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Contramodules for algebraic groups: the existence of mock projectives

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Abstract. Let G be an affine algebraic group over an algebraically closed field of positive characteristic. Recent work of Hardesty, Nakano, and Sobaje gives necessary and sufficient conditions for the existence of so-called mock injective G-modules, that is, modules which are injective upon restriction to all Frobenius kernels of G. In this article, we give analogous results for contramodules, including showing that the same necessary and sufficient conditions on G guarantee the existence of mock projective contramodules. In order to do this, we first develop contramodule analogs to many well-known (co)module constructions.

1 Introduction

Let G be an affine algebraic group defined over an algebraically closed field of characteristic p which splits over the subfield $\mathbb{F}_p$. Then, G admits a Frobenius morphism $F: G \to G$. Let $G_r = \ker(F^r)$ denote the r^{th} Frobenius kernel. In 2015, Friedlander defined a support theory for rational G-modules for many important classes of groups G and showed that a G-module has trivial support if and only if it is a mock injective module, that is, a module which is injective when restricted to all Frobenius kernels [Fri15]. It is also shown that mock injectivity of a module is a weaker condition than injectivity, i.e., there are mock injective modules which are not injective as a G-module. Such modules are called proper mock injective modules. Recent work of Hardesty, Nakano, and Sobaje gives an explicit description of when G admits proper mock injective modules [HNS17].

In this article, we consider the contramodule analog of the work of Hardesty, Nakano, and Sobaje. That is, we aim to give a description of when G admits proper mock projective contramodules, i.e., contramodules which are not projective as a $k[G]$-contramodule, but which are projective when restricted to $k[G_r]$.

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Many of the results in this article look strikingly similar to those of the work of the aforementioned authors, suggesting that looking through the lens of contramodules may be another useful way to investigate properties of algebraic groups.

The contents are as follows. In Section 2, we will give the definition of contramodules, and discuss important families of them. We will also describe the induction and restriction functors, before finishing by giving some additional constructions in the case of contramodules over a Hopf algebra.

The remaining sections are highly motivated by the work of Hardesty, Nakano, and Sobaje. In Section 3, we show that the same conditions on G to ensure the existence of proper mock injective modules will also ensure the existence of mock projective contramodules.

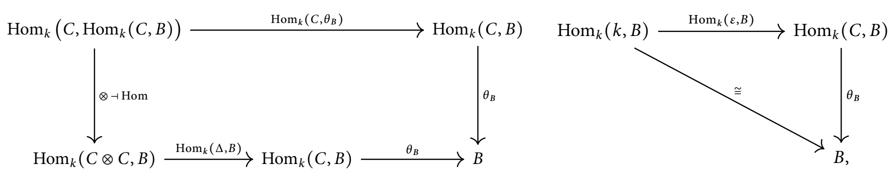
In Section 4, we investigate mock projective contramodules with certain conditions on the radical, including but not limited to finite co-dimensionally.

2 Contramodules

In this section, we define contramodules over a coalgebra and introduce their associated induction and restriction functors. We also present several results and constructions in the case, where our coalgebra has the additional structure of a Hopf algebra. For more details about contramodules, see [Pos10] or [Pos21].

2.1 First definitions

Let (C, Δ, ε) be a coalgebra over a field k, where Δ denotes the comultiplication map and ε denotes the counit. A (left) C-contramodule (B, θ_B), or just B, is a k-vector space B equipped with a linear map $\theta_B : \mathrm{Hom}_k(C, B) \longrightarrow B$, called the contra-action, satisfying contra-associativity and contra-unity conditions. That is, the following two diagrams commute:



where "$\otimes \dashv \mathrm{Hom}$" denotes the tensor-hom adjunction, which for any vector spaces U, V, W is given by identifying $\mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W))$ and $\mathrm{Hom}_k(V \otimes_k U, W)$. We remark that using instead the identification $\mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)) \cong \mathrm{Hom}_k(U \otimes_k V, W)$ gives the definition of a right C-contramodule. Unless stated otherwise, contramodules will be left contramodules.

