Proceedings pf the Edinburgh Mathematical Society (1986) 29, 359–365 ©

MAXIMAL SUBALGEBRAS OF HEYTING ALGEBRAS

by M. E. Adams*

(Received 17th April 1985)

1. Introduction

A *Heyting algebra* is an algebra $(H; \vee, \wedge, \rightarrow, 0, 1)$ of type $(2,2,2,0,0)$ for which $(H; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\rightarrow$ is the binary operation of relative pseudocomplementation (i.e., for $a, b, c \in H$, $a \wedge c \leqq b$ iff $c \leqq a \rightarrow b$). Associated with every subalgebra of a Heyting algebra is a separating set. Those corresponding to maximal subalgebras are characterized in Proposition 8 and, subsequently, are used in an investigation of Heyting algebras.

Heyting algebras are a generalization of Boolean algebras. In D. Sachs [**13**] (see also G. Grätzer, K. M. Koh, and M. Makkai [**8**]) it is shown *inter alia* that, for a Boolean algebra with at least eight elements, every non-trivial element is both included and excluded by maximal proper subalgebras. Furthermore, every proper subalgebra is the intersection of maximal subalgebras. The *Frattini subalgebra* of an algebra A, denoted $\Phi(A)$, is the intersection of the maximal subalgebras. As seen by the above, for a Boolean algebra B, it is always the case that $\Phi(B) = \{0, 1\}$.

It is interesting to compare Heyting with Boolean algebras. That, as for Boolean algebras, maximal subalgebras occur freely in Heyting algebras is indicated in the following:

Theorem 1. *For a Heyting algebra H the following are equivalent:*

(i) *H has a meet irreducible zero.*

(ii) *$H \cong \Phi(K)$ for some Heyting algebra K.*

(iii) *For infinite $\kappa \geqq |H|$, there is a family of non-isomorphic Heyting algebras $(H_i : i < 2^\kappa)$ such that, for $i < 2^\kappa$, $|H_i| = \kappa$ and $\Phi(H_i) \cong H$. If H is finite there are, in addition, infinitely many finite non-isomorphic Heyting algebras $(H_i : i < \omega)$ such that $\Phi(H_i) \cong H$.*

By contrast, Theorem 2 shows that the ubiquity of maximal subalgebras suggested in Theorem 1 is after all illusory.

Theorem 2. *For $\kappa \geqq \omega$, there is a family $(H_i : i < 2^\kappa)$ of non-isomorphic Heyting algebras such that, for $i < \kappa$, $|H_i| = \kappa$ and H_i has no proper maximal subalgebras.*

*The support for the NSF (grant # 8402909) is gratefully acknowledged.

The presentation uses topological duality for Heyting algebras. A brief summary of the necessary required facts is given in Section 2: for further information see either of the survey papers B. A. Davey and D. Duffus [6] or H. A. Priestley [12].

2. Preliminaries

For a poset P and $Q \subseteq P$, let $(Q] = \{x : x \leq y$ for some $y \in Q\}$ and $[Q) = \{x : x \geq y$ for some $y \in Q\}$. If $Q = \{x\}$ denote $(Q]$, $[Q)$ by $(x]$, $[x)$, respectively. Say Q is *decreasing* if $Q = (Q]$, *increasing* if $Q = [Q)$, and *convex* if $Q = [Q) \cap (Q]$. A mapping $\varphi : P \to P'$ is *order preserving* if $\varphi(x) \leq \varphi(y)$ whenever $x \leq y \in P$.

A pair (P, τ) is a *totally order disconnected space* if P is a poset, τ a topology on P, and, for $x, y \in P$, if $x \not\leq y$ then there exists a clopen decreasing $Q \subseteq P$ such that $x \notin Q$ and $y \in Q$. A compact totally order disconnected space is called a *Priestley space.*

In [11], H. A. Priestley showed that the category of distributive (0, 1)-lattices with all (0, 1)-lattice homomorphisms is dually equivalent to the category of Priestley spaces and continuous order preserving functions. Under the duality elements of a distributive (0, 1)-lattice L correspond to the clopen decreasing subsets of the associated Priestley space (P, τ). For $a \in L$, let $A \subseteq P$ denote the clopen decreasing set that represents a. Then, for a (0, 1)-lattice homomorphism $f : L \to L'$ there corresponds a continuous order preserving mapping $\varphi : P' \to P$, and $f(a)$ is represented by $\varphi^{-1}(A)$. In addition, f is an isomorphism iff φ is a homeomorphism and an order isomorphism.

