P), \tag{1.3}$$

which is a more specific form of the well-known theorem of Wintner (cf. [24]) asserting that the convolution of two symmetric unimodal distributions on R is unimodal.

EXAMPLE 4. *n-symmetric convolutions* $\square_n$ $(n = 2,3, \ldots)$: These convolutions appear in the contex of α-symmetric distributions (cf. [4]). We have $\kappa = 1$,

$$\Omega(t) = E\Lambda_s(t/\sqrt{D}), \tag{1.4}$$

with $n = 2(s+1)$ and D being a random variable with Beta $\left(\frac{1}{2}, \frac{n-1}{2}\right)$ distribution, and

$$g_\kappa(x) = \frac{2\Gamma\left(s + \frac{3}{2}\right)(2x^{2s+1})}{\sqrt{\pi}\,\Gamma(s+1)(1+x^2)^{2s+1}}.$$

The paper is organized as follows: in §2 we introduce generalized independent increments processes ($\circ$-i.i. processes) and $\circ$-Lévy processes. We prove that $\circ$-Lévy processes are strong Markov Feller processes. In §3 the infinitesimal genarators associated with $\circ$-Lévy processes are studied. Generalized Bernstein functions are discussed in §4. Finally, in §5 we obtain analogues of some of Sato's and Lamperti's results on self-similar processes (cf. [13], [15]).

2. Generalized independent increments processes

Suppose that $\mu_{s,t}$ $(0 \leq s < t)$ is a family of p.m.'s on $\bar{R}_+$ such that the following eguation is satisfied:

$$\mu_{s,t} \circ \mu_{t,u} = \mu_{s,u} \quad (0 \leq s < t < u). \tag{2.1}$$

For every x in $\bar{R}_+$ and $B \in \bar{\mathfrak{B}}$, $\bar{\mathfrak{B}}$ being the Borel σ-field of $\bar{R}_+$, we put

(2.2)
$$P_{s,t}(x, B) = \delta_x \circ \mu_{s,t}(B).$$

This definition and (2.1) imply the Chapman-Kolmogorov equation

$$\int P_{s,t}(x, dy) P_{t,u}(y, B) = P_{s,u}(x, B) \quad (0 \leq s < t < u),$$

which can be proved by characteristic functions. Hence, there exists a $\bar{R}_+$-valued Markov process $\{X_t\}$ with transition probability $P_{s,t}$ given by (2.2), that is

$$P(X_t \in B \mid X_u, u \leq s) = P_{s,t}(X_s, B).$$

The probability measure under the initial condition $X_0 = x$ is denoted by P^x. As usual the expectation with respect to P^x is denoted by E^x.

If $\circ$ is the ordinary convolution then $\{X_t\}$ is a process with independent increments. Therefore, in general case, $\{X_t\}$ will be referred to as a *generalized independent increments process,* or more precisely, $\circ$-*independent increments process* ($\circ$-i.i. process).

We say that a family of p.m.'s $\{\mu_t\}$ in $\bar{P}$ is a *generalized convolution semigroup* (shortly, $\circ$-semigroup), if the following conditions are satisfied:

(2.3)
$$\mu_t \circ \mu_s = \mu_{t+s} \quad (t, s \geq 0)$$
$$\mu_t \xrightarrow{w} \delta_0 \quad \text{as} \quad t \to 0.$$

It follows that $\mu_0 = \delta_0$.

It is easily seen that if $\{\mu_t\}$ is an $\circ$-semigroup then the family $\{\mu_{s,t}\}$ given by

$$\mu_{s,t} = \mu_{t-s} \quad (0 \leq s < t)$$

satisfies (2.1) and induces a time-homogenous $\circ$-i.i. process $\{X_t\}$ which will be called in the sequel an $\circ$-*Lévy process.*

For an extended generalized convolution algebra $(\bar{P}, \circ)$ define *generalized translation operators* by

(2.4)
$$(\tau^a f)(x) = \int^{-} f(u) \delta_a \circ \delta_x(du),$$

where $a, x \in \bar{R}_+$ and f is a continuous function on $\bar{R}_+$. Here and in the sequel $\int^{-}$ denotes the integral over $\bar{R}_+$. The operators τ^a, $a \in \bar{R}_+$, will be called $\circ$-*translation operators* (cf. Levitan [14]). Using these operators, Volkovich [23] obtained an analytic characterization of generalized convolutions.

Let μ be a finite measure on $\bar{R}_+$. We put

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$$(2.5) \qquad (\tau^{\mu} f)(x) = \int^{-} f(u)\mu\circ\delta_x(du) = \int^{-} (\tau^{a} f)(x)\mu(da),$$

where $x \in \bar{R}_+$ and f is a continuous function on $\bar{R}_+$.

LEMMA 2.1. *For every finite measure μ the operator τ^{μ} transforms C_0 into C_0.*

Proof. The assertion follows from the fact that the extended generalized convolution $\circ$ is continuous in each variable separately (cf. Urbanik [21], Proposition 2.4). □

Proofs of Lemmas 2.2, 2.3 and 2.4 below are similar to those for the ordinary convolution and will be omitted.

