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ELLIPTIC UNITS AND CLASS FIELDS OF GLOBAL FUNCTION FIELDS

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ABSTRACT. Elliptic units of global function fields were first studied by D. Hayes in the case that $\deg\infty$ is assumed to be 1, and he obtained some class number formulas using elliptic units. We generalize Hayes' results to the case that $\deg\infty$ is arbitrary.

0. **Introduction.** Let K be a global function field over a finite field $\mathbb{F}_q$. Let ∞ be a fixed place of degree δ, and A the subring of K consisting of those elements which are regular outside ∞. For a nontrivial character Ψ of $\operatorname{Pic} A$ the value $L_K(0,\Psi)$ can be expressed using the invariants $\xi(\mathfrak{c})$ of ideals $\mathfrak{c}$ of A. (See Hayes [5] for the case $\delta = 1$ and Gross and Rosen [2] for arbitrary δ.)

In this note we define elements $\langle\mathfrak{a} \mid \mathfrak{b}\rangle$ and $[\mathfrak{a} \mid \mathfrak{b}]$ for some pair of ideals $\mathfrak{a}$ and $\mathfrak{b}$ which generalize those in [4] for the case $\delta = 1$. Then we show that $[\mathfrak{a} \mid A]$ (resp. $\langle\mathfrak{a} \mid A\rangle$) not only lies in the Hilbert class field H_A (resp. normalizing field $\tilde{H}_A$) of A, but also generate the extension H_A (resp. $\tilde{H}_A$) over K. This is nothing but the analogue of the fact that the ring class field of an imaginary quadratic field is generated by the quotient $\Delta(\mathfrak{a})/\Delta(R)$ of discriminant functions ([10]). Finally using the elliptic units we get class number formulas generalizing those obtained by Hayes in [5]. Oukhaba ([7], [8], [9]) also studied the elliptic units of function fields assuming that ∞ is totally split.

1. **Preliminaries.** By an elliptic A-module we mean a Drinfeld module of rank one on A. Let H_A be the *Hilbert class field* of A as defined in [3]. Let K_∞ be the completion of K at ∞ and C the completion of the algebraic closure of K_∞. Then H_A is the smallest extension field of K with the property that every elliptic A-module defined over C is isomorphic to an elliptic A-module defined over H_A. We denote by $\operatorname{Pic} A$ the group of all the isomorphism classes of fractional ideals of A and h_A its order. Let h_K be the class number of the field K. Then $h_A = h_K\delta$. Denote by $\kappa(\infty)$ the residue field at ∞.

Let ρ be an elliptic A-module. We say that ρ is *normalized* if the leading coefficient $s_\rho(x)$ of ρ_x belongs to $\kappa(\infty)$ for any $x \in A\setminus\{0\}$. Fix a sign function $\operatorname{sgn}\colon K_\infty^* \longrightarrow \kappa(\infty)^*$. We say that an elliptic A-module ρ is sgn-*normalized* if ρ is normalized and s_ρ is equal to a twisting of sgn. Then every elliptic A-module is isomorphic to a sgn-normalized elliptic A-module. For details see [6]. Let Γ be an A-lattice of rank 1 in C. We say that an A-lattice Γ is *special* if its associated elliptic A-module ρ^Γ is sgn-normalized. For an

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A-lattice Γ in C define $\xi(\Gamma)$ to be an element of C^* so that $\xi(\Gamma)\Gamma$ is special. Then $\xi(\Gamma)$ is determined up to the multiplication by elements of $\kappa(\infty)^*$.

For an integral ideal $\mathfrak{a}$ of A, let $\rho_{\mathfrak{a}}$ be the $\mathfrak{a}$-isogeny defined in [3]. Then the elliptic module $\mathfrak{a} * \rho$ is defined to be the unique elliptic module satisfying $(\mathfrak{a} * \rho)_x \cdot \rho_{\mathfrak{a}} = \rho_{\mathfrak{a}} \cdot \rho_x$. Then we have the following lemma whose proof is straightforward.

LEMMA 1.1. *i) For $x \in R$, we have $(x) * \rho = s_\rho(x)^{-1} \rho s_\rho(x)$.*
ii) $(\omega^{-1}\rho\omega)_{\mathfrak{a}} = \omega^{-q^{\deg \mathfrak{a}}} \rho_{\mathfrak{a}} \omega$, for any $\omega \in C$ and any integral ideal $\mathfrak{a}$ of A.
iii) $s_{\mathfrak{a}*\rho} = \sigma^{\deg \mathfrak{a}} \circ s_\rho$, where σ is the qth power map and $\mathfrak{a}$ is an ideal of A.

LEMMA 1.2. *Let ρ_1 and ρ_2 be two isomorphic* sgn-*normalized elliptic A-modules. Then*

$$s_{\rho_1} = s_{\rho_2}.$$

PROOF. Pick $c \in C^*$ such that $\rho_2 = c^{-1}\rho_1 c$. Then $c^{q^\delta - 1} \in \kappa(\infty)^*$. Write $a = c^{q^\delta - 1}$. Then $s_{\rho_2}(x) = a^{\deg x/\delta} s_{\rho_1}(x)$. Since their corresponding sign functions are the same, a must be 1 by Lemma 4.2 of [6].

