SPACES OF DIMENSION ZERO

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1. Introduction. In a recent paper **(1)** it was remarked that the theory of zero-dimensional spaces is exactly that part of general topology which can be described in terms of equivalence relations. Here, it will be shown how this idea can be used to obtain the following characterizations of certain types of zero-dimensional spaces:

Any compact zero-dimensional space which has a denumerable basis for its open sets and is dense in itself is homeomorphic to the space of 2-adic integers.

Any locally compact zero-dimensional space which is non-compact, has a denumerable basis for its open sets and is dense in itself, is homeomorphic to the space of 2-adic numbers.

The first of these statements is a well-known theorem **(6**; vol. 2, §40, II**)**, whilst the second one, here occurring as a simple consequence of the former, was proved in **(5)**. However, in both cases, the methods employed are rather different from ours which are, in fact, no more than a refinement of arguments used in **(1)**. In similar ways, the following assertions concerning non-archimedean metric spaces will be proved:

The number of inequivalent non-archimedean metrics on a non-compact zero-dimensional space which has a denumerable basis for its open sets and is dense in itself, is at least equal to the power of the continuum.

Any separable non-archimedean metric space can be mapped by a metric equivalence into the space of all formal power series with integral coefficients, taken with its so-called topology of formal convergence.

Any two n-adic metric spaces are metrically equivalent to each other.

2. Preliminaries. Topological terms, unless otherwise stated, will be used in the sense of **(3)**. The term "space" will always be taken to mean "Hausdorff space which has a denumerable basis for its open sets." Zero-dimensionality means the existence of a basis for the open sets consisting of open-closed sets. Two metric spaces E and F are called metrically equivalent if there exists a homeomorphism from E onto F which is uniformly continuous in both directions. Two metrics on the same space E are called equivalent if the identity mapping of E onto itself is a metric equivalence with respect to these metrics. The topology of formal convergence (a term due to E. Witt) in the ring of all formal power series

$$p = \sum_{n\geqslant 0} c_n z^n,$$

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c_n arbitrary elements from a given ring and z an indeterminate, is obtained by taking the ideals (z^k) as a system of neighbourhoods of $p = 0$. The space of n-adic numbers, defined by completing the rational numbers with respect to the ring topology given by the ideals (n^k), taken in its natural metric, will be referred to as the n-adic metric space. A uniform structure of a space E is here a "système fondamentale des entourages" in the sense of **(3**; chap. II**)**, compatible with the topology of E.

Equivalence relations on a set will be denoted by $\alpha, \beta, \gamma, \ldots$. All equivalence relations considered here will be relations on some topological space E. The α-class to which $x \in E$ belongs will be called $\alpha(x)$. The α for which $\alpha(x) = E$ is called the all-relation. The number of α-classes into which E decomposes will be denoted by $|\alpha|$ and called the index of α. If each α-class is an open-closed set in E, α will be called open-closed. The expression $\alpha \leqslant \beta$ ("α is finer than β") means that each α-class is contained in some β-class. If $\alpha \leqslant \beta$ and each β-class contains the same number of α-classes, this number will be denoted by $(\beta : \alpha)$, called the index of α in β. Writing down $(\beta : \alpha)$ will always be meant to imply the existence of this number.

As an immediate consequence of **(1**, Satz 10**)**, one has:

LEMMA 1. *If a compact zero-dimensional space E possesses a uniform structure consisting of a decreasing sequence* $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ *of open-closed equivalence relations for which* α_1 *is the all-relation on E and each* $(\alpha_{i-1} - 1 : \alpha_i)$ *equals* 2, *then E is homeomorphic to the space of* 2*-adic integers.*

According to **(1)**, this homeomorphism is given by the following method: The α_k-classes in each α_{k-1}-class are taken to be numbered, in a fixed manner, by 0 and 1; and for each $x \in E$, $c_k(x)$ is defined as the number $\alpha_{k+2}(x)$ in $\alpha_{k+1}(x)$. Then, E can be mapped by

$$x \rightarrow p(x) = \sum_{k \geqslant 0} c_k(x) z^k$$

into the ring $\mathfrak{P}$ of all power series in an indeterminate z with coefficients 0 and 1 from the prime field of characteristic 2. This mapping is a homeomorphism of E onto $\mathfrak{P}$ if $\mathfrak{P}$ is taken with its topology of formal convergence. In this topology, however, $\mathfrak{P}$ is homeomorphic to the space of 2-adic integers.

