J. Austral. Math. Soc. 19 (Series A) (1975), 371–375.

# NATURAL EXTENSIONS OF PROBABILITY MEASURE IN FUNCTION SPACE

R. Z. YEH

(Received 21 December 1971; revised 25 March 1974)

Communicated by C. C. Heyde

## 1. Background

Let $\{X_t\}_{t\in T}$ be a family of real ($R$) random variables defined on a probability space $(\Omega, \mathscr{A}, P)$ and having the ranges in a subset $S$ of $R$, that is, $X_t(\Omega) \subset S$ for all $t$. Let $X$ be the mapping of $\Omega$ into the function space $S^T$

$$X : \Omega \to S^T$$

such that for any $\omega \in \Omega$

$$[X(\omega)](t) = X_t(\omega)$$

We shall write $X = \{X_t\}_{t\in T}$ and call $X$ the random function arising from $\{X_t\}_{t\in T}$. It is well-known that any finite subfamily of $\{X_t\}_{t\in T}$ induces a "finite joint distribution" in $S^T$, and according to Kolmogorov (1933) these finite joint distributions can be simultaneously extended to a probability measure $P_0$ on the Borel class $\mathscr{B}_0$ of subsets of $S^T$. This extension is *natural* in the sense that for any $B \in \mathscr{B}_0$ $P_0(B)$ turns out to be exactly $P[X^{-1}(B)]$. The Kolmogorov extension $P_0$ has however a shortcoming in that its domain $\mathscr{B}_0$ is not broad enough to include many events of practical interest.

Following Kakutani (1943), Nelson (1959) has formulated a regular Borel measure $P_1$, which extends the Kolmogorov extension $P_9$ to the topological Borel class $\mathscr{B}_1$ containing $\mathscr{B}_0$ provided that $S$ is a compact subset of $R$. The Kakutani extension $P_1$ has the following *regularity* property: for any $B \in \mathscr{B}_1$

$$\sup_{F\subset B} P_1(F) = P_1(B) = \inf_{G\supset B} P_1(G)$$

where $F$ and $G$ respectively denote closed and open subsets of $S^T$. The definitions of $\mathscr{B}_1$ and $\mathscr{B}_0$ are given in section 2.

The purpose of this paper is to show by a simple well-known example of Doob (1953) that the Kakutani extension is not natural in that given $B \in \mathscr{B}_1$ $P_1(B)$

may not be the same as $P[X^{-1}(B)]$. In contrast the naturalness of Doob's extension (1937) will be revealed by a suitable formulation.

## 2. Doob's example

Let $(\Omega, \mathscr{A}, P)$ be the probability space where $\Omega = [0,1]$, $\mathscr{A}$ is the Borel subsets of $[0,1]$, and $P$ is the Lebesgue measure. Let $T = [0,1]$, and for each $t \in [0,1]$ let $X_t: \Omega \to R$ be defined by $X_t(\omega) = \delta_{t\omega}$ (Kronecker delta taking the value 1 if $\omega = t$ and 0 if $\omega \neq t$). Clearly, the resulting random function $X$ takes $\omega$ to a function on the interval $[0,1]$ which assumes 0 everywhere except at $t = \omega$, where it assumes 1. Thus, $X: \Omega \to S^T$ with $S = [0,1]$, a compact subset of $R$.

It is not difficult to see that all the finite joint distributions induced by $X$ are one-point distributions concentrated at the origins of finite dimensional Euclidean spaces. According to Nelson (1959) these finite joint distributions can now be extended to the regular Borel measyre $P_1$, whose domain is the *topological Borel class* $\mathscr{B}_1$, which is the sigma-algebra of subsets of $S^T$ generated by the topology (that is, all the open sets) generated by the (open) neighborhoods of the form

$$N = \{x \in S^T: a_i < x(t_i) < b_i\}$$

where $t_i \in T$, and $-\infty \leqq a_i < b_i < \infty$ are pairs of real numbers for $i = 1, 2, \cdots, n$ with $n$ finite but otherwise arbitrary. The *Borel class* $\mathscr{B}_0$ is simply the sigma-algebra of subsets of $S^T$ generated by the subsets $N$ described above. Clearly, $\mathscr{B}_1 \supset \mathscr{B}_0$ since $\mathscr{B}_1$ is generated by a larger collection of subsets of $S^T$.

## 3. Unnaturalness of Kakutani extension

The unnaturalness of $P_1$ is shown in regard to the simple Doob process described above. Specifically, we will show that the elementary event $\{\theta\}$ where $\theta$ is the zero function in $S^T$, that is, $\theta(t) \equiv 0$, receives $P_1$ measure 1 while $X^{-1}(\{\theta\})$ being the empty subset of $\Omega$ receives $P$ measure 0.