LEMMA 1.3. *For each elliptic A-module ρ there exist exactly $\frac{q^\delta - 1}{q-1}$ distinct* sgn-*normalized elliptic A-modules which are isomorphic to ρ.*

PROOF. Let ρ be a sgn-normalized elliptic A-module. For each $\alpha \in \kappa(\infty)^*$, $\alpha^{-1}\rho\alpha$ is sgn-normalized. From the proof of the above lemma any sgn-normalized elliptic A-module isomorphic to ρ is of this form. Now the result follows from the fact that $\alpha^{-1}\rho\alpha = \beta^{-1}\rho\beta$ if and only if $\alpha/\beta \in \mathbb{F}_q^*$.

Let ρ be a sgn-normalized elliptic A-module. Then there exists $w \in C^*$ such that $\rho' = w\rho w^{-1}$ is defined over H_A. Then $w^{q^\delta - 1} \in H_A$. Let $w_0 = w^{q-1}$, and $\tilde{H}_A = H_A(w_0)$. We call $\tilde{H}_A$ the *normalizing field* with respect to $(A, \mathrm{sgn}, \infty)$. Then every elliptic A-module over C is isomorphic to a sgn-normalized module defined over $\tilde{H}_A$. Let $\widetilde{\mathrm{Pic}}\, A$ be the quotient group of the group of fractional ideals modulo the subgroup of principal ideals generated by an element $x \in K$ with $\mathrm{sgn}(x) = 1$.

THEOREM 1.4 ([6] SECTION 4). *i) $\mathrm{Gal}(\tilde{H}_A/K)$ is isomorphic to $\widetilde{\mathrm{Pic}}\, A$, and*

$$[\tilde{H}_A : K] = \frac{q^\delta - 1}{q - 1} \cdot h_A.$$

ii) $\tilde{H}_A/K$ is unramified at any finite places.
iii) $\tilde{H}_A/H_A$ is totally ramified at ∞ with the inertia group isomorphic to $\kappa(\infty)^*/\mathbb{F}_q^*$.
iv) A finite place $\mathfrak{p}$ splits completely in $\tilde{H}_A/K$ if and only if $\mathfrak{p} = xA$ with $\mathrm{sgn}(x) \in \mathbb{F}_q^*$.
v) Let $\tilde{B}$ be the integral closure of A in $\tilde{H}_A$. Then for a sgn-*normalized elliptic A-module ρ and an ideal $\mathfrak{a}$ of A, the extended ideal $\mathfrak{a}\tilde{B}$ is a principal ideal and generated by the constant term $D(\rho_{\mathfrak{a}})$ of $\rho_{\mathfrak{a}}$.*

Let $\mathfrak{m}$ be an ideal of A and ρ a sgn-normalized module. Let $\Lambda_{\mathfrak{m}}$ be the set of $\mathfrak{m}$-torsion points of ρ. Put $\tilde{K}_{\mathfrak{m}} = \tilde{H}_A(\Lambda_{\mathfrak{m}})$ be the field generated by $\mathfrak{m}$-torsion points of ρ over $\tilde{H}_A$.

THEOREM 1.5 ([6] SECTION 4). *i)* $\tilde{K}_{\mathfrak{m}}$ *is abelian over* K*, and independent of the choice of the* sgn*-normalized module.*

ii) $\mathrm{Gal}(\tilde{K}_{\mathfrak{m}}/\tilde{H}_A) \simeq (A/\mathfrak{m})^*$.

iii) Let $\lambda \in \Lambda_{\mathfrak{m}}$ *and* $\sigma_{\mathfrak{a}}$ *be the Artin automorphism of* $\mathrm{Gal}(\tilde{K}_{\mathfrak{m}}/K)$ *associated to the ideal* $\mathfrak{a}$*. Then*

$$\lambda^{\sigma_{\mathfrak{a}}} = \rho_{\mathfrak{a}}(\lambda).$$

iv) Let G_∞ *be the decomposition group of* $\tilde{K}_{\mathfrak{m}}/K$ *at* ∞*. Then* G_∞ *is the inertia group at* ∞ *and isomorphic to* $\kappa(\infty)^*$.

v) Let $H_{\mathfrak{m}}$ *be the fixed field of* $\tilde{K}_{\mathfrak{m}}$ *under* G_∞ *and* $N^-_{\mathfrak{m}}\colon \tilde{K}_{\mathfrak{m}} \longrightarrow H_{\mathfrak{m}}$ *be the corresponding norm map. Then* $N^-_{\mathfrak{m}}(\tilde{K}^*_{\mathfrak{m}})$ *consists of totally positive elements. Here an element* x *is said to be totally positive if* $\mathrm{sgn}\big(\sigma(x)\big) = 1$*, for any automorphism* σ *over* K.

vi) For $\lambda \in \Lambda_{\mathfrak{m}}$ *and* $\sigma \in \mathrm{Gal}(\tilde{K}_{\mathfrak{m}}/K)$*,* $\lambda^{\sigma-1}$ *is a unit in the ring of integers of* $\tilde{H}_{\mathfrak{m}} = \tilde{H}_A H_{\mathfrak{m}}$*, the fixed field of* $\mathbb{F}_q^* \subset \mathrm{Gal}(\tilde{K}_{\mathfrak{m}}/\tilde{H}_A)$.