Given two C-contramodules B and D, let $\mathrm{Hom}^C(B, D)$ denote the space of contramodule homomorphisms from B to D. That is, the linear maps $f : B \longrightarrow D$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_k(C,B) & \xrightarrow{\theta_B} & B \\ \Big\downarrow{\scriptstyle \mathrm{Hom}_k(C,f)} & & \Big\downarrow{\scriptstyle f} \\ \mathrm{Hom}_k(C,D) & \xrightarrow{\theta_D} & D. \end{array}$$

Now, consider the space $\mathrm{Hom}_k(C, k)$. It can be given the structure of a C-contramodule by applying the comultiplication of the coalgebra C in the first factor. That is, $\mathrm{Hom}_k(C, k)$ has structure map $\theta : \mathrm{Hom}_k\left(C, \mathrm{Hom}_k(C, k)\right) \longrightarrow \mathrm{Hom}(C, k)$ given by the composition:

$$\mathrm{Hom}_k(C, \mathrm{Hom}_k(C, k)) \cong \mathrm{Hom}_k(C \otimes C, k) \xrightarrow{\mathrm{Hom}_k(\Delta, k)} \mathrm{Hom}_k(C, k).$$

More generally, one may replace k with any vector space V, obtaining what we will call the *free contramodule on* V. One can show that there is an isomorphism of vector spaces

$$\mathrm{Hom}^C\left(\mathrm{Hom}_k(C, V), W\right) \cong \mathrm{Hom}_k(V, W)$$

for each C-contramodule W. In particular, free contramodules are projective. It follows that any contramodule B is projective if and only if it is a direct summand of a free contramodule. We now see that one may construct projective contramodules by taking the dual of injective comodules.

Lemma 1.1 *Let C be a coalgebra, (M, Δ_M) an injective right C-comodule, and V a vector space. Then, $\mathrm{Hom}_k(M, V)$ is a projective C-contramodule, with contra-action given by the composition:*

$$\mathrm{Hom}_k\left(C, \mathrm{Hom}_k(M, V)\right) \cong \mathrm{Hom}_k(M \otimes C, V) \xrightarrow{\mathrm{Hom}_k(\Delta_M, V)} \mathrm{Hom}_k(M, V).$$

Proof As M is an injective comodule, the coaction map $\Delta_M : M \longrightarrow M \otimes C$ splits. Applying the additive functor $\mathrm{Hom}_k(-, V)$ yields a split map of contramodules $\mathrm{Hom}_k(M \otimes C, V) \longrightarrow \mathrm{Hom}_k(M, V)$ with contra-actions induced from the coaction on the relevant comodules. However, as $M \otimes C$ is the cofree comodule on M we have an isomorphism $\mathrm{Hom}_k(M \otimes C, V) \cong \mathrm{Hom}_k\left(C, \mathrm{Hom}_k(M, V)\right)$ of contramodules, where the latter is the free contramodule on $\mathrm{Hom}_k(M, V)$. Thus, $\mathrm{Hom}_k(M, V)$ is a direct summand of a free contramodule and is therefore projective. ∎

Given a coalgebra C, we denote the category of left C-comodules by C-Comod, and the category of left C-contramodules by C-Contra. Moreover, we denote the category of right C-comodules by Comod-C, and the category of right C-contramodules by Contra-C.

2.2 Induction and restriction

Let $\pi : C \to D$ be a map of coalgebras. Then, given a C-contramodule V, one obtains a D-contramodule structure on V via the composition:

$$\mathrm{Hom}_k(D,V) \xrightarrow{\mathrm{Hom}_k(\pi,V)} \mathrm{Hom}_k(C,V) \longrightarrow V.$$

We call this the restriction to D and denote it by $\mathrm{Res}^C_D(V)$, $\mathrm{Res}(V)$, or $V|_D$.[1]