Since Heyting algebras are distributive (0, 1)-lattices, the category of Heyting algebras is isomorphic to a subcategory of distributive (0, 1)-lattices. An *h-space* is a Priestley space (P, τ) such that $[Q)$ is clopen for every convex clopen $Q \subseteq P$. For h-spaces P, P', an *h-map* is a continuous order preserving map $\varphi : P \to P'$ for which $\varphi((x]) = (\varphi(x)]$. The following is now folklore: see, for example, H. A. Priestley [12].

Proposition 3. *The category of Heyting algebras with all homomorphisms is dually equivalent to the category of all h-spaces and h-maps.*

For a Heyting algebra H, if $a, b \in H$ then, under the duality, $a \to b$ corresponds to the clopen decreasing set $P \backslash [A \backslash B)$.

Let L be a distributive (0, 1)-lattice with Priestley space (P, τ), and L_1 a (0, 1)-sublattice of L. The *separating set* of L_1 is

$$S = \{(x, y) \in P \times P : x \not\geq y \text{ and, for all } a \in L_1,\ x \in A \text{ implies } y \in A\}.$$

A set $X \subseteq P$ is *compatible* with S if, for all $(x, y) \in S$, $x \in X$ implies $y \in X$. It was shown in [1] (see also J. Hashimoto [9]) that $L_1 = \{a \in L : A$ is compatible with $S\}$.

3. Separating sets for Heyting algebras

Since any subalgebra of a Heyting algebra is a (0, 1)-sublattice, it is determined by an appropriate separating set. The aim of this section is to characterize the separating sets of maximal subalgebras of Heyting algebras. Throughout let H be a Heyting algebra with h-space (P, τ).

For distinct $p,q\in P$, define

$$S_{p,q}=\{(x,p)\in P\times\{p\}:x\ngeq p \quad\text{and}\quad x\geqq q\}\cup\{(x,q)\in P\times\{q\}:x\geqq p \quad\text{and}\quad x\ngeq q\}.$$

Lemma 4. *For distinct $p,q\in P$, if $(p]\cup\{q\}=(q]\cup\{p\}$, then $S=S_{p,q}$ is the separating set of a proper subalgebra of H.*

Proof. First it must be shown that S is a separating set. Let $C(S)$ denote the family of clopen decreasing subsets of P compatible with S. Clearly, for a clopen decreasing set $A\subseteq P$, $A\in C(S)$ iff $p\in A$ is equivalent to $q\in A$. Thus $C(S)$ is a $(0,1)$-sublattice of H with a separating set that contains S. For every $x\ngeq y\in P$ and $(x,y)\notin S$, it is required to find $A\in C(S)$ such that $x\in A$ and $y\notin A$. By total order disconnectedness, this is routine if either $x\ngeq p,q$ or $x\geqq p,q$. Suppose, for example, $x\geqq p$ and $x\ngeq q$. If $q\geqq y$ then, since $(x,y)\notin S$, $q>y$. It follows that $p\geqq y$ and, hence, $x\geqq y$. Consequently, $q\ngeq y$ and there is a clopen decreasing set A such that $x,q\in A$ and $y\notin A$. A similar argument in the event that $x\ngeq p$ and $x\geqq q$ completes the proof that S is a separating set.