2. **Elliptic units.** We know that $\mathrm{Gal}(\tilde{H}_A/K)$ acts transitively on the set S of all the sgn-normalized elliptic A-modules via $\rho \longmapsto \rho^\sigma$, for $\sigma \in \mathrm{Gal}(\tilde{H}_A/K)$. Now fix a sgn-normalized elliptic A-module ρ from $\frac{q^\delta-1}{q-1}$ sgn-normalized elliptic A-modules associated to the lattice A. Then the map $\sigma \longmapsto \rho^\sigma$ sets up a one-to-one correspondence between $\mathrm{Gal}(\tilde{H}_A/K)$ and S. If we identify $\widetilde{\mathrm{Pic}}\,A$ with $\mathrm{Gal}(\tilde{H}_A/K)$ via the Artin map $\mathfrak{a} \longmapsto \tau_{\mathfrak{a}}$, then it is shown in [3] that $\rho^{\tau_{\mathfrak{a}}} = \mathfrak{a} * \rho$ for integral ideals $\mathfrak{a}$ of A. One can define $\mathfrak{a} * \rho$ for any fractional ideal $\mathfrak{a}$ of A from this property. This sets up a one-to-one correspondence between $\widetilde{\mathrm{Pic}}\,A$ and S.

For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{a}$ integral, we define

$$\langle \mathfrak{a} \mid \mathfrak{b} \rangle = D(\rho_{\mathfrak{a}}^{\tau^{-1}_{\mathfrak{a}\mathfrak{b}}}) = D\Big(\big((\mathfrak{a}\mathfrak{b}) * \rho\big)_{\mathfrak{a}}\Big),$$

and

$$[\mathfrak{a} \mid \mathfrak{b}] = \langle \mathfrak{a} \mid \mathfrak{b} \rangle^{\frac{q^\delta-1}{q-1}},$$

where $D(\rho_{\mathfrak{a}})$ is the constant term of $\rho_{\mathfrak{a}}$.

PROPOSITION 2.1. *i)* $\langle \mathfrak{a} \mid \mathfrak{b} \rangle \in \tilde{H}_A$ *and generates the ideal* $\mathfrak{a}\tilde{B}$*.*

ii) For $x \in K$*, we have*

$$\langle \mathfrak{a} \mid x\mathfrak{b} \rangle = s_{(\mathfrak{a}\mathfrak{b})^{-1}*\rho}(x)^{q^{\deg \mathfrak{a}}-1} \langle \mathfrak{a} \mid \mathfrak{b} \rangle,$$

and

$$[\mathfrak{a} \mid x\mathfrak{b}] = [\mathfrak{a} \mid \mathfrak{b}].$$

iii) If $\mathfrak{c}$ *is an integral ideal, then*

$$\langle \mathfrak{a}\mathfrak{c} \mid \mathfrak{b} \rangle = \langle \mathfrak{a} \mid \mathfrak{b} \rangle \langle \mathfrak{c} \mid \mathfrak{a}\mathfrak{b} \rangle.$$

iv) For an ideal $\mathfrak{c}$*,*

$$\langle \mathfrak{a} \mid \mathfrak{b} \rangle^{\tau_{\mathfrak{c}}} = \langle \mathfrak{a} \mid \mathfrak{b}\mathfrak{c}^{-1} \rangle.$$

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v) $[\mathfrak{a} \mid \mathfrak{b}]$ *lies in* H_A^*, *in fact,* $[\mathfrak{a} \mid \mathfrak{b}] = N_{\tilde{H}_A/H_A}(\langle \mathfrak{a} \mid \mathfrak{b} \rangle)$.

vi) If $x \in \mathfrak{a}^{-1}$, *then*

$$\langle x\mathfrak{a} \mid \mathfrak{b} \rangle = \frac{x}{s_{(\mathfrak{a}\mathfrak{b})^{-1}*\rho}(x)} \langle \mathfrak{a} \mid \mathfrak{b} \rangle,$$

and

$$[x\mathfrak{a} \mid \mathfrak{b}] = \bar{x}^{\frac{q^\delta - 1}{q-1}} [\mathfrak{a} \mid \mathfrak{b}],$$

where $\bar{x} = \frac{x}{\operatorname{sgn}(x)}$.