Now, let (M,Δ_M) be a left D-comodule and let (B,θ_B) be a left D-contramodule. Then, $\mathrm{Cohom}_D(M,B)$ denotes the cohomomorphisms between M and B. It is a quotient vector space of $\mathrm{Hom}_k(M,B)$ given by the following coequaliser:

$$\mathrm{Cohom}_D(M,B) = \mathrm{coeq}\left(\mathrm{Hom}_k(D\otimes M,B) \underset{\mathrm{Hom}_k(M,\theta_B)}{\overset{\mathrm{Hom}_k(\Delta_M,B)}{\rightrightarrows}} \mathrm{Hom}_k(M,B)\right)$$

with $\mathrm{Hom}_k(M,\theta_B) : \mathrm{Hom}_k(D\otimes M,B) \cong \mathrm{Hom}_k\left(M,\mathrm{Hom}_k(D,B)\right) \to \mathrm{Hom}_k(M,B)$. In particular, when $M = C$ with D-comodule structure given by $(\pi\otimes \mathrm{id})\circ\Delta_C$ we can equip $\mathrm{Cohom}_D(C,B)$ with a C-contramodule structure by observing it is nothing more than a quotient of the free contramodule $\mathrm{Hom}_k(C,B)$. This is the induction from D-contramodules to C-contramodules. We denote the resulting contramodule by $\mathrm{Ind}^C_D(B)$. One can show that induction and restriction form an adjoint pair.

Lemma 1.2 *The functor Ind : D-Contra $\longrightarrow$ C-Contra is left adjoint to the functor Res : C-Contra $\longrightarrow$ D-Contra, that is, for all $B \in$ D-Contra and $V \in$ C-Contra we have*

$$\mathrm{Hom}^C\left(Ind^C_D(B),V\right) \cong \mathrm{Hom}^D\left(B, Res^C_D(V)\right).$$

2.3 Contramodules over a Hopf algebra

In the case of (co-)modules over a (co-)algebra, if one in fact has a Hopf algebra structure then one may equip the relevant module category with a monoidal structure. For contramodules this is not quite the case. Instead, we can produce new contramodules via a bifunctor which takes a right comodule and left contramodule as arguments. In this section, we explicitly describe this bifunctor and give some properties of it.

Let $(H,\nabla,\eta,\Delta,\varepsilon,S)$ be a Hopf algebra. Recall that one may write the comultiplication using Sweedler's notation, that is, given $c\in H$ we write the comultiplication as $\Delta(c) = \sum c_{(1)}\otimes c_{(2)}$, with coassociativity implying that we may write $(\mathrm{id}\otimes\Delta)\circ\Delta(c) = (\Delta\otimes\mathrm{id})\circ\Delta(c) = \sum c_{(1)}\otimes c_{(2)}\otimes c_{(3)}$.

Given a right H-comodule M and a left H-contramodule B, we may equip $\mathrm{Hom}_k(M,B)$ with a "diagonal" contramodule structure via the following composition:

$$\begin{aligned}
&\mathrm{Hom}_k\left(H,\mathrm{Hom}_k(M,B)\right) \xrightarrow{\mathrm{Hom}_k(\nabla,\mathrm{Hom}_k(M,B))} \mathrm{Hom}_k\left(H\otimes H,\mathrm{Hom}_k(M,B)\right)\\
&\cong \mathrm{Hom}_k\left(M\otimes H,\mathrm{Hom}_k(H,B)\right) \xrightarrow{\mathrm{Hom}_k(\Delta_M,\mathrm{Hom}_k(H,B))} \mathrm{Hom}_k(M,\mathrm{Hom}_k(H,B))\\
&\xrightarrow{\mathrm{Hom}_k(M,\theta_B)} \mathrm{Hom}_k(M,B),
\end{aligned}$$

[1] We omit explicit reference to the map π in the notation as it will always be clear from context.

where the intermediate identification is given by

$$\mathrm{Hom}_k\left(T\otimes U, \mathrm{Hom}_k(V,W)\right) \cong \mathrm{Hom}_k\left(V\otimes T, \mathrm{Hom}_k(U,W)\right).$$

One readily checks that this gives $\mathrm{Hom}_k(M,B)$ the structure of a H-contramodule. That is, given a Hopf algebra H, we have a bifunctor

$$\mathrm{Hom}_k(-,-) : \text{Comod-}H^{\mathrm{op}} \times H\text{-Contra} \longrightarrow H\text{-Contra}.$$

Let G be an algebraic group over a field k, and let $k[G]$ denote its coordinate ring. Then, $k[G]$ is a Hopf algebra. Moreover, let $T\subset G$ denote a maximal torus, and $k[T]$ be its coordinate ring. