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Characterizing Two-Dimensional Maps Whose Jacobians Have Constant Eigenvalues

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Abstract. Recent papers have shown that C^1 maps $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ whose Jacobians have constant eigenvalues can be completely characterized if either the eigenvalues are equal or F is a polynomial. Specifically, $F = (u, v)$ must take the form

$$u = ax + by + \beta\phi(\alpha x + \beta y) + e$$
$$v = cx + dy - \alpha\phi(\alpha x + \beta y) + f$$

for some constants $a, b, c, d, e, f, \alpha, \beta$ and a C^1 function ϕ in one variable. If, in addition, the function ϕ is not affine, then

$$\alpha\beta(d - a) + b\alpha^2 - c\beta^2 = 0. \tag{1}$$

This paper shows how these theorems cannot be extended by constructing a real-analytic map whose Jacobian eigenvalues are $\pm 1/2$ and does not fit the previous form. This example is also used to construct non-obvious solutions to nonlinear PDEs, including the Monge–Ampère equation.

1 Introduction

A C^1 function $f\colon k^2 \to k^2$ ($k = \mathbb{R}$ or $\mathbb{C}$) is *unipotent* if the eigenvalues of the Jacobian matrix $J(f)$ all equal one for all points in k. Under various scenarios, it has been shown that such functions are always invertible and can be classified explicitly. Chen [4] proved the case when $k = \mathbb{C}$ and f is holomorphic, Chamberland [2] when $k = \mathbb{R}$ and f is real-analytic, and most impressively, Campbell [1] when $k = \mathbb{R}$ and f is C^1. Specifically, we have

***Theorem 1.1* (Campbell)** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be C^1. Then $J(f)$ is unipotent if f is of the form*

$$f(x, y) = \bigl(x + b\phi(ax + by) + c,\ y - a\phi(ax + by) + d\bigr) \tag{2}$$

for some constants $a, b, c, d \in \mathbb{R}$ and some function ϕ of a single variable. If that is the case, then f has an explicit global inverse. Conversely, if f is C^1 and $J(f)$ is unipotent, then f is of the form above for a ϕ that is C^1.

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Of course, one may extend this result to maps whose Jacobians have equal eigenvalues. To see this, add a multiple of the identity map to the map (2) to obtain

$$(u, v) = \big(sx + b\phi(ax + by) + c, sy - a\phi(ax + by) + d\big),$$

whose Jacobian eigenvalues are both s. If $s \neq 0$, the map is invertible since $au + bv = s(ax + by) + ac + bd$. If $s = 0$, this forces $au + bv$ to be a constant, so the map is not surjective.

In a different setting, Cima *et al.* [5] considered stability questions surrounding the iteration of *polynomial* maps $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. They showed that if the eigenvalues of the Jacobian were always less the one in magnitude, then there exists a fixed point of f which is globally asymptotically stable. An important lemma they use is that since the eigenvalues are bounded, they must actually be constant. Though the authors pursue questions regarding stability, they essentially prove

Theorem 1.2 *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map. Then the eigenvalues of the $J(F)$ are constant if and only if $F = (u, v)$ takes the form*

$$u = ax + by + \beta\phi(\alpha x + \beta y) + e \tag{3}$$

$$v = cx + dy - \alpha\phi(\alpha x + \beta y) + f \tag{4}$$

for some constants a, b, c, d, e, f, α, β and a polynomial ϕ in one variable. If, in addition, the function ϕ is not affine, then

$$\alpha\beta(d - a) + b\alpha^2 - c\beta^2 = 0. \tag{5}$$

If the two eigenvalues are non-zero, then F has an explicit polynomial inverse.

From their proof, it is clear that F must take the form (3)–(4). If ϕ is not affine, the fact that the eigenvalues are constant forces condition (5) to hold.

It seems logical to try and extend Theorem 1.2 to C^1 maps, thus also generalizing Theorem 1.1. Unfortunately, such structure does not exist, as is demonstrated by the following result.