LEMMA 2.2. *Every τ^{μ} is a positive bounded operator on C_0 commuting with $\circ$-translation operators.*

In the sequel, any operator on a function space commuting with $\circ$-translation operators will be called *$\circ$-translation invariant.*

LEMMA 2.3. *Let A be a positive bounded $\circ$-translation invariant operator on C_0. There exists a uniquely determined finite measure μ on $\bar{R}_+$ such that*

$$A = \tau^{\mu}.$$

LEMMA 2.4. *For any $\mu, \nu \in \bar{P}$*

$$(2.6) \qquad \tau^{\mu}\tau^{\nu} = \tau^{\nu}\tau^{\mu} = \tau^{\mu\circ\nu}.$$

We note that

$$\int^{-} f(u)(\mu\circ\nu)(du) = \int^{-}\int^{-} (\tau^{u} f)(v)\mu(du)\nu(dv),$$

where $\mu, \nu \in \bar{P}$ and f is a continuous function on $\bar{R}_+$.

THEOREM 2.5. *Let $\{\mu_t\}$ be an $\circ$-semigroup of p.m.'s on $\bar{R}_+$. The formula*

$$(2.7) \qquad S_t = \tau^{\mu_t} \quad (t \geq 0)$$

defines a strongly continuous $\circ$-translation invariant contraction semigroup on C_0.

Conversely, if $\{S_t\}$ *is a strongly continuous* $\circ$*-translation invariant contraction semigroup of positive operators on* C_0*, then it is given by* (2.7) *with the same* $\circ$*-semigroup of p.m.'s on* $\bar{R}_+$*. The correspondence* $\{\mu_t\} \leftrightarrow \{S_t\}$ *is one-to-one.*

Proof. From Lemmas 2.1, 2.2 and 2.4 it follows that $\{S_t\}$ defined by (2.7) is an $\circ$-translation invariant contraction semigroup. Its strong continuity follows from Chung's remark (cf. Chung [5], p. 49). The converse statement follows from Lemma 2.3. Finally, the one-to-one correspondence $\{\mu_t\} \leftrightarrow \{S_t\}$ is a consequence of Lemma 2.2. □

Let $\{X_t\}$ be an $\circ$-Lévy process with the transition probability given by

$$P_t(x,.) = \mu_t \circ \delta_x \quad (t \geq 0,\ x \in \bar{R}_+).$$

The corresponding semigroup $\{S_t\}$ can be written in the form

$$(S_t f)(x) = E^x f(X_t). \tag{2.8}$$

By Theorem 2.5 $\{S_t\}$ is a strongly continuous semigroup on C_0, which implies that $\{X_t\}$ is a Feller process. Moreover, since the function $(t, x, f) \mapsto (S_t f)(x)$ is continuous (cf. Chung [5]), it follows that the process is a strong Markov process (cf. Blumenthal and Getoor [3], p.41). Thus we have the following theorem (cf. Chung [5], Proposition 2, p.50 and Theorem 6, p.54):

THEOREM 2.6. *Every* $\circ$*-Lévy process is a strong Markov Feller process. Consequently, it is stochastically continuous and has a version with right continuons paths having left limits.*

Remark 2.7. For some generalized convolution $\circ$, there exist $\circ$-Lévy processes with continuous paths. For example, the absolute value of the Brownian motion is a $*_{1,1}$-Lévy process having continuous paths.

3. Infinitesimal generators

The aim of this section is to study the infintesimal generators of the semigroups associated with $\circ$-Lévy processes.

To begin with we introduce the following generalized differential operator:

$$D^{\circ} f(x) = \lim_{y \to 0+} \frac{\tau^x f(y) - f(x)}{w(y)}, \tag{3.1}$$

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where f is a function in C_0 and the limit is taken in C_0-norm and the function $w(.)$ is defined by

$$w(y) = 1 - \Omega(y), \quad 0 \le y \le x_0 \tag{3.2}$$
$$= 1 - \Omega(x_0), \quad y > x_0$$

x_0 being a number such that $0 < \Omega(y) < 1$ for $0 < y \le x_0$. The domain of D° is denoted by $\mathfrak{D}(D^\circ)$.

As in Klosowska [11] and Bingham [2] we shall assume that

$$V^{-1} = \int x^\varkappa \sigma_\varkappa(dx) < \infty, \tag{3.3}$$

which holds true for all known examples of regular generalized convolutions.