Lemma 1 can be strengthened slightly: One can replace the hypothesis $(\alpha_i : \alpha_{i+1}) = 2$ by the weaker condition

$$(\alpha_i : \alpha_{i+1}) = 2^{n_i},$$

with some natural numbers n_i, for in this case there are, for each i, sequences

$$\alpha_i = \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_{n_i} = \alpha_{i+1}$$

between α_i and α_{i+1} satisfying $(\beta_j : \beta_{j+1}) = 2$.

A further result from **(1)** needed here is:

LEMMA 2. *A space E is zero-dimensional if and only if its open-closed equivalence relations form a uniform structure of E. A compact E is zero-*

dimensional if and only if it has a uniform structure consisting of a decreasing sequence of open-closed equivalence relations of finite index.

Of course, the second part of this statement would no longer be true if the condition, always implicitly assumed here, that E have a denumerable basis were not satisfied.

Finally, a metric $|x, y|$ on a set E is called non-archimedean, if it satisfies the condition $|x, y| \leqslant \max\{|x, z|, |z, y|\}$ for any x, y and z from E. It is well known that any non-archimedean metric space is zero-dimensional.

3. The compact case. In order to prove the first statement in §1 it is now sufficient to show that any compact E of dimension zero and dense in itself possesses a uniform structure consisting of open-closed α_i such that $\alpha_i \geqslant \alpha_{i+1}$ and $(\alpha_i : \alpha_{i+1})$ is always a power of 2.

Let α_i $(i = 1, 2, \ldots)$ be a decreasing sequence of relations on E as given by Lemma 2. Since E is dense in itself, any α_i-class must consist of more than just one point. Therefore, any fixed α_i-class contains an arbitrarily large number of α_k-classes for suitably large k. From this, it can be deduced that there is also a decreasing sequence of open-closed equivalence relations β_i such that $\beta_i \leqslant \alpha_i$, $\beta_i \geqslant \alpha_{n(i)}$ for some suitable $n(i)$, and $(\beta_i : \beta_{i+1})$ is a power of 2.

Suppose that the first k members $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_k$ of this new sequence have already been determined. Then, by assumption, $\beta_k \geqslant \alpha_{n(k)}$. If $(\beta_k : \alpha_{n(k)})$ is defined and a power of 2, one can take

$$\beta_{k+1} = \alpha_{n(k)}, \quad n(k+1) = n(k) + 1.$$

Otherwise, let m be the largest number of $\alpha_{n(k)}$-classes contained in any β_k-class and $2^s \geqslant m$. In any of the finitely many β_k-classes B, let m_B be the number of $\alpha_{n(k)}$-classes and C a fixed one of these. Now, for a sufficiently large l_B, C contains more than $2^s - m_B + 1$ α_{l_B}-classes. By forming, if necessary, unions of these, one can obtain a decomposition of C into exactly $2^s - m_B + 1$ open-closed sets. These, together with the $m_B - 1$ $\alpha_{n(k)}$-classes in B other than C decompose B into 2^s open-closed sets, and taking this for each B, one has a decomposition of this kind for E. The corresponding relation β is open-closed, satisfies $\beta \leqslant \alpha_{k+1}$ because of $\beta \leqslant \alpha_{n(k)} < \beta_k \leqslant \alpha_k$ and also $\beta \geqslant \alpha_l$ for any l greater than all l_B. Hence, one can put $\beta_{k+1} = \beta$ and $n(k+1)$ equal to, say, the first number greater than the l_B.