vii) Let $\mathfrak{P}$ *be a prime ideal of* H_A *and* $\tau_{\mathfrak{P}}$ *be the Artin automorphism in* $\operatorname{Gal}(\tilde{H}_A/H_A)$ *associated to the ideal* $\mathfrak{P}$. *Let* $x_{\mathfrak{P}} \in A$ *be any generator of the principal ideal* $N(\mathfrak{P})$ *of* A. *Then*

$$\langle \mathfrak{a} \mid \mathfrak{b} \rangle^{\tau_{\mathfrak{P}}} = s_{(\mathfrak{a}\mathfrak{b})^{-1}*\rho}(x_{\mathfrak{P}})^{1-q^{\deg \mathfrak{a}}} \langle \mathfrak{a} \mid \mathfrak{b} \rangle,$$

and

$$[\mathfrak{a} \mid \mathfrak{b}]^{\tau_{\mathfrak{P}}} = [\mathfrak{a} \mid \mathfrak{b}].$$

PROOF. i) is clear from definition. ii) follows from

$$\begin{aligned} \big((x\mathfrak{a}\mathfrak{b})^{-1} * \rho\big)_{\mathfrak{a}} &= \Big((x^{-1}) * \big((\mathfrak{a}\mathfrak{b})^{-1} * \rho\big)\Big)_{\mathfrak{a}} \\ &= s_{(\mathfrak{a}\mathfrak{b})^{-1}*\rho}(x)^{q^{\deg \mathfrak{a}}} \big((\mathfrak{a}\mathfrak{b})^{-1} * \rho\big)_{\mathfrak{a}} s_{(\mathfrak{a}\mathfrak{b})^{-1}*\rho}(x)^{-1}. \end{aligned}$$

Since

$$\begin{aligned} \big((\mathfrak{a}\mathfrak{b}\mathfrak{c})^{-1} * \rho\big)_{\mathfrak{a}\mathfrak{c}} &= \big(\mathfrak{c} * (\mathfrak{a}\mathfrak{b}\mathfrak{c})^{-1} * \rho\big)_{\mathfrak{a}} \big((\mathfrak{a}\mathfrak{b}\mathfrak{c})^{-1} * \rho\big)_{\mathfrak{c}} \\ &= \big((\mathfrak{a}\mathfrak{b})^{-1} * \rho\big)_{\mathfrak{a}} \big((\mathfrak{a}\mathfrak{b}\mathfrak{c})^{-1} * \rho\big)_{\mathfrak{c}}, \end{aligned}$$

we get iii). iv) follows from

$$\big((\mathfrak{a}\mathfrak{b})^{-1} * \rho\big)_{\mathfrak{a}}^{\tau_{\mathfrak{c}}} = \big(\mathfrak{c} * (\mathfrak{a}\mathfrak{b})^{-1} * \rho\big)_{\mathfrak{a}} = \big((\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-1})^{-1} * \rho\big)_{\mathfrak{a}}.$$

v) follows from the properties ii) and iv). The first statement of vi) follows easily from the definitions and Lemma 1.1. For the second statement, let $s_{(\mathfrak{a}\mathfrak{b})^{-1}*\rho}(x) = \operatorname{sgn}(x)^{q^i}$, for some i. Then

$$\big(\operatorname{sgn}(x)^{q^i}\big)^{\frac{q^\delta-1}{q-1}} = \big(\operatorname{sgn}(x)^{q^\delta - 1}\big)^{\frac{q^i-1}{q-1}} \big(\operatorname{sgn}(x)\big)^{\frac{q^\delta-1}{q-1}} = \operatorname{sgn}(x)^{\frac{q^\delta-1}{q-1}},$$

since $\operatorname{sgn}(x)^{q^\delta - 1} = 1$. vii) follows easily from the fact that $\tau_{\mathfrak{P}}(\rho) = s_\rho(x_{\mathfrak{P}})^{-1} \rho s_\rho(x_{\mathfrak{P}})$ ([6], Proposition 4.7).

For an ideal $\mathfrak{a}$ of A one can define the invariant $\xi(\mathfrak{a})$ to be an element of C^* such that $\xi(\mathfrak{a})\mathfrak{a}$ is the lattice associated to the elliptic A-module $\mathfrak{a}^{-1} * \rho$. Then this $\xi(\mathfrak{a})$ is well-defined up to the multiplication by $\mathbb{F}_q^*$. Fix $\xi(A)$ from $q-1$ possible values so that $\xi(A)A$ is the lattice associated to the elliptic module ρ and define $\eta(A) = \xi(A)^{\frac{q^\delta-1}{q-1}}$. We can fix $\xi(\mathfrak{a})$ (resp. $\eta(\mathfrak{a})$) to be the element of C^* such that

$$\frac{\xi(A)}{\xi(\mathfrak{a})} = \langle \mathfrak{a} \mid A \rangle \quad \Big(\text{resp. } \frac{\eta(A)}{\eta(\mathfrak{a})} = [\mathfrak{a} \mid A]\Big),$$

for each ideal $\mathfrak{a}$ of A.