Theorem 1.3 *Let $F = (u, v)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$u = \frac{x}{2} + s \tag{6}$$

$$v = -\frac{y}{2} + \tan^{-1}(x + s) \tag{7}$$

where the function $s\colon \mathbb{R}^2 \to \mathbb{R}$ is defined implicitly by

$$0 = \tan^{-1}(x + s) - \tan^{-1}(s) + s - y. \tag{8}$$

The eigenvalues of this map's Jacobian are $\pm 1/2$, yet this map does not have the form (3)–(4) for any C^1 function ϕ. Lastly, F is globally invertible.

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These results are relevant with respect to Jacobian conjectures. It has been shown that if all unipotent polynomial functions are invertible in all dimensions, then the Keller Jacobian Conjecture is true, that is, any polynomial map from k^n to k^n whose Jacobian determinant is a non-zero constant is invertible; an excellent resource is the recent book of van den Essen [8].

Section 2 gives a proof of Theorem 1.3. Sections 3 and 4 show the limitations of earlier techniques to obtain this theorem and how special solutions to some nonlinear PDEs exist. Specifically, Section 3 considers the approaches of Campbell and Cima *et al.* and summarizes results for a class of Monge–Ampère equations, while Section 4 considers Chamberland's approach and proves the existence of non-obvious solutions to a planar PDE. Section 5 offers a new approach and shows how Theorem 1.3 is naturally produced.

2 Proof of Theorem 1.3

First note that the function s is well-defined since for each $(x, y) \in \mathbb{R}^2$ the function $m(s) := \tan^{-1}(x+s) - \tan^{-1}(s) + s - y$ is bijective in the variable s. The injectivity is shown by proving $m'(s) > 0$, while the the boundedness of the first two terms yields the surjectivity. The implicit function theorem guarantees that s is real-analytic. A straight-forward calculation using implicit differentiation verifies that the eigenvalues of $D(F)$ are $\pm 1/2$.

To prove that the function F cannot take the form (3)–(4), we argue by contradiction. Suppose there exists a C^1 function ϕ and constants $a, b, c, d, e, f, \alpha, \beta$ such that

$$\frac{x}{2} + s = ax + by + \beta\phi(\alpha x + \beta y) + e$$

$$-\frac{y}{2} + \tan^{-1}(x+s) = cx + dy - \alpha\phi(\alpha x + \beta y) + f$$

Setting $x = 0$ implies via (8) that $s = y$ and

$$y = by + \beta\phi(\beta y) + e \tag{9}$$

$$-\frac{y}{2} + \tan^{-1}(y) = dy - \alpha\phi(\beta y) + f \tag{10}$$

Multiplying (9) by α and (10) by β then adding gives

$$\alpha y + \beta\Big(-\frac{y}{2} + \tan^{-1}(y)\Big) = \alpha(by + e) + \beta(dy + f). \tag{11}$$

Equation (11) implies $\beta = 0$, which yields the desired contradiction in (10).

The easiest way to show that the function F is globally invertible is to construct the inverse. Solving for x in (6), use this with (7) to obtain

$$x = 2(u - s) \tag{12}$$

$$y = 2\big(\tan^{-1}(2u - s) - v\big) \tag{13}$$

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Substituting these into (8) yields

$$0 = n(s) := -\tan^{-1}(2u - s) - \tan^{-1}(s) + s + 2v.$$

As was done for the function m, one may show that n is bijective in s for any given u and v, therefore we have constructed the inverse function. This completes the proof.

An important recent result actually gives the injectivity of the map F very easily. Cobo *et al.* [6] have proven that any two-dimensional C^1 map whose Jacobian eigenvalues do not intersect the interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ must be injective. Since the Jacobian eigenvalues are $\pm 1/2$, the injectivity follows immediately. This general theorem also proves the so-called Chamberland conjecture [3] in dimension two.

3 Approaches of Campbell and Cima et al. and the Monge–Ampère Equation

We wish to consider the approaches used by Campbell and Cima *et al.* and see why their results cannot be extended. The latter approach allows us to use earlier results of this paper to state a general theorem for the Monge–Ampère equation.

The approach used by Campbell [1] to prove Theorem 1.1 is highly dependent on the fact that the eigenvalues of the Jacobian both equal one. He subtracts the identity map from the original map to give a new function whose Jacobian matrix is nilpotent. This plays a crucial role in the rest of the proof, therefore this proof admits no obvious modification if the eigenvalues differ.