This completes the proof, since it was assumed that α_1 is the all-relation and β_1, therefore, can be taken as α_1. That the sequence β_i forms a uniform structure of E is, of course, an immediate consequence of $\beta_i \leqslant \alpha_i$.

As a corollary one has: *Any totally bounded non-archimedean metric space which is dense in itself is metrically equivalent to a subspace of the metric space of 2-adic integers.* For a space E of this type has a zero-dimensional compact $\tilde{E}$ as its metric completion which is also dense in itself, therefore homeomorphic to the space of 2-adic integers and hence metrically equivalent to it with respect to its metric induced from E and the natural metric for the 2-adic

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integers in the latter. Also, since the space of $\mathfrak{p}$-adic integers for any prime ideal $\mathfrak{p}$ of any number field is compact, dense in itself and has a denumerable basis, one obtains as a further consequence: *For any* $\mathfrak{p}$, *the space of* $\mathfrak{p}$-*adic integers is homeomorphic to the space of* 2-*adic integers.*

The first of these corollaries can be regarded as a partial strengthening of a theorem by Urysohn (**6**; vol. I, §23) according to which any zero-dimensional space is homeomorphic to a subspace of Cantor's compact zero-dimensional space which is, of course, homeomorphic to the space of 2-adic integers.

4. The non-compact locally compact case. If E is zero-dimensional, and not compact but locally compact, then it is the union of denumerably many disjoint open-closed compact sets: Since E has a denumerable basis for its open sets, it also has such a basis $\mathfrak{B}$ consisting of open-closed sets. If, then, for each $x \in E$, V_x is an open neighbourhood with compact closure and $B_x \in \mathfrak{B}$ such that $x \in B_x \subseteq V_x$, these B_x are compact open-closed and have E as their union. Furthermore, there are only denumerably many of them and, hence, they can be arranged in a sequence B_i, $i = 1, 2, \ldots$. Now, the disjoint sets

$$B_k^* = B_k - \bigcup_{i<k} B_i, \qquad k = 1, 2, \ldots$$

still have E as their union and are open-closed compact.

The considered space E being dense in itself, each of these B^*_k, since it is open in E, must also be dense in itself and therefore homeomorphic to the space of 2-adic integers. Furthermore, as the B^*_k are open-closed, E is the topological sum in the sense of (**3**; chap. I) of its compact subspaces B^*_k, hence homeomorphic to the sum of denumerably many spaces of 2-adic integers. Finally, as the space of the 2-adic numbers is itself a space of this type, E is homeomorphic to it.

In exactly the same manner as above one obtains the corollary: *For any* $\mathfrak{p}$, *the space of* $\mathfrak{p}$-*adic numbers is homeomorphic to the space of* 2-*adic numbers.*

5. The number of distinct non-archimedean metrics of a zero-dimensional space. Let E be the space in question. It possesses a sequence $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ of open-closed equivalence relations where $\bigcap \alpha_i(x) = x$ and the $\alpha_i(x)$ form a neighbourhood basis for each $x \in E$, originating from one of its non-archimedean metrics which are known to exist (**1**). The connection between the α_i and the metric is such that

$$(*) \qquad |x, y| = 2^{-k} \text{ if } x\alpha_i y, \qquad i = 1, 2, \ldots, k; \quad i \neq k+1.$$

defines an equivalent metric (**1**). If, now, each $|\alpha_i|$ is finite, E is totally bounded with respect to this metric. Therefore, under the hypothesis that E is not totally bounded in each of its non-archimedean metrics (the other case will be considered later on) one can assume that, for some i, E decomposes into infinitely many α_i-classes. Obviously, no generality is lost by taking $i = 1$.