PROPOSITION 2.2. *We have*

$$\frac{\xi(\mathfrak{a})}{\xi(\mathfrak{a}\mathfrak{b})} = \langle \mathfrak{b} \mid \mathfrak{a} \rangle \quad (\textit{resp. } \frac{\eta(\mathfrak{a})}{\eta(\mathfrak{a}\mathfrak{b})} = [\mathfrak{b} \mid \mathfrak{a}]),$$

and

$$\left(\frac{\xi(\mathfrak{a})}{\xi(\mathfrak{b})}\right)^{\tau_{\mathfrak{c}}} = \frac{\xi(\mathfrak{a}\mathfrak{c}^{-1})}{\xi(\mathfrak{b}\mathfrak{c}^{-1})} \quad (\textit{resp. } \left(\frac{\eta(\mathfrak{a})}{\eta(\mathfrak{b})}\right)^{\tau_{\mathfrak{c}}} = \frac{\eta(\mathfrak{a}\mathfrak{c}^{-1})}{\eta(\mathfrak{b}\mathfrak{c}^{-1})}).$$

Let N be a subgroup of $G = \mathrm{Gal}(H_A/K)$ of order n. Let $L = H_A^N$ and q_L be the number of constants in L. Define I_N to be the group of ideals $\mathfrak{a}$ of A with associated Artin automorphism $\tau_{\mathfrak{a}} \in N$, P_N the G-submodule of H_A^* generated by $\eta(A)/\eta(\mathfrak{a})$ with $\mathfrak{a} \in N$, and $E_N = P_N \cap B^*$, where B is the integral closure of A in H_A. Put $P = P_G$ and $E = E_G$. We call the elements of P_N the *elliptic numbers of level* N and the elements of E_N the *elliptic units of level* N. The map $\mathfrak{a} \longmapsto \tau_{\mathfrak{a}}^{-1}: I_N \longrightarrow G$ makes H_A^* into an I_N-module. Define

$$f_N: I_N \longrightarrow P_N$$

by $f_N(\mathfrak{a}) = \eta(R)/\eta(\mathfrak{a})$. Then it is easy to see that $f_N(\mathfrak{a}\mathfrak{b}) = f_N(\mathfrak{a})f_N(\mathfrak{b})^{\tau_{\mathfrak{a}}^{-1}}$. Let

$$M = \{x^{\frac{q^{\delta}-1}{q-1}} : x \in K^*\}.$$

It is clear from the definition that M is a subgroup of K^* and contained in P_N for every N. Then it is not hard to see that $E_N \cap M = \{1\}$ and so the natural map $E_N \longrightarrow P_N/M$ is injective. Let S_N be a set of $n-1$ prime ideals of A which maps bijectively onto $N \setminus \{1\}$ via the Artin map, and P_N' be the subgroup of P_N generated by $f_N(\mathfrak{p})$ with $\mathfrak{p} \in S_N$. The following are simple generalizations of those given in [5];

N1. For $\pi \in P_N$ and $\sigma \in G$, $\pi^{\sigma-1} \in E_N$, and so the composition f_N^* of f_N with the natural map $P_N \longrightarrow P_N/E_N$ is a group homomorphism.

N2. $P_N = P_N' M E_N$ and $P = P_G' M$.

N3. P_N' is a free group freely generated by $f_N(\mathfrak{p})$, $\mathfrak{p} \in S_N$.

N4. $P_N/ME_N \simeq N$.

N5. The elliptic numbers are totally positive, and so $P \cap \mathbb{F}_{q^{\delta}}^* = E \cap \mathbb{F}_{q^{\delta}}^* = \{1\}$ and $P \cap K^* = M$.

N6. Each element of $P^{\sigma-1}$ is the $(q-1)$-st power of a unit in H_A for any $\sigma \in G$.

The proofs are mostly the same as in [5], so we only prove N6. Let $\mathfrak{p}$ be a prime ideal of A. Let λ be any root of $\rho_{\mathfrak{p}}^{\tau_{\mathfrak{p}}^{-1}}$. Let $\tilde{N}_{\mathfrak{p}}: \tilde{K}_{\mathfrak{p}} \longrightarrow \tilde{H}_A$, $N_{\mathfrak{p}}: \tilde{K}_{\mathfrak{p}} \longrightarrow H_A$, $N_{\mathfrak{p}}^-: \tilde{K}_{\mathfrak{p}} \longrightarrow \tilde{H}_{\mathfrak{p}}$, $N_{\mathfrak{p}}^+: \longrightarrow H_A$, and $N: \tilde{H}_A \longrightarrow H_A$ be the norm maps. Then from the definition,

$$\langle \mathfrak{p} \mid A \rangle = \tilde{N}_{\mathfrak{p}}(\lambda).$$

From v) of Proposition 2.1 we have $[\mathfrak{p} \mid A] = N(\langle \mathfrak{p} \mid A \rangle)$. Thus

$$f(\mathfrak{p}) = N\big(\tilde{N}_{\mathfrak{p}}(\lambda)\big) = N_{\mathfrak{p}}(\lambda) = N_{\mathfrak{p}}^+\big(N_{\mathfrak{p}}^-(\lambda)\big) = N_{\mathfrak{p}}^+(\lambda^{q-1}).$$

Hence $f(\mathfrak{p})^{\sigma-1} = N_{\mathfrak{p}}^+(\lambda^{\sigma-1})^{q-1}$, since $\lambda^{\sigma-1}$ lies in $\tilde{H}_{\mathfrak{p}}$. Therefore N6 follows.