LEMMA 3.1. *Let $\{\mu_t\}$ be an $\circ$-semigroup in $(P, \circ)$. There exists a finite measure m on R_+ such that*

$$\frac{w(x)}{t}\mu_t(dx) \xrightarrow{w} m \quad \text{as} \quad t \to 0. \tag{3.4}$$

Proof. Since μ_1 is $\circ$-infinitely divisible, there is a unique finite measure m on R_+ such that

$$\hat{\mu}_1(u) = \exp \int \frac{\Omega(ux) - 1}{w(x)} m(dx)$$

by [16] Theorem 13 and [17] Theorem 1. Hence

$$\hat{\mu}_t(u) = \exp\Big(t \int \frac{\Omega(ux) - 1}{w(x)} m(dx)\Big).$$

Let $m_t(dx) = t^{-1}w(x)\mu_t(dx)$ for $t > 0$. Then

$$\int \frac{\Omega(ux) - 1}{w(x)} m_t(dx) = t^{-1}(\hat{\mu}_t(u) - 1) \to \int \frac{\Omega(ux) - 1}{w(x)} m(dx) \quad (t \to 0)$$

uniformly on every finite interval. Now the argument in the proof of [16] Theorem 13 applies and we get $m_t \xrightarrow{w} m$ as $t \to 0$. □

LEMMA 3.2. *Suppose that* (3.3) *holds. Define*

$$\beta_y(u) = Vy^{-\varkappa}u^\varkappa T_y\sigma_\varkappa(du) \quad (y > 0).$$

Then every β_y *is a p.m. on* R_+ *and*

(3.5) $$\beta_y \xrightarrow{w} \delta_0 \quad (y \to 0).$$

Proof. We have

$$\hat{\beta}_y(t) = \int u^\kappa \Omega(tuy) V_{\sigma_\kappa}(du),$$

which implies that $\hat{\beta}_y(0) = 1$ and therefore β_y is a p.m. Moreover, lettng y tend to zero we have $\hat{\beta}_y(t) \to 1$. Consequently, (3.5) holds. □

Let H be the class of functions of the form

$$f_a(x) = \exp(-a^\kappa x^\kappa) \quad (a > 0,\ x \in R_+).$$

LEMMA 3.3. *Suppose that* (3.3) *holds. The operator* D° *is densely defined in* C_0, *and the domain* $\mathfrak{D}(D^\circ)$ *contains the class* H, (3.1) *is equivalent to the following*

(3.1′) $$D^\circ f(x) = \lim_{y \to 0} \frac{\tau^x f(y) - f(x)}{Vy^\kappa}.$$

Proof. When (3.3) holds, Klosowska ([11], Lemma 1) shows that

(3.6) $$\frac{w(y)}{y^\kappa} \to V \quad (y \to 0),$$

which implies that (3.1) is equivalent to (3.1′). The linear combinations of elements of H are dense in C_0. Let us prove that $D^\circ f_a$ is defined for any $a > 0$. By (1.2), (2.4) and (3.3) we have

$$\left| \frac{\tau^x f_a(y) - f_a(x)}{Vy^\kappa} + \int \Omega(axv) a^\kappa v^\kappa \sigma_\kappa(dv) \right| =$$

$$= \left| \frac{\int\int \Omega(auv)\sigma_\kappa(dv)\delta_x \circ \delta_y(du) - \int \Omega(axv)\sigma_\kappa(dv)}{Vy^\kappa} + \int \Omega(axv) a^\kappa v^\kappa \sigma_\kappa(dv) \right|$$

$$= \left| \int \Omega(axv) \left\{ \frac{\Omega(ayv) - 1}{Vy^\kappa} + a^\kappa v^\kappa \right\} \sigma_\kappa(dv) \right|$$

$$= \left| \int \Omega(axuy^{-1}) \frac{\Omega(au) - 1 + Va^\kappa u^\kappa}{Vy^\kappa} T_y \sigma_k(du) \right|$$

$$\leq \int | \Omega(au) - 1 + Va^{\varkappa}u^{\varkappa} | V^{-1}y^{-\varkappa}T_y\sigma_{\varkappa}(du)$$

$$= V^{-2}\int \left| \frac{\Omega(au) - 1}{u^{\varkappa}} + Va^{\varkappa} \right| Vy^{-\varkappa}u^{\varkappa}T_y\sigma_{\varkappa}(du),$$

where the integrand is a continuous bounded function of u and vanishes at $u = 0$. By Lemma 3.2, the last expression tends to zero as $y \to 0$, which implies that $H \subset \mathfrak{D}(D^{\circ})$ and

$$\text{(3.7)} \qquad \lim_{y\to 0} \frac{\tau^x f_a(y) - f_a(x)}{Vy^{\varkappa}} = -\int \Omega(axv)a^{\varkappa}v^{\varkappa}\sigma_{\varkappa}(dv)$$

uniformly in x for every positive number a. □

THEOREM 3.4. *Suppose that* (3.3) *holds. Let A be the infinitesimal generator of the semigroup associated with an $\circ$-Lévy process on $\bar{R}_+$ with domain $\mathfrak{D}(A)$. Then $\mathfrak{D}(D^{\circ}) \subset \mathfrak{D}(A)$ and*

$$\text{(3.8)} \qquad Af(x) = \int \frac{\tau^x f(u) - f(x)}{w(u)} \nu(du - \rho f(x)$$

for $f \in \mathfrak{D}(D^{\circ})$, where ρ is a nonnegative constant and ν is a finite measure on R_+. The integrand assumes the value $D^{\circ}f(x)$ at $u = 0$. The pair (ν, ρ) is uniquely determined by A.