Since E is separable, the open-closed α_1-classes are a denumerable collection of sets, say, $C_1, C_2, \ldots$. Then, with respect to a given increasing sequence $k_1, k_2, \ldots$ of natural numbers, one can decompose C_i into its α_{k_i}-classes. The decomposition of E into open-closed sets thus obtained gives an open-closed equivalence relation β_1 which can be used to define the sequence $\beta_k = \beta_1 \wedge \alpha_k$ where $\wedge$ denotes taking the lattice theoretic meet of two equivalence relations **(2)**. In the manner given by the formula (*), the sequence $\beta_1 \geqslant \beta_2 \geqslant \ldots$ defines a new non-archimedean metric on E. The number of metrics that can be obtained in this way is equal to the number of increasing sequences of natural integers, hence equal to the cardinal number $\mathfrak{c}$ of the continuum. However, these $\mathfrak{c}$ different metrics need not all be inequivalent to each other. In order to prove the assertion stated in §1 it will now be shown that this set of metrics splits into $\mathfrak{c}$ different equivalence classes.

Let k^*_i and k_i be two different increasing sequences of natural integers and β^*_k, β_k the two corresponding sequences of open-closed equivalence relations. The metrics defined by β^*_k and β_k will be equivalent if and only if to each β_k there exists a $\beta^*_e \leqslant \beta_k$ and vice versa. In particular, one then has a relation of the type $\beta_1 \geqslant \beta^*_i \geqslant \beta_k$ with suitable i and k. Now, by definition of β_e, the β_e-classes contained in C_m will be equal to the β_1-classes in C_m for all sufficiently large m: as $\beta_e = \beta_1 \wedge \alpha_e$, the β_e-classes on C_m are intersections of α_{k_m}-classes and α_e-classes. If m is large enough, one has $\alpha_{k_m} \leqslant \alpha_e$ anyway, so these intersections will merely be α_{k_m}-classes, and these are also the β_1-classes in C_m. The relation $\beta_1 \geqslant \beta^*_i \geqslant \beta_k$ therefore implies that the β^*_i-classes in all C_m for sufficiently large m are also equal to the β_1-classes, and this then gives the result: From a certain $m = m_0$ onwards β_1 and β^*_1 are equal in C_m.

Now, let the sequence $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ satisfy this further condition: Any α_i-class decomposes into more than one α_{i+1}-class. Then, β_1 and β^*_1 can only be equal on C_m if $k_m = k^*_m$. In this case, therefore, one obtains that the two sequences k_i and k^*_i are equal from a suitable index onwards. Since to any increasing sequence of natural integers there exist only denumerably many other such sequences coinciding with it from a suitable index onwards, the $\mathfrak{c}$ different metrics defined above group into equivalence classes of at most denumerably many metrics each; the number of these classes will then still be $\mathfrak{c}$.

The restriction just placed on the sequence $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ can be shown to be satisfied if not by the α_i themselves, then at least by suitable modifications of them. The basis for this will be that each open-closed set in E, being infinite since E is dense in itself, can be decomposed into two open-closed sets. Using this, one may define new relations α^*_i in the following way: $\alpha^*_1 = \alpha_1$. If α^*_n is already defined such that $\alpha^*_n \leqslant \alpha_n$, decompose each $\alpha^*_n \wedge \alpha_{n+1}$-class into two open-closed sets and define α^*_{n+1} by the resulting decomposition of E. The sequence $\alpha^*_1 \geqslant \alpha^*_2 \geqslant \ldots$ has all desired properties and using it, if necessary, in place of the original α_i one obtains the existence of $\mathfrak{c}$ inequivalent non-archimedean metrics on E.