It remains to show that S is the separating set of a subalgebra of H and not simply of a $(0,1)$-sublattice. Suppose this is not the case and $P\backslash[A\backslash B)\notin C(S)$ for some $A,B\in C(S)$. With no loss of generality, assume $p\in P\backslash[A\backslash B)$ and $q\notin P\backslash[A\backslash B)$. Then $p\notin[A\backslash B)$ and $q\in[A\backslash B)$. By hypothesis, $q\in A\backslash B$ and, hence, $p\in A\cap B$. In particular, $p\in B$ and $q\notin B$ which contradicts the choice of B. □

Lemma 5. *For distinct $p,q\in P$, if $(p]\cup\{q\}=(q]\cup\{p\}$, then the subalgebra of H with separating set $S=S_{p,q}$ is maximal.*

Proof. It is required that the subalgebra generated by the addition of any clopen decreasing set $A\subseteq P$ to $C(S)$ be H. It is enough to show that every element of S may be separated. Since $A\notin C(S)$, it separates p and q. Suppose, with no loss in generality, $p\in A$ and $q\notin A$.

Consider $(x,q)\in S$. If $x\in A$ then A separates x and q. If $x\notin A$, let $Q=((x]\cap[p))\backslash A$. By hypothesis, $q\notin[Q)$ and, since Q is closed, it follows from total order disconnectedness that there exists a clopen decreasing set B with $p,q\in B$ and $B\cap Q=\emptyset$. Observe that $y\ngeq p,q$ for any $y\in R=(x]\cap(B\backslash A)$. As R is closed, $p,q\notin C$ for some clopen decreasing set $C\supseteq R$. Observe that, by construction, $B,C\in C(S)$. Consider $P\backslash[B\backslash(A\cup C))$ (which corresponds to $b\rightarrow(a\vee c)$ in H). Since $(x]\cap(B\backslash(A\cup C))=\emptyset$ and $q\in[B\backslash(A\cup C))$, $x\in P\backslash[B\backslash(A\cup C))$ and $q\notin P\backslash[B\backslash(A\cup C))$. The pair (x,q) has been separated.

Suppose $q\ngeq p$. Then $x\ngeq p,q$ for any $x\in Q=A\cap(q]$. As Q is closed there is a clopen decreasing set B such that $p,q\notin B$ and $Q\subseteq B$. Clearly, $p\notin P\backslash[A\backslash B)$ and $q\in P\backslash[A\backslash B)$. Any element $(x,p)\in S$ can now be separated by arguing as above. □

For a subalgebra H_1 of H and $a\in H$, let $[H_1,a]$ denote the subalgebra generated by $H_1\cup\{a\}$. For $b\in[H_1,a]$ there is a Heyting polynomial $p(x_1,\dots,x_n)$ and $a_i\in H_1\cup\{a\}$ for $1\leqq i\leqq n$ such that $b=p(a_1,\dots,a_n)$. Let $b'=p(a'_1,\dots,a'_n)$ where $a'_i=a_i$ for $a_i\in H_1$ and 1 otherwise.

Lemma 6. *For $x \in P$ and $b \in [H_1, a]$, if $x \in A$ then $(x] \cap B = (x] \cap B'$.*

Proof. Use induction on the length n of the polynomial $p(x_1, \dots, x_n)$. Obviously $b \in H_1 \cup \{a\}$ satisfies the inductive hypothesis. An inductive step where $b = c \vee d$ or $c \wedge d$ is clear. Suppose then $b = c \rightarrow d$. It must be shown that $(x] \cap (P \backslash [C \backslash D)) = (x] \cap (P \backslash [C' \backslash D'))$. Let $y \in (x]$. Then $y \in P \backslash [C \backslash D)$ iff $y \notin [C \backslash D)$. This is equivalent to $(y] \cap (C \backslash D) = \emptyset$ which holds iff $((y] \cap C) \backslash ((y] \cap D) = \emptyset$. By hypothesis, $((y] \cap C) \backslash ((y] \cap D) = ((y] \cap C') \backslash ((y] \cap D')$. Thus $y \in P \backslash [C \backslash D)$ iff $y \in P \backslash [C' \backslash D')$. □

Lemma 7. *Let H_1 be a proper subalgebra with separating set S. If H_1 is maximal then $S = S_{p,q}$ for some $p, q \in P$ such that $(p] \cup \{q\} = (q] \cup \{p\}$.*

Proof. Choose $(x, y) \in S$. Let $q = y$ and p be a minimal element of $\{x : (x, q) \in S\}$.