The approach of Cima *et al.* [5] takes a totally different route. As mentioned earlier, restricting attention to polynomial maps whose Jacobian eigenvalues are bounded forces these eigenvalues to be constant. Supposing $F = (P, Q)$ is such a map, we have

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = t_1, \quad \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x} = t_2 \tag{14}$$

for some constants t_1 and t_2. Letting $(\bar{P}, \bar{Q}) = F - (t_1/2)I$ implies

$$\frac{\partial \bar{P}}{\partial x} + \frac{\partial \bar{Q}}{\partial y} = 0, \quad \frac{\partial \bar{P}}{\partial x}\frac{\partial \bar{Q}}{\partial y} - \frac{\partial \bar{P}}{\partial y}\frac{\partial \bar{Q}}{\partial x} = \bar{t}_2.$$

where $\bar{t}_2 = t_2 - t_1^2/4$. The first equation implies there is a function $H\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\bar{P} = -\frac{\partial H}{\partial y}, \quad \bar{Q} = \frac{\partial H}{\partial x}.$$

Putting this into the second equation yields

$$\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 = \bar{t}_2. \tag{15}$$

The authors then cite a result of Dillen [7] which states that *polynomials H* satisfying this equation with $\bar{t}_2 < 0$ must (up to a complex affine transformation) take the form

$$H(u, v) = \sqrt{-\bar{t}_2}\,uv + h(u) \tag{16}$$

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where h is a polynomial in one variable u. Dillen's proof exploits the fact that H is a polynomial and there is no obvious way to generalize this. We now compile several theorems together to obtain the following general result.

Theorem 3.1 *Let H be a C^2 function. The Monge–Ampère equation*

$$\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \Big(\frac{\partial^2 H}{\partial x \partial y}\Big)^2 = c$$

has solutions as follows:

c positive:
H is quadratic.

$c = 0$:
H takes the form $H(x, y) = \psi(ax + by) + cx + dy + e$ for some constants a, b, c, d, e and a C^2 function ψ in one variable.

c negative:
If H is a polynomial, H takes the form $H(x, y) = \sqrt{-c}xy + h(x)$ up to an affine transformation. There is a non-polynomial function H not of this form satisfying

$$\frac{\partial H}{\partial x} = -y + 2\tan^{-1}(x + s)$$
$$\frac{\partial H}{\partial y} = -x - 2s$$

where the function $s\colon \mathbb{R}^2 \to \mathbb{R}$ is defined implicitly by

$$0 = \tan^{-1}(x + s) - \tan^{-1}(s) + s - y.$$

Proof The case $c > 0$ is due to Jörgens [10]. For the case $c < 0$, the polynomial subcase is due to Dillen [7], while the non-polynomial example is simply Theorem 1.3 reworked for this setting (and normalized by a constant factor to yield $c = -1$). The only part left to prove is the case $c = 0$.

Suppose H satisfies

$$\frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2} - \Big(\frac{\partial^2 H}{\partial x \partial y}\Big)^2 = 0.$$

Let $u = x + H_y$ and $v = y - H_x$. Then

$$u_x + v_y = 2, \quad u_x v_y - u_y v_x = 1.$$

This implies that the map $f = (u, v)$ is C^1 and unipotent, so by Theorem 1.1, we have

$$H_y = b\phi(ax + by) + d, \quad H_x = -a\phi(ax + by) + c$$

for some constants a, b, c, d and some C^1 function ϕ. Integrating gives the desired result. ∎

As noted by Kusano and Swanson [11], there is virtually nothing known about solutions of Monge–Ampère equations in unbounded domains.