It suffices to show that every open set containing $\{\theta\}$ has $P_1$ measure 1, for then by the regularity of $P_1$ $P_1(\{\theta\})$ must be 1. Let $G$ be any open subset of $S^T$ containing $\{\theta\}$, then there must be an open neighborhood $N$ containing $\theta$ contained in $G$, $\theta \in N \subset G$. Now to contain $\theta$, $N$ must be of the form

$$N = \{x \in S^T: a_i < x(t_i) < b_i\}$$

where $a_i < 0 < b_i$ for $i = 1, 2, \cdots, n$. But since each finite joint distribution is concentrated at the origin, this means $P_1(N) = 1$; and consequently $P_1(G) = 1$, completing the proof that

$$P_1(B) \neq P[X^{-1}(B)]$$

for $B = \{\theta\}$.

https://doi.org/10.1017/S1446788700031578 Published online by Cambridge University Press

## 4. Formulation of Doob extension

Let $\{X_t\}_{t \in T}$ be a family of real random variables on $(\Omega, \mathscr{A}, P)$ constituting the random function $X: \Omega \to R^T$ (see section 1), and let $\mathscr{X} = X(\Omega)$ be the range of the random function $X$, then we have the following

PROPOSITION 1. *The Kolmogorov outer measure of* $\mathscr{X}$ *is* 1, *that is,*

$$\bar{P}_0(\mathscr{X}) = \inf_{B \supset \mathscr{X}} P_0(B) = 1$$

*where* $B$ *are members of* $\mathscr{B}_0$.

PROOF. In view of naturalness of Kolmogorov extension $P_0$ we have for any $B \supset \mathscr{X}$ and $B \in \mathscr{B}_0$

$$P_0(B) = P[X^{-1}(B)] = P(\Omega) = 1$$

and hence $\bar{P}_0(\mathscr{X}) = 1$.

DEFINITION 1. By the *Doob class* $\mathscr{B}_{\mathscr{X}}$ of subsets of $R^T$ relatibve to $\mathscr{X} \subset R^T$ we mean the totality of subsets of $R^T$ of the form

$$D = (B \cap \mathscr{X}) \cup H$$

where $B$ belongs to the Borel class $\mathscr{B}_0$ and $H$ is a subset of $\mathscr{X}^c = R^T - \mathscr{X}$.

PROPOSITION 2. A *Doob class* $\mathscr{B}_{\mathscr{X}}$ *relative to any* $\mathscr{X}$ *is a sigma-algebra containing the Borel class* $\mathscr{B}_0$.

PROOF. To see $\mathscr{B}_{\mathscr{X}} \supset \mathscr{B}_0$ it suffices to show that any $B \in \mathscr{B}_0$ can be expressed in the form

$$B = (B \cap \mathscr{X}) \cup (B \cap \mathscr{X}^c)$$

That $\mathscr{B}_{\mathscr{X}}$ is a sigma-algebra is just as obvious.

DEFINITION 2. We define the *Doob extension* $P_{\mathscr{X}}$ of Kolmogorov extension $P_0$ of finite joint distributions for $R^T$ as follows: for any $D \in \mathscr{B}_{\mathscr{X}}$

$$P_{\mathscr{X}}(D) = P_0(B)$$

where $D = (B \cap \mathscr{X}) \cup H$ with $B \in \mathscr{B}_0$ and $H \subset \mathscr{X}^c$.

Obviously $P_{\mathscr{X}}$ is an extension of $P_0$ since for any $B \in \mathscr{B}_0$ we have $B = (B \cap \mathscr{X}) \cup (B \cap \mathscr{X}^c)$ and $P_{\mathscr{X}}(B) = P_0(B)$. To see that $P_{\mathscr{X}}$ is well-defined we must show the following

PROPOSITION 3. *If* $D = (B' \cap \mathscr{X}) \cup H' = (B'' \cap \mathscr{X}) \cup H''$, *then*

$$P_0(B') = P_0(B'')$$

PROOF. We need only show $P_0(B' - B'') = P_0(B'' - B') = 0$. We will merely assume $P_0(B' - B'') > 0$ to derive a contradiction, the other assumption leading to a similar contradiction.

https://doi.org/10.1017/S1446788700031578 Published online by Cambridge University Press

Since by Proposition 1 $\inf_{B\supset\mathscr{X}} P_0(B) = 1$, and $\mathscr{B}_0$ is a sigma-algebra, there actually exists a $B_0 \in \mathscr{B}_0$ such that $B_0 \supset \mathscr{X}$ and $P_0(B_0) = 1$. Now from $H' = D - \mathscr{X} = H''$ follows $B' \cap \mathscr{X} = B'' \cap \mathscr{X}$ so that $B' - B'' \subset \mathscr{X}^c$, therefore

$$B_0 - (B' - B'') \supset \mathscr{X}$$

We will derive the contradiction by showing $P_0[B_0 - (B' - B'')] < 1$.