If $p \not\geq x$ for some $x < q$, then there is a clopen decreasing set $A \subseteq P$ such that $p, x \in A$ and $q \notin A$. Clearly, $A \notin C(S)$ and, by Lemma 6, (p, x) is an element of the separating set for $[H_1, a]$. Since H_1 is a maximal subalgebra, this is not possible. Thus, $(q] \subseteq (p] \cup \{q\}$.

There are two cases to consider.

First, $q > p$. Then $S_{p,q} \subseteq S$ and $(p] \subseteq (q]$.

Second, $q \not\geq p$. If $(q, p) \notin S$, then $q \in A$ and $p \notin A$ for some $A \in C(S)$. Let $Q = (p] \cap A$. Since Q is closed, the choice of p implies that $Q \subseteq B$ and $q \notin B$ for some $B \in C(S)$. Consequently, $p \in P \backslash [A \backslash B)$ and $q \notin P \backslash [A \backslash B)$. This is inconsistent with $(p, q) \in S$. Thus $(q, p) \in S$ and $S_{p,q} \subseteq S$. As above, $(q, p) \in S$ and H_1 maximal yields $(p] \subseteq (q] \cup \{p\}$.

In either case, $(p] \cup \{q\} = (q] \cup \{p\}$ and $S_{p,q} \subseteq S$. By Lemma 4, $S = S_{p,q}$. □

The above is combined in the following proposition.

Proposition 8. *For a Heyting algebra H, S is the separating set of a maximal proper subalgebra of H iff $S = S_{p,q}$ for distinct $p, q \in P$ with $(p] \cup \{q\} = (q] \cup \{p\}$.*

4. Proof of theorem 1

Lemma 9. *For a Heyting algebra H, $\Phi(H)$ has a meet irreducible zero.*

Proof. Suppose $a \wedge b = 0$ for non-trivial $a, b \in H$. Then $A \cap B = \emptyset$ and, since $A, B \neq \emptyset$, it is possible to choose minimal elements $p \in A$ and $q \in B$. By Proposition 8, $S_{p,q}$ is the separating set of a maximal subalgebra that contains neither a nor b. □

Since, in Theorem 1, (iii) implies (ii), it remains to show that (i) implies (iii). For the remainder of this section let H be a Heyting algebra with a meet irreducible zero and h-space (P, τ). By the duality, P has a minimum element m.

Let B denote a Boolean algebra with at least one atom (and at least eight elements). Any atom of B corresponds to an isolated point of its Stone space (Q, σ): distinguish one such point $q \in Q$. (A *Stone space* is a Priestley space in which the order is trivial.) Endow $R = P \times Q$ with the product topology and let (S, ρ) be the one point compactification of $R \backslash (\{m\} \times Q)$ by an element M.

Define a partial order on S as the transitive closure of the following relation:

(i) $M \leqq (x,y)$ for all $(x,y) \in S$;

(ii) $(x,q) \leqq (y,q)$ for $x \leqq y$ in P;

(iiii) $(x,y) \leqq (x,q)$ for all $x \in P$ and $y \in Q$.

It is readily verified that this is indeed a partial order on S.

To see that (S,ρ) is totally order disconnected consider $(x,y) \not\leqq (u,v)$. If $x \not\leqq u$ then there is a clopen increasing set $X \subseteq P$ such that $x \in Y$ and $u \notin X$. The set $X \times Q$ is clopen increasing in S and separates the elements in question. Otherwise $x < u$ and $v \neq q$. Choose clopen sets $Y \subseteq X \subseteq P$ such that X is increasing, $x \in Y$, $m \notin X$, and $u \notin Y$. Then $Y \times Q \cup (X \times \{q\})$ is a suitable clopen increasing set. Or $x = u$, $y \neq v$, and $v \neq q$. Choose clopen sets $X \subseteq P$ and $Y \subseteq Q$ such that X is increasing, $x \in X$, $m \notin X$, $y \in Y$, $q \in Y$, and $v \notin Y$. Then $X \times Y$ is a suitable clopen increasing set.