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3. $v_\infty\big(\xi(\mathfrak{c})\big)$ **and the value of** L**-function at** 0 **and generators of class fields.** Fix a valuation v_∞ on C extending the normalized valuation of K at ∞. For an integral ideal $\mathfrak{c}$ of A define the partial zeta function

$$\zeta_{\mathfrak{c}}(s) = \sum_{x\in\mathfrak{c}} |x|_\infty^{-s}.$$

Put $S = q^{-s}$. Then

$$\zeta_{\mathfrak{c}}(s) = Z_{\mathfrak{c}}(S) = \sum_{x\in\mathfrak{c}} S^{\deg x}.$$

It is shown in ([1], (4.10)) that

$$v_\infty\big(\xi(\mathfrak{c})\big) = -Z'_{\mathfrak{c}}(1)/\delta.$$

Now we are going to evaluate $Z'_{\mathfrak{c}}(1)$ for any integral ideal $\mathfrak{c}$ of A with degree c.

For each integer i we define

$$i^* = inf\{n : n \geq i, n \equiv 0(\delta)\}$$

and

$$i_* = sup\{n : n \leq i, n \equiv 0(\delta)\}.$$

Let $m = m_{\mathfrak{c}} = (c + 2g - 1)^*$ and $n = n_{\mathfrak{c}} = 1 - g + m - c$, where g is the genus of the smooth curve associated to K. Let

$$\ell(\mathfrak{c}) = -\sum_{t=0}^{\frac{m-c_*}{\delta}} t\delta|F_t(\mathfrak{c})|,$$

where $F_t(\mathfrak{c}) = \{x \in \mathfrak{c} : \deg x = t\delta + c_*\}$. Using the equation (2.5), Chapter III of [1],

$$Z'_{\mathfrak{c}}(1) = -\ell(\mathfrak{c}) - c_* - m_{\mathfrak{c}} q^{n_{\mathfrak{c}}} + \frac{\delta q^{n_{\mathfrak{c}}}}{q^\delta - 1}.$$

Therefore we get

PROPOSITION 3.2. *We have*

$$\delta v_\infty\big(\xi(\mathfrak{c})\big) = \ell(\mathfrak{c}) + c_* + m_{\mathfrak{c}} q^{n_{\mathfrak{c}}} - \frac{\delta q^{n_{\mathfrak{c}}}}{q^\delta - 1}.$$

Now let Ψ be a nontrivial character of $\mathrm{Gal}(H_A/K)$. Then we can view Ψ as a function on the ideals $\mathfrak{a}$ of A. Let

$$L_A(s,\Psi) = \prod_{\mathfrak{p}\text{ prime}} \left(1 - \frac{\Psi(\mathfrak{p})}{N(\mathfrak{p})^{-s}}\right)^{-1}.$$

Then $L_A(s,\Psi) = (1 - q^{-\delta s})L_K(s,\Psi)$. It is shown in [2] Proposition 7.9 that

$$L'_A(0,\Psi) = -\frac{1}{q-1}\sum_{\mathfrak{c}} \overline{\Psi(\mathfrak{c})}\Big(\deg\mathfrak{c} - \delta v_\infty\big(\xi(\mathfrak{c})\big)\Big).$$

Then using L'hospital's rule we see that

$$L_K(0,\Psi) = \frac{1}{\delta(q-1)}\sum_{\mathfrak{c}} \overline{\Psi(\mathfrak{c})}\Big(\delta v_\infty\big(\xi(\mathfrak{c})\big) - \deg\mathfrak{c}\Big).$$

Here $\mathfrak{c}$ runs over any set of representatives of $\mathrm{Pic}\,A$. Define $\lambda(\mathfrak{c}) = \delta v_\infty\big(\xi(\mathfrak{c})\big) - \deg\mathfrak{c}$. Then $\lambda(\mathfrak{c})$ depends only on the class of $\mathrm{Pic}\,A$.

THEOREM 3.3. *Let* Ψ *be a nontrivial character on* Pic A*. Then we have*

$$L(0,\Psi)=\frac{1}{\delta(q-1)}\sum_{\mathfrak{c}}\overline{\Psi(\mathfrak{c})}\lambda(\mathfrak{c}),$$

where the sum runs over a complete set of representatives of Pic A*.*

Now following the same methods in the proof of Satz 2 of [10] replacing log by $\log_q$, $A_f(\Psi)$ by $\sum_{\mathfrak{c}}\overline{\Psi(\mathfrak{c})}\lambda(\mathfrak{c})$, and $\frac{\Delta(\mathfrak{a})}{\Delta(R_f)}$ by $\frac{\eta(\mathfrak{a})}{\eta(A)}$, we can get without difficulty the following theorem.