Conversely, for any pair (ν, ρ), there exists a unique $\circ$-Lévy process on $\bar{R}_+$ satisfying (3.8) *for all $f \in \mathfrak{D}(D^{\circ})$.*

Proof. Let A be the infinitesimal generator for the semigroup $\{S_t\}$ given by (2.7) and (2.8). Putting

$$\rho(t) = \mu_t(R_+) \quad (t \geq 0)$$

and taking into account the continuity of $\{\mu_t\}$, we have

$$\rho(t) = \exp(-\rho t) \quad (t \geq 0)$$

with some $\rho \geq 0$. Let $f \in \mathfrak{D}(D^{\circ})$. We have

$$Af(x) = \lim_{t\to 0} \frac{S_t f(x) - f(x)}{t}$$
$$= \lim_{t\to 0} \int^{-} [\tau^x f(y) - f(x)] \frac{1}{t} \mu_t(dy)$$

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$$= \lim_{t\to 0}\left\{\int [\tau^x f(y) - f(x)]\,\frac{1}{w(y)}\,t^{-1}w(y)\mu_t(dy) - \frac{1-\rho(t)}{t}f(x)\right\}$$

$$= \int \frac{\tau^x f(y) - f(x)}{w(y)}\,\nu(dy) - \rho f(x),$$

where ν is the weak limit of $t^{-1}w(y)\mu_t(dy)$ as $t\to 0$ (Lemma 3.1), and the integrand in the last expression assumes the value $D^\circ f(x)$ (Lemma 3.3).

Since the last expression of the above equalities belongs to C_0 and since the convergence is boundedly pointwise, the limit can be taken in C_0-norm by the use of a general theory (Dynkin [7] Lemma 2.11). This shows that $\mathfrak{D}(D^\circ) \subset \mathfrak{D}(A)$ and (3.8) holds.

To prove the uniqueness of representation (3.8), use the fact $H \subset \mathfrak{D}(D^\circ)$ in Lemma 3.8. By (3.7) we have $D^\circ f_a(0) = -V^{-1}a^\kappa$. Hence

$$Af_a(0) = -V^{-1}a^\kappa\nu(\{0\}) + \int_{(0,\infty)} \frac{\exp(-a^\kappa y^\kappa) - 1}{w(y)}\,\nu(dy) - \rho.$$

Since $Af_a(0) \to -\rho$ as $a\to 0$, ρ is unique. Since $a^{-\kappa}(Af_a(0)+\rho) \to -V^{-1}\nu(\{0\})$ as $a\to\infty$, $\nu(\{0\})$ is unique. Moreover, if finite measures ν and ν' satisfy

$$\int_{(0,\infty)} \frac{\exp(-a^\kappa y^\kappa) - 1}{w(y)}\,\nu(dy) = \int_{(0,\infty)} \frac{\exp(-a^\kappa y^\kappa) - 1}{w(y)}\,\nu'(dy)$$

for all $a > 0$, then $\nu = \nu'$ on $(0, \infty)$ by the uniqueness theorem for Laplace transforms, because the above equality is written to

$$\int_0^\infty e^{-a^\kappa s}ds \int_s^\infty \frac{\nu(dy)}{w(y)} = \int_0^\infty e^{-a^\kappa s}ds \int_s^\infty \frac{\nu'(dy)}{w(y)}.$$

Conversely, given a pair (ν, ρ), let γ be an $\circ$-infinitely divisible p.m. on R_+ satisfying

$$\tilde{\gamma}(u) = \exp\int \frac{\Omega(ux) - 1}{w(x)}\,\nu(dx)$$

(cf. Urbanik [16]). Then the infinitesimal generator A for the semigroup $\{S_t\}$ given by (2.7) with

$$\mu_t(t) = \exp(-\rho t)\gamma^{\circ t} + (1 - \exp(-\rho t))\delta_\infty$$

satisfies (3.8). It is easy to see that this μ_t is uniquely determined by (ν, ρ). □

A particular but very important case of $\circ$-Lévy processes is the processes induced by the characteristic measure $\sigma_\varkappa$.

THEOREM 3.5. *Suppose that* (3.3) *holds. Let A be the infinitesimal generator for the $\circ$-Lévy process $\{X_t\}$ such that the P^0-distribution of X_1 is equal to $\sigma_\varkappa$. Then $Af = D^\circ f$ for every $f \in \mathfrak{D}(D^\circ)$.*

Proof. Apply Theorem 3.4. The measure ν there must satisfy

$$\int \frac{\Omega(ux) - 1}{w(x)} \nu(dx) = - u^\varkappa$$

in this case by virtue of (1.2). Since the integrand assumes the value $u^\varkappa$ at $x = 0$, we have $\nu = \delta_0$. □

Now, by virtue of formulas (3.1′) and (3.7), we get the following examples of D°:

α-convolutions: $D^\circ f(x) = a^{-1} x^{1-\alpha} f'(x)$.
Symmetric convolution: $D^\circ f = f''$.
Kingman convolution $*_{1,\beta}$ $(\beta = 2(s+1) > 1)$: By Gradshteyn and Ryzhik ([8], 3.381 (4)), the constant V in (3.3) is given by

$$V = \frac{1}{4(s+1)}.$$

Next, for $f \in C_0$ and $x, y \geq 0$ we have (cf. Urbanik [16])

$$\tau^x f(y) = \frac{\Gamma(s+1)}{\sqrt{\pi}\,\Gamma\left(s + \frac{1}{2}\right)} \int_{-1}^{1} f((x^2 + 2uxy + y^2)^{\frac{1}{2}}(1 - u^2)^{s-\frac{1}{2}} du,$$

which together with Lemma 3.3 leads to the following formula (cf. Gradshteyn and Ryzhik [8], 3.251 (1) and 3.249 (5)):

$$D^\circ f(x) = f''(x) + (2s+1)x^{-1} f'(x).$$

4. Generalized Bernstein functions

We say that the family $\{\nu_t\}$ of sub-p.m.'s on R_+ is an $\circ$-*semigroup* if the following conditions are satisfied:

$$\nu_t \circ \nu_s = \nu_{t+s} \quad (t, s \geq 0).$$

$$\nu_t \to \delta_0 \quad \text{vaguely as } t \text{ tends to } 0,$$

that is, $\int f(x)\nu_t(dx) \to f(0)$ as $t \to 0$ for every continuous function f on R_+ with compact support.

Clearly, these conditions imply that $\nu_0 = \delta_0$ and $\nu_t \xrightarrow{w} \delta_0$ as $t \to 0$. Let $\{X_t\}$ be an $\circ$-Lévy process on $\bar{R}_+$ induced by an $\circ$-semigroup $\{\mu_t\}$ of p.m.'s (cf. §2). The restriction of $\{\mu_t\}$ to R_+, denoted by $\{\nu_t\}$, is an $\circ$-semigroup of sub-p.m.'s. Since every measure ν_t is infinitely divisible with respect to $\circ$, the characteristic function of ν_t is of the form (cf. Urbanik [16], [17])

$$\hat{\nu}_t(u) = \exp(-tf(u)),\ (u,\ t \geq 0), \tag{4.1}$$

where f is given by

$$f(u) = a + bu^{\varkappa} + \int (1 - \Omega(ux))m(dx), \tag{4.2}$$

a, b being nonnegative constants and m being a measure on R_+ vanishing at the origin such that

$$\int w(x)m(dx) < \infty, \tag{4.3}$$

where $w(.)$ is a function defined by (3.2).

Let $F(\circ)$ denote the set of all functions of the form (4.2). Let $S(\circ)$ denote the set of all functions in $F(\circ)$ corresponding to $\circ$-self-decomposable sub-p.m.'s (cf. Urbanik [17]). For the ordinary convolution the set $F(\circ)$ coincides with the set of all Bernstein functions (cf. Berg & Forst [1], p. 61). Hence in general case the functions in $F(\circ)$ will be called *generalized Bernstein functions,* shortly $\circ$-*Bernstein functions.*

It is evident that the set $F(\circ)$ is a cone which does not depend upon the choice of the system of characteristic functions and is closed under the convergence that is uniform on every compact set.

PROPOSITION 4.1. *Let* $\{\mu_t\}$ *be an* $\circ$-*semigroup* (*of sub-p.m.'s*) *and* $\{\nu_t\}$ *a* $*_\alpha$-*semigroup* $(\alpha > 0)$. *Then the integral*

$$\tau_t = \int \mu_{s^\alpha}\nu_t(ds) \quad (t \geq 0)$$

defines an $\circ$-*semigroup.*

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Proof. We have, for $t, u \geq 0$,

$$\hat{\tau}_t(u) = \int \exp(-s^\alpha f(u))\nu_t(ds)$$
$$= \exp(-tg(f^{\alpha^{-1}}(u))),$$

f, g being generalized Bernstein functions associated with $\{\mu_t\}$ and $\{\nu_t\}$, respectively. □

As an immediate consequence of the above proposition we have

COROLLARY 4.2. *If $f \in F(\circ)$ and $g \in F(*_\alpha)$, then $g(f^{\alpha^{-1}}) \in F(\circ)$. In particular, if h is a Bernstein function, then $h(f)$ is an $\circ$-Bernstein function.*

The converse statement is also true. Namely, we have

PROPOSITION 4.3. *Let g be a function such that for every generalized convolution $\circ$ and for every $f \in F(\circ)$ the composite function $g(f^{\alpha^{-1}})$ belongs to $F(\circ)$. Then g is $*_\alpha$-Bernstein function.*

Proof. It follows from the fact that the function $f(x) = x^\alpha$ belongs to $F(*_\alpha)$. □

Let $\circ$ and $\circ'$ be regular generalized convolutions. Let us denote $G(\circ) = \{\hat{\mu} : \mu \in Q\}$, which is independent of the choice of the system of characteristic functions. Then we have the following inclusions:

$$G(*_\alpha) \subset G(\circ)$$
$$F(*_\alpha) \subset F(\circ) \tag{4.4}$$
$$S(*_\alpha) \subset S(\circ)$$

where $0 < \alpha \leq \kappa(\circ)$, $\kappa(\circ)$ being the characteristic exponent of $\circ$. Moreover, Theorem 2.2 in Urbanik [18] can be formulated as follows:

THEOREM 4.4. *If $G(\circ) = G(\circ')$, then $\circ = \circ'$.*

Similarly, we have the following:

THEOREM 4.5. *The following equalities are equivalent*:

(i) $\circ = \circ'$,

(ii) $F(\circ) \subset F(\circ')$,

(iii) $S(\circ) \subset S(\circ')$.