The remaining case to be considered is that E is totally bounded in all its non-archimedean metrics. This property implies, as will be shown now, the compactness of E. Let $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ be chosen as above and take any open-closed equivalence relation α on E. As the sequence α_i defines a non-archimedean metric on E by (*) so does the sequence $\alpha'_i = \alpha \wedge \alpha_i$. E being totally bounded in this metric, there are only finitely many α-classes. Hence, any decomposition of E into open-closed sets must be finite. From this it follows that any denumerable open-closed covering of E contains a finite covering, for if

$$\bigcup_{i=1}^{\infty} B_i = E,$$

B_i open-closed, then the

$$B_i^* = B_i - \bigcup_{k<i} B_k$$

give a decomposition of E into open-closed sets which will, of course, only be finite if $B^*_i = \phi$ and therefore

$$\bigcup_{k<i} B_k \supseteq B_i$$

from some $i = i_0$ onwards; this gives

$$\bigcup_{k<i_0} B_k = E.$$

Finally, as E is zero-dimensional, one concludes from this that any open covering of E, having a denumerable open-closed covering as a refinement, contains a finite covering. E therefore is compact.

In all, it is then proved that a zero-dimensional space which is dense in itself either has at least $\mathfrak{c}$ inequivalent non-archimedean metrics or is compact, in which case, of course, all its metrics are equivalent.

6. Imbeddings by metric equivalences. Let P now be the ring of all formal power series

$$\sum_{n\geqslant 0} c_n z^n$$

with integral coefficients, taken with its topology of formal convergence. This space $\mathfrak{P}$ is a universal space for the separable non-archimedean metric spaces in the sense that any such space can be mapped into $\mathfrak{P}$ by a metric equivalence. The mapping which will do this can again be defined as follows **(1)**:

Let $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ be a sequence of open-closed equivalence relations on E resulting from its given metric. Then, the α_i-classes in the different α_{i-1}-classes can be regarded as numbered in a fixed manner. With respect to this numbering, let $c_n(x)$ be the number of $\alpha_n(x)$ in $\alpha_{n-1}(x)$ for $x \in E$; then, in exactly the same way as in §1 in the special case of the compact spaces (Lemma 1) the mapping

$$p: x \to p(x) = \sum c_n(x) z^n$$

is a metric equivalence of E into $\mathfrak{P}$. Of course, in special cases, p may even be a metric isomorphism, but since the sequence $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ determines the metric only up to equivalence, this will not be so in general.

$\mathfrak{P}$ is obviously itself a separable non-archimedean metric space, since the denumerable set of all integral polynomials is dense in $\mathfrak{P}$. Therefore $\mathfrak{P}$ is, in a sense, a minimal universal space for this type of space and, of course, characterized by this property.

For a more restricted class of metric spaces than the one just considered, one can obtain a universal space of an even simpler nature than the space $\mathfrak{P}$. A non-archimedean metric on a separable space E will be called *evenly locally compact* if it can be represented—up to equivalence—by a decreasing sequence $\alpha_1 \geqslant \alpha_2 \geqslant \ldots$ of open-closed equivalence relations such that $\alpha_1(x)$ is compact for each $x \in E$ and the (necessarily finite) indices $(\alpha_{n-1}: \alpha_n)$ exist (see §2). Then, the following holds: *Each separable space with an evenly locally compact non-archimedean metric is metrically equivalent to a subspace of the metric space of* 2*-adic numbers*. The class of spaces admitted here includes, of course, all the n-adic metric spaces for any natural integer n and, more generally, the spaces of all separable locally compact groups whose topologies are given by a denumerable decreasing sequence of invariant subgroups as neighbourhood basis for the unit element.

By the preceding construction, a space E of the type now considered is metrically equivalent to a subset of $\mathfrak{P}$ contained in the set given by all

$$\sum_{n \geqslant 0} a_n z^n, \qquad 0 \leqslant a_n < j_n,$$

where j_0 denotes the (possibly infinite) number of α_1-classes in E and j_n, $n \geqslant 1$, the index $(\alpha_n: \alpha_{n+1})$. Now, for any natural integer a one has

$$a = \varphi_0(a) + \varphi_1(a)\, 2 + \ldots + \varphi_s(a)\, 2^s,$$

$\varphi_i(a)$ equal to 0 or 1, with some suitable s. In particular, then, any a, $0 \leqslant a < j_n$, can be written as

$$\varphi_0(a) + \varphi_1(a)\, 2 + \ldots + \varphi_{s_n}(a)\, 2^{s_n} \text{ for } n \geqslant 1.$$