THEOREM 3.4. *Let* Ω *be a subfield of* H_A *containing* K *and let* $\mathfrak{A}$ *be the subgroup of* Pic A *corresponding to* Ω*. If* $\mathfrak{t}\in$ Pic $A\setminus\mathfrak{A}$*, then*

$$\Omega=K\Big(N_{\Omega}^{H_A}([\mathfrak{a}\mid A]^n)\Big),$$

for any integral ideal $\mathfrak{a}\in\mathfrak{t}$ *and any positive integer* n*.*

COROLLARY 3.5. *We have*

$$H_A=K([\mathfrak{a}\mid A]),$$

where $\mathfrak{a}$ *is any integral ideal of* A *which is not principal.*

COROLLARY 3.6. *Let* $\mathfrak{a}$ *be an integral ideal of* A *of degree prime to* δ*. Then*

$$\tilde{H}_A=K(\langle\mathfrak{a}\mid A\rangle).$$

PROOF. Clearly $\mathfrak{a}$ is not principal. Since sign functions are surjective, part vii) of Proposition 2.1 implies that

$$[K(\langle\mathfrak{a}\mid A\rangle):K([\mathfrak{a}\mid A])]=\frac{q^{\delta}-1}{q-1}.$$

Since $K(\langle\mathfrak{a}\mid A\rangle)\subset\tilde{H}_A$ and $H_A=K([\mathfrak{a}\mid A])$, we get the result.

4. **Class number formulas.** For a subgroup N of G, define $s(N)=\sum_{\sigma\in N}\sigma$ and $e_N=\frac{s(N)}{n}$. Let I_N be the augmentation ideal of $\mathbb{Z}[N]$ and $I=I_G$. Define

$$\ell\colon H_A^*\longrightarrow\mathbb{Z}[G]$$

by $x\longmapsto\sum_{\sigma\in G}v_\infty(x^\sigma)\sigma^{-1}$, and

$$\ell^*\colon H_A^*\longrightarrow\mathbb{Q}\otimes I$$

by $x\longmapsto(1-e_G)\ell(x)$. Then for $x\in B^*$ we have $\ell(x)=\ell^*(x)\in I$. Define

$$\omega=\sum_{\mathfrak{c}}\Big(\lambda(\mathfrak{c})-\lambda(A)\Big)\tau_{\mathfrak{c}}.$$

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PROPOSITION 4.1. *We have*

$$\ell^*(P_N) = \frac{q^\delta - 1}{\delta(q-1)} \omega I_N \mathbb{Z}[G],$$

and

$$\mathrm{Ker}(\ell^*) \cap P = M.$$

Therefore ℓ^* *gives an isomorphism of* P/M *onto* $\frac{q^\delta-1}{\delta(q-1)}\omega I_N \mathbb{Z}[G]$.

Let q_F be the number of constants of a function field F. Then it is well-known that the function $Z(s) = (q_F^{-s} - 1)\zeta(s)$ has the value $h_F/(q_F - 1)$ at $s = 0$. Thus for Galois extension L of K we have

$$\frac{(q_K - 1)h_L}{d(q_L - 1)h_K} = \prod_{\chi \neq 1} L_K(0, \chi),$$

where χ runs through the nontrivial characters of $\mathrm{Gal}(L/K)$ and d is the dimension of $\mathbb{F}_{q_L}$ over $\mathbb{F}_{q_K}$. Thus we get

$$\det \omega = \big(\delta(q-1)\big)^{h_A - 1} \frac{(q-1)h_{H_A}}{\delta(q^\delta - 1)h_K}.$$

by Theorem 3.3 viewing ω as an endomorphism on the free group I of rank $h_A - 1$. Then we have the following theorems whose proofs are exactly the same as in [5] up to the factor $\frac{q^\delta-1}{\delta(q-1)}$.

THEOREM 4.2. *P_N/M is G-isomorphic to $I_N\mathbb{Z}[G]$ and E_N is G-isomorphic to $I_N I$, and so $E_N = P_N^I$.*

THEOREM 4.3. *Every elliptic unit is the $(q-1)$-st power of a unit in $\tilde{H}_A$.*

With the aid of Theorem 4.2 we can show that $E_N = \mathrm{Ker}\, N_{H_A/L}$ on E and $L \cap E/N_{H_A/L}(E) \simeq \mu_n(G)/N$, where $\mu_n(G)$ is the subgroup of elements G whose orders divide n. Theorem 4.3 enables us to define

$$\bar{E}_N = \{x \in C : x^{q-1} \in E_N\} \subset B^*.$$

Now we are able to give several class number formulas.