Proof. We shall prove that (ii) implies (i). Suppose that (ii) is true. Let Ω and Ω' be the kernels of $\circ$ and $\circ'$, respectively. By (4.2) there exist a', b' and m' such that

$$1 - \Omega(u) = a' + b'u^{\varkappa(\circ')} + \int (1 - \Omega'(ux))m'(dx).$$

Since $\Omega(0) = 1$ and $\Omega(u)$ is bounded, we have $a' = b' = 0$. Similarly, there is a measure m such that

$$1 - \Omega'(u) = \int (1 - \Omega(uy))m(dy).$$

Hence

$$\begin{aligned} 1 - \Omega(u) &= \int\int (1 - \Omega(uxy))m'(dx)m(dy) \\ &= \int (1 - \Omega(ux))H(dx), \end{aligned} \tag{4.5}$$

where

$$H(dx) = \int m'(dx/y)m(dy).$$

In particular, we have the equation

$$\begin{aligned} \int_0^{x_0} (1 - \Omega(x))H(dx) &= \int_0^{x_0} w(x)H(dx) \\ &\leq 1 - \Omega(1), \end{aligned} \tag{4.6}$$

where x_0 is the same as in (3.2). On the other hand, by formula (41) in Urbanik [16] and by Fatou's lemma

$$1 \geq \liminf_{t \to 0} \int \frac{1 - \Omega(tx)}{1 - \Omega(t)} H(dx) \geq \int x^{\varkappa(\circ)} H(dx).$$

Consequently, H is finite on every half-liine $[A, \infty)$ $(A > 0)$, which together with (4.6) implies that H satisfies the condition (4.3). Therefore, by (4.5) and by

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uniqueness of the representation (4.2), it follows that $H = \delta_1$ and consequently, $m' = b\delta_c$ for some positive b, c, which implies that

$$\Omega(u) = b\Omega'(cu) + 1 - b, \quad (u \geq 0). \tag{4.7}$$

Let p be a positive number less than $\min(\kappa(\circ), \kappa(\circ'))$. Let σ_p and σ_p' be $\circ$-stable and $\circ'$-stable measures, respectively, with the same exponent p (cf. Urbanik [16]). Integrating both sides of (4.7) with respect to σ_p and σ_p' and using Fubini's theorem, we get the equation

$$\int \exp(-y^p u^p)\sigma_p'(dy) = b\int \exp(-c^p x^p u^p)\sigma_p(dx) + 1 - b.$$

Notice that σ_p and σ_p' do not have point mass at 0 (cf. Urbanik [19] Lemma 2.2; the proof becomes simpler since our generalized convolutions are regular).

Letting $t \to 0$ in the last equation, we get $b = 1$ and $\Omega(u) = \Omega'(cu)$ $(u \geq 0)$. Consequently, $\circ = \circ'$ which completes the proof that (ii) implies (i). The proof that (iii) implies (i) is similar and is omitted. □

As a consequence of the above theorem we have the following characterization of α-convolutions:

THEOREM 4.6. *Let $0 < \alpha \leq \kappa(\circ)$. Then the equality $\circ = *_\alpha$ (and necessarily $\alpha = \kappa(\circ)$) holds if and only if, for any $\circ'$, $g \in F(\circ)$, and $f \in F(\circ')$, the composite function $g(f^{1/\alpha})$ belongs to $F(\circ')$.*

Proof. The "only if" part follows from Corollary 4.2. To prove the "if" part let us take g from $F(\circ)$, $\circ' = *_\alpha$, and $f(x) = x^\alpha$. By the assumption the composite function $g(f^{1/\alpha}) = g$ belongs to $F(*_\alpha)$, which implies $F(\circ) \subset F(*_\alpha)$ and, by Theorem 4.5, $\circ = *_\alpha$. □

We conclude this section by giving a sufficient condition for transience of $\circ$-Lévy processes.

THEOREM 4.7. *Suppose that the kernel Ω is nonnegative. Then every non-constant $\circ$-Lévy process on R_+ is transient.*

Proof. Let μ_t and f be the $\circ$-semigroup and the $\circ$-Bernstein function associated with a non-constant $\circ$-Lévy process $\{X_t\}$. Thus f is not identically zero. By Lemma 2.1 in Urbanik [20] the set of zeros of f has Lebesgue measure zero.