4 Chamberland's Approach

This section considers generalizing the approach of Chamberland [2] to maps where the two eigenvalues of $D(f)$, λ_1 and λ_2, are real and distinct. We shall assume that the map $f = (u, v)$ is real-analytic. By Schur's Theorem of matrix analysis (see, for example, [12, p. 308]), there exist real-analytic functions A and θ from $\mathbb{R}^2$ to $\mathbb{R}$ such that (using the abbreviation $s = \sin(\theta)$ and $c = \cos(\theta)$)

$$\begin{aligned}\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} &= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \lambda_1 & A \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 + s^2(\lambda_2 - \lambda_1) + csA & cs(\lambda_2 - \lambda_1) + c^2 A \\ cs(\lambda_2 - \lambda_1) - s^2 A & \lambda_1 + c^2(\lambda_2 - \lambda_1) - csA \end{bmatrix}.\end{aligned} \tag{17}$$

Note that although one may add any multiple of π to a solution θ, this function will be unique after it is specified at one point. Using the identities $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$, terms from (17) imply

$$\frac{\partial}{\partial y}\big(s^2(\lambda_2 - \lambda_1) + csA\big) = \frac{\partial}{\partial x}\big(cs(\lambda_2 - \lambda_1) + c^2 A\big)$$

$$\frac{\partial}{\partial y}\big(cs(\lambda_2 - \lambda_1) - s^2 A\big) = \frac{\partial}{\partial x}\big(c^2(\lambda_2 - \lambda_1) - csA\big)$$

which may be expanded as

$$(\lambda_2 - \lambda_1)2sc\theta_y + (c^2 - s^2)A\theta_y + csA_y = (\lambda_2 - \lambda_1)(c^2 - s^2)\theta_x - 2scA\theta_x + c^2 A_x \tag{18}$$

$$(\lambda_2 - \lambda_1)(c^2 - s^2)\theta_y - 2scA\theta_y - s^2 A_y = -(\lambda_2 - \lambda_1)2cs\theta_x - (c^2 - s^2)A\theta_x - csA_x \tag{19}$$

Multiplying (18) by $2sc$ and (19) by $c^2 - s^2$ then adding yields

$$(\lambda_2 - \lambda_1)\theta_y + s^2 A_y = -A\theta_x + csA_x. \tag{20}$$

Similarly, multiplying (18) by $c^2 - s^2$ and (19) by $2sc$ then subtracting yields

$$A\theta_y + csA_y = (\lambda_2 - \lambda_1)\theta_x + c^2 A_x. \tag{21}$$

Multiplying (20) by θ_x and (21) by θ_y then adding simplifies to

$$A(\theta_x^2 + \theta_y^2) = (cA_x - sA_y)(c\theta_y + s\theta_x). \tag{22}$$

Similarly, multiplying (20) by θ_y and (21) by θ_x then subtracting yields

$$(\lambda_2 - \lambda_1)(\theta_x^2 + \theta_y^2) = (cA_x - sA_y)(s\theta_y - c\theta_x). \tag{23}$$

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Since θ is real-analytic, so is $\theta_x^2 + \theta_y^2$, so either θ is a constant function (a trivial case), or $\theta_x^2 + \theta_y^2$ is never zero on an open set. On any open set where this is the case, (23) implies $s\theta_y - c\theta_x \neq 0$, so (22) and (23) may be combined to give

$$A = (\lambda_2 - \lambda_1)\frac{c\theta_y + s\theta_x}{s\theta_y - c\theta_x}. \tag{24}$$

Using this form of A in equation (23) simplifies to

$$0 = s(\theta_x\theta_{yy} - \theta_y\theta_{xy}) - c(\theta_x\theta_{xy} - \theta_y\theta_{xx}) \tag{25}$$

Solving such an equation is formidable, but it is tempting to think that θ must take the form

$$\theta = \phi(\alpha x + \beta y)$$

for some one-variable function ϕ and constants α and β. However, if this were the case, A must also take this form, forcing u and v to take the form

$$u = ax + by + m(\alpha x + \beta y) + e$$

$$v = cx + dy + n(\alpha x + \beta y) + f$$

for some constants $a, b, c, d, e, f, \alpha, \beta$ and one-variable functions m and n. Since the eigenvalues of the Jacobian are constant, we have $\alpha m' + \beta n'$ is constant. This forces (u, v) to take the form (3)–(4). Since the function from Theorem 1.3 is not of this form, we conclude with

Theorem 4.1 *There is a real-analytic solution $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ to the equation*

$$0 = (\theta_x\theta_{yy} - \theta_y\theta_{xy})\sin(\theta) - (\theta_x\theta_{xy} - \theta_y\theta_{xx})\cos(\theta)$$

which is not of the form $\phi(\alpha x + \beta y)$ for some constants α and β.