THEOREM 4.4. *We have*

$$[O_L^* : \mathbb{F}_{q_L}^*(L \cap E)] = (q^\delta - 1)^{[L:K]-1} \frac{h_{O_L}}{|\mu_n(G)|} \frac{q-1}{q_L - 1} \tag{4.4.1}$$

$$[O_L^* : \mathbb{F}_{q_L}^* N_{H_A/L}(E)] = (q^\delta - 1)^{[L:K]-1} \frac{h_{O_L}}{n} \frac{q-1}{q_L - 1}, \tag{4.4.2}$$

$$[O_L^* : \mathbb{F}_{q_L}^* N_{H_A/L}(\bar{E})] = \left(\frac{q^\delta - 1}{q-1}\right)^{[L:K]-1} \frac{h_{O_L}}{n} \frac{q-1}{q_L - 1}, \tag{4.4.3}$$

$$[B^* : \mathbb{F}_{q^\delta}^* E_N E^N] = (q^\delta - 1)^{h_A - 1} \frac{n^{[L:K]} h_B}{|\mu_n(G)|} \frac{q-1}{q^\delta - 1}, \tag{4.4.3}$$

$$[B^* : \mathbb{F}_{q^\delta}^* E_N O_L^*] = (q^\delta - 1)^{h_A - [L:K]} \frac{n^{[L:K]} h_B}{h_{O_L}} \frac{q-1}{q^\delta - 1}, \tag{4.4.5}$$

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and

$$(4.4.6)\qquad [B^* : \mathbb{F}^*_{q^\delta}\bar{E}_N O^*_L] = \Big(\frac{q^\delta - 1}{q-1}\Big)^{h_A - [L:K]} \frac{n^{[L:K]} h_B}{h_{O_L}} \frac{q-1}{q^\delta - 1}.$$

PROOF. Let $q_L = q^e$ and $d = \frac{\delta}{e}$. We first note that $\delta h_K = h_A$, $dh_L = R_L h_{O_L}$ and $\det \omega|_{I^N} = \big(\delta(q-1)\big)^{[L:K]-1} \frac{(q-1)h_L}{e(q_L-1)h_K}$, where R_L is the regulator of O_L. Then

$$\begin{aligned}
[\ell(O^*_L) : \ell(L\cap E)] &= \frac{1}{R_L}\Big[I^N : \frac{q^\delta - 1}{\delta(q-1)}\omega(I^2)^N\Big] \\
&= \frac{1}{R_L}[I^N : \omega I^N][\omega I^N : \omega(I^2)^N]\Big[\omega(I^2)^N : \frac{q^\delta - 1}{\delta(q-1)}\omega(I^2)^N\Big] \\
&= \frac{1}{R_L}\det \omega|_{I^N}[I^N : (I^2)^N]\Big[(I^2)^N : \frac{q^\delta - 1}{\delta(q-1)}(I^2)^N\Big] \\
&= \frac{1}{R_L}\det \omega|_{I^N}|G^n|\Big(\frac{q^\delta - 1}{\delta(q-1)}\Big)^{[L:K]-1} \\
&= (q^\delta - 1)^{[L:K]-1}\frac{h_{O_L}}{|\mu_n(G)|}\frac{q-1}{q_L - 1}.
\end{aligned}$$

Thus we get (4.4.1) and (4.4.2) is an immediate consequence of (4.4.1) and the fact that $L\cap E/N_{H_A/L}(E) \simeq \mu_n(G)/N$. It is known in the proof of Corollary 4.5 of [5] that

$$[\mathbb{F}^*_q N_{H_A/L}(\bar{E}) : \mathbb{F}^*_q N_{H_A/L}(E)] = (q-1)^{[L:K]-1}.$$

But it is easy to see that

$$[\mathbb{F}^*_{q_L} N_{H_A/L}(\bar{E}) : \mathbb{F}^*_q N_{H_A/L}(\bar{E})] = \frac{q_L - 1}{q-1}$$

and

$$[\mathbb{F}^*_{q_L} N_{H_A/L}(E) : \mathbb{F}^*_q N_{H_A/L}(E)] = \frac{q_L - 1}{q-1}.$$

Hence we get (4.4.3) from (4.4.2). Exactly the same proof of Proposition 4.6 of [5] would give (4.4.4). (4.4.5) follows from (4.4.1) and (4.4.4) with the equality that

$$[\mathbb{F}^*_{q^\delta} O^*_L : \mathbb{F}^*_{q^\delta} E^N] = [O^*_L : \mathbb{F}^*_{q_L} E^N].$$

(4.4.6) is an immediate consequence of (4.4.5) using the fact that

$$\begin{aligned}
[\mathbb{F}^*_{q^\delta}\bar{E}_N O^*_L : \mathbb{F}^*_{q^\delta} E_N O^*_L] &= [\mathbb{F}^*_{q^\delta}\bar{E}_N : \mathbb{F}^*_{q^\delta} E_N] \\
&= \frac{1}{q-1}[\bar{E}_N : E_N] \\
&= (q-1)^{h_A - [L:K]}.
\end{aligned}$$

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