Further, for every continuous nonnegative function g on R_+ with compact support there exist positive constants a and b such $f(b) > 0$ and for every $u \geq 0$

$$g(u) \leq a\Omega(bu)$$

which implies that

$$\begin{aligned}\int E^x g(X_t)dt &\leq a\int E^x \Omega(bX_t)dt \\ &= a\int\int \Omega(bu)(\delta_x \circ \mu_t)(du)dt \\ &= a\Omega(bx)\int \exp(-tf(b))dt \\ &= a/f(b) < \infty.\end{aligned}$$

□

Remark 4.8. For some generalized convolution $\circ$, there exist non-constant recurrent $\circ$-Lévy processes. In such a case the kernel Ω must take negative values somewhere (see Kingman [10], Theorem 10, for a transience criterion for $*_{1,\beta}$-Lévy processes).

5. Self-similar $\circ$-i.i. processes

This section continuous the line of research of Lamperti [13] and Sato [15].

Consider an $\circ$-i.i. process $\{X_t\}$ on $\bar{R}_+$ with transition probability $P_{s,t}$ given by (2.2). We say that the process $\{X_t\}$ is *H-self-similar* $(H > 0)$, if it is H-self-similar as a Markov process, namely, if for any $a > 0$ and $x \in \bar{R}_+$ the finite-dimensional P^x-distributions of $\{X_t\}$ are identical with the finite-dimensional $P^{a^H x}$-distribution of $\{a^{-H}X_{at}\}$.

The following theorems stand for analogues of Sato's results [15]:

THEOREM 5.1. *If $\{X_t\}$ is an H-self-similar $\circ$-i.i. process, then for every t the P^0-distribution of X_t is $\circ$-decomposable.*

THEOREM 5.2. *Suppose that μ is an $\circ$-self-decomposable measure in $\bar{P}$ and $\mu \neq \delta_\infty$. Then for any $H > 0$ and $t_0 > 0$ there exists a unique H-self-similar $\circ$-i.i. process $\{X_t\}$ such that μ is the P^0-distribution of X_{t_0}. The uniqueness here is in the sense of finite-dimensional distributions.*

A natural question arises: What can be said about the P^x-distribution of X_t

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for $x > 0$? And, more generally, what can be said about the P^{ν}-distribution of X_t for $\nu \in \bar{P}$? The following theorem answers these questions and gives a characterization of α-convolutions by self-similarity.

THEOREM 5.3. *Let $\{X_t\}$ be H-self-similar $\circ$-i.i. process such that $\mu_{0,t} \neq \delta_\infty$ for every $t > 0$. Let $\nu \in \bar{P}$. Then, the P^{ν}-distribution of X_t is $\circ$-self-decomposable for every $t > 0$, if and only if ν is $\circ$-self-decomposable.*

Consequently, the following two statements are equivalent:

(i) *There exists an H-self-similar $\circ$-i.i. process $\{X_t\}$ and a point x $(0 < x < \infty)$ such that $\mu_{0,t} \neq \delta_\infty$ for every $t > 0$ and the P^{x}-distribution of X_t is $\circ$-self-decomposable for every $t > 0$.*

(ii) *$\circ$ is an α-convolution for some α $(0 < \alpha < \infty)$.*

A p.m. $\mu \in \bar{P}$ is said to be $\circ$-stable if, for any pair a, b in $(0, \infty)$, there exists $c \in (0, \infty)$ such that $T_a\mu \circ T_b\mu = T_c\mu$. If $\mu \in \bar{P}$ is $\circ$-stable, then $\mu = \delta_\infty$ or $\mu \in P$.

THEOREM 5.4. *Let $\{X_t\}$ be a non-constant $\circ$-Lévy process. Then it is self-similar if and only if the P^0-distribution of X_1 is $\circ$-stable. If the stable index is α, then the order H of self-similarity is α^{-1}.*

Proof of Theorem 5.1. Note that for any $t > 0$ and $x \in \bar{R}_+$ the P^x-distribution of X_t is equal to $\mu_{0,t} \circ \delta_x$. Hence and by H-self-similarity of the process we have, for every $c = \dfrac{s}{t} > 1$ and $a = c^{-H}$,

$$\begin{aligned}\mu_{0,t} &= \text{the } P^0\text{-distribution of } c^{-H}X_{ct}\\ &= T_a\mu_{0,s} = T_a\mu_{0,t} \circ T_a\mu_{t,s},\end{aligned}$$

which proves that the P^0-distribution of X_t is $\circ$-self-decomposable. □

Proof of Theorem 5.2. Suppose that μ is $\circ$-self-decomposable in $\bar{P}$. Then for any $0 \leq s < t$ there exist a unique p.m. $\mu_{s,t}$ from $\bar{P}$ such that

$$T_t\mu = T_s\mu \circ \mu_{s,t},$$

which implies the following equality

$$T_c\mu_{s,t} = \mu_{cs,ct}, \quad (0 \leq s < t,\ c < 0). \tag{5.1}$$

Then the family $\{\mu_{s,t}\}$ satisfies (2.1) and induces an $\circ$-i.i. process $\{Y_t\}$ with tran-

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sition probability (2.2). We claim that the process is 1-self-similar.