Now

$$P_0[B_0 - (B'-B'')] = P_0(B_0) - P_0[B_0 \cap (B'-B'')],$$

but

$$P_0[B_0 \cap (B'-B'')] = P_0(B'-B'') - P_0[B_0^c \cap (B'-B'')],$$

and

$$P_0[B_0^c \cap (B'-B'')] \leqq P_0(B_0^c) = 1 - P_0(B_0) = 0;$$

therefore $P_0[B_0 \cap (B'-B'')] = P_0(B'-B'') > 0$. Hence

$$P_0[B_0 - (B'-B'')] < P_0(B_0) = 1$$

and the proof is complete.

The following theorem shows the naturalness of Doob extension. Essentially, it is inherited from the naturalness of Kolmogorov extension.

THEOREM. *Let $X = \{X_t\}_{t\in T}$ be a real ($R$) random function defined on a probability space $(\Omega, \mathscr{A}, P)$. Let $\mathscr{X} = X(\Omega) \subset R^T$ and $\mathscr{B}_{\mathscr{X}}$ be the Doob class of subsets of $R^T$, and let $P_{\mathscr{X}}$ be the Doob extension of the Kolmogorov extension $P_0$ of finite joint distributions in $R^T$. Then for any $D \in \mathscr{B}_{\mathscr{X}}$, $P_{\mathscr{X}}(D) = P[X^{-1}(D)]$.*

PROOF. Let $D = (B \cap \mathscr{X}) \cup H$, where $B$ is a member of the Borel class $\mathscr{B}_0$ and $H \subset \mathscr{X}^c$. By Definition 2 and the naturalness of Kolmogorov extension

$$P_{\mathscr{X}}(D) = P_0(B) = P[X^{-1}(B)]$$

Now

$$X^{-1}(D) = X^{-1}(B \cap \mathscr{X}) \cup X^{-1}(H) = X^{-1}(B) \cap \Omega \cup \varnothing$$

hence

$$P[X^{-1}(D)] = P[X^{-1}(B)]$$

and

$$P_{\mathscr{X}}(D) = P[X^{-1}(D)]$$

Finally we add that Doob extension can be strengthened by replacing $\mathscr{X}$ in $\mathscr{B}_{\mathscr{X}}$ by $\mathscr{X}_0 = \mathscr{X}(\Omega_0)$ where $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$. This is due to the following

PROPOSITION 4. *Let $\mathscr{X}_0 \subset \mathscr{X} \subset R^T$, then the Doob class $\mathscr{B}_{\mathscr{X}_0}$ contains the Doob class $\mathscr{B}_{\mathscr{X}}$.*

https://doi.org/10.1017/S1446788700031578 Published online by Cambridge University Press

PROOF. To see $\mathscr{B}_{\mathscr{X}} \subset \mathscr{B}_{\mathscr{X}_0}$ let $E \in \mathscr{B}_{\mathscr{X}}$, then by Definition 1, $E = (B \cap \mathscr{X}) \cup H$ with $B \in \mathscr{B}_0$ and $H \subset \mathscr{X}^c$. Now

$$E = (B \cap \mathscr{X}_0) \cup [B \cap (\mathscr{X} - \mathscr{X}_0)] \cup H$$

but since $[B \cap (\mathscr{X} - \mathscr{X}_0)] \cup H$ is clearly contained in $\mathscr{X}_0^c$, t follows that $E \in \mathscr{B}_{\mathscr{X}_0}$.

## References

J. L. Doob (1953), *Stochastic Processes*, (John Wiley, New York).

J. L. Doob (1947), 'Probability in function space', *Bull. Amer. Math. Soc.* **53,** 15–30.

J. L. Doob (1937), Stochastic processes depending on continuous parameter, *Trans. Amer. Math. Soc.* **42,** 107–140.

Shizuo Kakutani (1943), Notes on infinite product spaces II, *Proc. Imp. Acad. Tokto* **19,** 184–188.

A. Kolmogorov (1933), (Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer Berlin).

Edward Nelson (1959), Regular probability mesaures on function space, *Ann. of Math.* (2) **69,** 630–643.

University of Hawaii
Honolulu
Hawaii 96822