Using this, we can define a mapping φ of

$$w = \sum a_n z^n, \qquad 0 \leqslant a_n < j_n,\ n = 0, 1, 2, \ldots,$$

by

$$\varphi(w) = \varphi_{s_0}(a_0)\, z^{-s_0} + \varphi_{s_0-1}(a_0)\, z^{-s_0+1} + \ldots + \varphi_1(a_0)\, z^{-1} + \varphi_0(a_0)$$
$$+ \sum_{n=1}^{\infty} (\varphi_1(a_n)\, z + \varphi_2(a_n)\, z^2 + \ldots + \varphi_{s_n}(a_n)\, z^{s_n})\, z^{s_1+s_2+\ldots+s_{n-1}}.$$

$\varphi(w)$ is an element of the ring $\mathfrak{L}$ of all Laurent forms

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$$\sum_{n \gg -\infty} a_n z^n$$

in z with integral coefficients, which, again, will be taken with the topology of formal convergence. Now, $w_1 \equiv w_2 (\text{mod } z^{e+1})$ implies $a_n = b_n$, $n = 0, 1, \ldots, e$, for the coefficients a_n of w_1, and b_n of w_2 and therefore $\varphi_i(a_n) = \varphi_i(b_n)$ for $n = 0, 1, \ldots, e$ and all corresponding i. From this it follows that

$$\varphi(w_1) \equiv \varphi(w_2) \ (\text{mod } z^{s_1+s_2+\ldots+s_e+1}).$$

Similarly, the converse holds, and since the

$$(z^{s_1+s_2+\ldots+s_e+1})$$

define a metric equivalent to that defined by the (z^e), φ is a metric equivalence. Furthermore, all $\varphi(w)$ lie in a part of $\mathfrak{K}$ which is itself metrically equivalent to the 2-adic metric space. This proves the above assertion.

In the case of non-compact separable evenly locally compact non-archimedean metric spaces which are dense in themselves one can easily obtain a much stronger result: *Any such space is metrically equivalent to the metric space of* 2-*adic numbers*. A space E of this type is, of course, the sum of its compact open-closed α_1-classes (see above) K_i, $i = 1, 2, \ldots$, and each of these is homeomorphic to the space of 2-adic integers. If C_i, $i = 1, 2, \ldots$, is the complete system of residue classes, with respect to addition, of all 2-adic numbers modulo the 2-adic integers, then K_i can be mapped homeomorphically onto C_i for each i. This gives a mapping ψ defined on E which is a metric equivalence. For, up to equivalence between metrics on E, the K_i can be taken to be the "unit spheres" in E, and this is what the C_i are in the 2-adic metric space. Then, since ψ carries unit spheres into unit spheres and is, in its restriction to these, a metric equivalence, it also is this for E as a whole.

In particular, this proves the last statement in §1 since the n-adic metric spaces are of the type just considered. More so, one can say that *there are only two essentially different*, i.e., *inequivalent*, *separable and evenly locally compact non-archimedean metric spaces which are dense in themselves:* The metric spaces of 2-adic integers and 2-adic numbers.

As a concluding remark, it may be pointed out that the metric product spaces of any finite number of p_i-adic metric spaces studied in **(5)** are all metrically equivalent to each other. This follows from the last remark and the fact that the metric product of p_i-adic metric spaces $(i = 1, 2, \ldots, r)$, is metrically equivalent to the $p_1 p_2 \ldots p_r$-adic metric space, which is an immediate consequence of (**4**; p. 96, Satz 5). There is a remarkable contrast between this and the results in **(5)** according to which an *isometric* mapping of one product of this type into another one can only exist if the product of the corresponding sets of prime numbers for the first space is less than or equal to that for the second space. This, then, goes to show that metric equivalence leads to much wider classes than the more restrictive concept of isometry.

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