Denote the indicator function of a set B by 1_B. Given $x \in \bar{R}_+$, $a > 0$, $0 \leq t_1 < \cdots < t_n$ and $B = B_1 \times \cdots \times B_n$, B_j's being Borel subsets of $\bar{R}_+$, we have by virtue of (2.2) and (5.1),

$$
\begin{aligned}
&P^x(Y_{t_1} \in B_1,\ldots,Y_{t_n} \in B_n) = \\
&= \int^{-} P_{0,t_1}(x, dx_1)\cdots \int^{-} P_{t_{n-1},t_n}(x_{n-1}, dx_n)1_B(x_1,\ldots,x_n) \\
&= \int^{-} \mu_{0,t_1}\circ\delta_x(dx_1)\cdots \int^{-} \mu_{t_{n-1},t_n}\circ\delta_{x_{n-1}}(dx_n)1_B(x_1,\ldots,x_n) \\
&= \int^{-} T_a[\mu_{0,t_1}\circ\delta_x](adx_1)\cdots \int^{-} T_a[\mu_{t_{n-1},t_n}\circ\delta_{x_{n-1}}](adx_n)1_B(x_1,\ldots,x_n) \\
&= \int^{-} \mu_{0,\,at_1}\circ\delta_{ax}(adx_1)\cdots \int^{-} \mu_{at_{n-1},\,at_n}\circ\delta_{ax_{n-1}}(adx_n)1_B(x_1,\ldots,x_n) \\
&= \int^{-} \mu_{0,\,at_1}\circ\delta_{ax}(adx_1)\cdots \int^{-} \mu_{at_{n-1},\,at_n}\circ\delta_{ax_{n-1}}(dx_n)1_B(a^{-1}x_1,\ldots,a^{-1}x_n) \\
&= P^{ax}(a^{-1}Y_{at_1} \in B_1,\ldots,a^{-1}Y_{at_n} \in B_n).
\end{aligned}
$$

This shows that $\{Y_t\}$ is a 1-self-similar Markov process. Moreover, we have $\mu = \mu_{0,1}$ and, therefore, μ is the P^0-distribution of Y_1.

Now let H and t_0 be arbitrary positive numbers. Putting $X_t = Y_{t_0^{-H}t^H}$ we get a required process.

The uniqueness of $\{X_t\}$ follows from the fact that the transition probability $P_{s,t}$ is uniquely determined by μ. Namely, for any $s < t$ and $x \in \bar{R}_+$ we have

$$T_{(t_0/t)^{-H}}\mu\circ\delta_x = T_{(t_0/t)^{-H}}\mu\circ P_{s,t}(x,.). \qquad \square$$

Proof of Theorem 5.3. Suppose that $\{X_t\}$ is an H-self-similar $\circ$-i.i. process such that $\mu_{0,t} \neq \delta_\infty$ for every $t > 0$. By Theorem 5.1 the P^0-distribution $\mu_{0,t}$ of X_t is $\circ$-self-decomposable for every $t \geq 0$. If $\nu \in \bar{P}$, then the P^ν-distribution of X_t equals $\nu\circ\mu_{0,t}$. Let $\mu_{0,1}(R_+) = a$. Then $\mu_{0,t}(R_+) = a$ for every $t > 0$, since $\mu_{0,t} = T_{t^H}\mu_{0,1}$. We have $\mu_{0,t} \to a\delta_0 + (1-a)\delta_\infty$ as $t \to 0$. Hence $\nu\circ\mu_{0,t}$ is $\circ$-self-decomposable for every $t > 0$ if and only if ν is $\circ$-self-decomposable. In particular, if there exists a point x $(0 < x < \infty)$ such that the P^x-distribution of X_t is $\circ$-self-decomposable for every $t > 0$, then the p.m. δ_x must be decomposable in the sense that there exist p.m.'s τ_1, τ_2 other than δ_0 such that $\delta_x = \tau_1\circ\tau_2$, and hence the generalized convolution $\circ$ is an α-convolution $(0 < \alpha < \infty)$ by a theorem of Kucharczak [12]. Conversely, if $\circ$ is an α-convolution and the process is H-self-similar and $\circ$-i.i., then, for every $x \in \bar{R}_+$, the p.m. δ_x is $\circ$-self-decomposable and the P^x-distribution of X_t $(t > 0)$ is $\circ$-self-decomposable. □

Proof of Theorem 5.4. Suppose that $\{X_t\}$ is a non-constant $\circ$-Lévy process induced by an $\circ$-semigroup $\{\mu_t\}$. Then $\mu_t \neq \delta_\infty$ for every $t > 0$. If the process is H-self-similar, then $\mu_t = T_{t^H}\mu_1$ and $\mu_t(R_+) = 1$ for every $t > 0$, and hence μ_1 is $\circ$-stable of index H^{-1}. Conversely, if μ_1 is $\circ$-stable of index α, then the process is α^{-1}-self-similar, which is proved by argument similar to the proof of Theorem 5.1. □

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