5 Generating Theorem 1.3

As seen in the last two sections, none of the three previous approaches hint at how to construct functions as seen in Theorem 1.3. This section presents a new approach which constructs the function in Theorem 1.3 naturally.

First, note that if the Jacobian of a C^1 map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ has constant eigenvalues which are complex, the map must be affine. This stems from the previously cited result of Jörgens [10] (see the $c > 0$ case of Theorem 3.1). Working backwards in the proof of Cima *et al.* (see the previous section), F is forced to be affine. We therefore assume that the eigenvalues are real. As in Cima *et al.*, having constant eigenvalues λ_1 and λ_2 of $D(F)$ implies

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \lambda_1 + \lambda_2$$

$$\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x} = \lambda_1\lambda_2.$$

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Letting $G = (\bar{P}, \bar{Q}) = F - \lambda_2 I$ gives

$$\frac{\partial \bar{P}}{\partial x} + \frac{\partial \bar{Q}}{\partial y} = k \tag{26}$$

$$\frac{\partial \bar{P}}{\partial x}\frac{\partial \bar{Q}}{\partial y} - \frac{\partial \bar{P}}{\partial y}\frac{\partial \bar{Q}}{\partial x} = 0$$

where $k = \lambda_1 - \lambda_2$. If $\bar{P}$ and $\bar{Q}$ are both constant functions, then (P, Q) is affine, so let us assume without loss of generality that $\nabla \bar{P}$ is non-zero at some point $(\bar{x}, \bar{y})$. A classical application of the implicit function theorem (see [13, Section 9.6]) allows us to write $\bar{Q} = g(\bar{P})$ for some C^1 function g in a neighbourhood of $(\bar{x}, \bar{y})$ (this same result was used by Campbell [1]). In the interest of constructing a function like that in Theorem 1.3, assume that g can be extended to the range of $\bar{P}$. We may also assume that g is not constant otherwise this leads to trivial cases. From equation (26) we have

$$\frac{\partial \bar{P}}{\partial x} + g'(\bar{P})\frac{\partial \bar{P}}{\partial y} = k. \tag{27}$$

Since this is a quasilinear PDE, the method of characteristics may be used here (this technique was used by Chamberland [2]). This gives a parametrization of the characteristics (with parameter t) as

$$\dot{x} = 1, \quad \dot{y} = g'(\bar{P}), \quad \dot{\bar{P}} = k. \tag{28}$$

The case $k = 0$, that is, $\lambda_1 = \lambda_2$, will not produce a function which we want by Theorem 1.1, so let let us suppose then that $k \neq 0$, that is, $\lambda_1 \neq \lambda_2$. At $t = 0$, let

$$x = 0, \quad y = s, \quad \bar{P} = h(s) \tag{29}$$

for a parameter s and some one-variable function h. The existence of a solution $\bar{P}$ is guaranteed by the Cauchy-Kowalewski theorem; see Garabedian [9]. Combining (28) and (29) forces $x = t$, and hence $\bar{P} = kx + h(s)$ and

$$0 = m(s) := \frac{1}{k} g\big(kx + h(s)\big) - \frac{1}{k} g\big(h(s)\big) + s - y. \tag{30}$$

For a function s to be implicitly defined globally, there must exist a unique s for each fixed $(x, y) \in \mathbb{R}^2$. This is accomplished if the function m is bijective in s. Perhaps the simplest way to satisfy this invertibility condition is to take $h(s) = s$, $g(z) = \tan^{-1}(z)$, $\lambda_1 = 1/2$ and $\lambda_2 = -1/2$ (which imply $k = 1$). This produces the function used in Theorem 1.3.

Obviously many other choices for the functions g, h and the constant k are possible to give the desired invertibility. Such functions show that the classification and explicit invertibility seen in the Theorems 1.1 and 1.2 are luxuries reserved for special classes of maps, so Jacobian Conjecture results, such as those of Cobo *et al.*, are truly valuable.

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