Suppose $X \subseteq S$ os clopen and consider $[X)$. Obviously, $[X)$ is clopen whenever $M \in X$. Suppose $M \notin X$. It is enough to consider $X = Y \times Z$ for clopen $Y \subseteq P$ and $Z \subseteq Q$. By hypothesis $[Y) \subseteq P$ is clopen and so $[X) = ([Y) \times \{q\}) \cup (Y \times Z)$ is too.

The above combines to the following.

Lemma 10. *(S,ρ) is an h-space.*

Let K denote the Heyting algebra with h-space (S,ρ).

Lemma 11. $\Phi(K) \cong H$.

Proof. Suppose $\emptyset \neq A \subseteq S$ is clopen decreasing. If $(x,y) \notin A$ for some $y \in Q \backslash \{q\}$, then $S_{M,(x,y)}$ is the separating set of a maximal subalgebra of K which shows that, in this case, $a \notin \Phi(K)$. Otherwise $A \supseteq P \times (Q \backslash \{q\})$. If $a \notin \Phi(K)$, then there is a maximal subalgebra with a separating set generated by some pair M or $(x,y) \in A$ together with $(u,q) \notin A$. However this is impossible since $(u,q) \not\leqq M$ or (x,y) implies that $(u,v) < (u,q)$ and $(u,v) \not\leqq M$ or (x,y) for some $v \in Q \backslash \{q\}$. Since, for any non-empty clopen decreasing $X \subseteq P$, $A = S \backslash ((P \backslash X) \times \{q\})$ is clopen decreasing in S, $\Phi(K) \cong H$. □

Lemma 12. *For Boolean algebras B, B', $B \cong B'$ iff $K \cong K'$.*

Proof. Let $X = \{(x,y) \in S : (x,y) \succ\!\!- M\}$. For $(x,y) \neq M$, $(x,y) \in X$ iff $y \neq q$. Further, for any $(x,q) \in S$, the closed subspace $\{(y,z) : (x,q) \succ\!\!- (y,z)\} \cap X$ is homeomorphic to the subspace $Q \backslash \{q\}$ of Q. □

The proof of Theorem 1 is concluded by observing that there are infinitely many non-isomorphic finite Boolean algebras and 2^κ non-isomorphic Boolean algebras of cardinality κ for any $\kappa \geqq \omega$.

5. Proof of Theorem 2

The construction of Section 4 clearly indicates one suitable for the proof of Theorem 2.

Let $Q=P\times(\omega+1)$ inherit the product topology where B is a Boolean algebra with Stone space (P,τ) and $\omega+1$ has the interval topology ($\omega+1$ under this topology is the Stone space of a finite co-finite Boolean algebra on a countable set). Define (R,σ) to be the one point compactification of $Q\backslash P\times\{\omega\}$ by an element M. Let $\leqq$ be a partial order on ω for which it is a connected downward directed binary tree and define a partial order on R as follows:

(i) $M\leqq(x,y)$ for all $(x,y)\in Q$;

(ii) $(x,y)\leqq(u,v)$ iff $x=u$ and $y\leqq v$.

Clearly R is partially ordered.

Observe that, for any clopen $X\subseteq P$, $[X\times\{y\})=X\times[y)$ where $[y)\subseteq\omega$ is a finite chain. Consider $(x,y)\nleqq(u,v)$. If $x\neq u$ there is a clopen set X with $x\in X$ and $u\notin X$. Thus $[X\times\{y\})$ is a clopen increasing set that separates the pair. Otherwise $y\nleqq v$ and $[P\times\{y\})$ will suffice. Thus (R,σ) is a Priestley space. Furthermore, for any clopen set $X\subseteq Q$ either $M\in X$ or $X=\bigcup(X_i\times\{y_i\}:1\leqq i\leqq n)$ where X_i is clopen. In either case $[X)$ is clopen and (R,σ) is an h-space.

Since ω is a downward directed binary tree under $\leqq$, $(p]\cup\{q\}$ and $(q]\cup\{p\}$ are distinct whenever $p,q\in R$ are. Thus if K is the Heyting algebra with h-space (R,σ), K has no maximal subalgebras.

The proof of Theorem 2 is concluded by observing that the subspace of maximal points of R is homeomorphic to P. Thus for non-isomorphic Boolean algebras B,B', the Heyting algebras K,K' are also non-isomorphic.

6. Concluding remarks

By K. M. Koh [**10**], for every lattice L there is a lattice K such that $L\cong\Phi(K)$. An analogous statement was shown to hold in the variety of distributive lattices [**1**] and, subsequently, in every non-trivial variety of lattices [**2**]. The variety of Boolean algebras is the smallest non-trivial variety of Heyting algebras and, since the only Boolean algebra with a meet irreducible zero is the two element Boolean algebra, Theorem 1 holds in this variety. However, it does not extend to every non-trivial variety. For example, let (P,τ) be the h-space of a member H of the variety of Heyting algebras $\mathbf{V}$ generated by all totally ordered sets. $\mathbf{V}$ is well known (see, for example, R. Balbes and Ph. Dwinger [**4**]). For $x\in P$, $(x]$ is totally ordered. Consider a clopen decreasing set $\emptyset\subset A\subset P$ and choose a minimal $q\in P\backslash A$. If $(q]\cap A=\emptyset$ then q is actually minimal in P. In this case let p be a minimal element of A. Otherwise $q\succ p$ for some $p\in A$. Either way $(p]\cup\{q\}=(q]\cup\{p\}$ and, hence, $a\notin\Phi(H)$. In short, $\Phi(H)=\{0,1\}$ for any $H\in\mathbf{V}$ although this variety clearly contains a proper class of non-isomorphic algebras with meet irreducible zeros. (Clearly, Theorem 2 is already invalid in the variety of Boolean algebras.)

By [**1**] and C. C. Chen, K. M. Koh, and S. K. Tan [**5**], there are finite distributive lattices that are only isomorphic to Frattini sublattices of infinite distributive lattices. By contrast, Theorem 1 shows that representation of Heyting algebras in this way may preserve finiteness whenever it is required. (A characterization of the Frattini sublattices of finite distributive lattices is still to be found.)

Some of the results presented here were first announced by the author in *Notices Amer. Math. Soc.* **21** (1974), A-524 (see also L. Vrancken-Mawet, *Bull. Soc. Roy. Sci. Liège* **51** (1982), 82–94).

REFERENCES

1. M. E. Adams, The Frattini sublattice of a distributive lattice, *Algebra Universalis* **3** (1973), 216–228.

2. M. E. Adams and J. Sichler, Frattini sublattices in varieties of lattices, *Colloq. Math.* **44** (1981), 271–278.

3. R. Balbes, Maximal and completely meet irreducible sublattices of a distributive lattice, *Algebra Universalis* **17** (1983), 317–328.

4. R. Balbes and Ph. Dwinger, *Distributive Lattices* (University of Missouri Press, 1975).

5. C. C. Chen, K. M. Koh and S. K. Tan, On the Frattini sublattice of a finite distributive lattice, *Algebra Universalis* **5** (1975), 88–97.

6. B. A. Davey and D. Duffus, Exponentiation and duality, in *Ordered Sets* (ed. I. Rival) (NATO Advanced Study Institutes Series, D. Reidel, Dordrecht, 1982) pp. 43–95.

7. G. Grätzer, *General Lattice Theory* (Birkhäuser Verlag, Basel, 1978).

8. G. Grätzer, K. M. Koh and M. Makkai, On the lattice of subalgebras of a Boolean algebra, *Proc. Amer. Math. Soc.* **36** (1972), 87–92.

9. J. Hashimoto, Ideal theory for lattices, *Math. Japon.* **2** (1952), 149–186.

10. K. M. Koh, On the Frattini sublattice of a lattice, *Algebra Universalis* **1** (1971), 104–116.

11. H. A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.* **2** (1970), 186–190.

12. H. A. Priestley, Ordered sets and duality for distributive lattices. *Ann. Discrete Math.* **23** (1984), 39–60.

13. D. Sachs, The lattice of subalgebras of a Boolean algebra, *Canad. J. Math.* **14** (1962), 451–460.

State University of New York
New Paltz, New York
U.S.A.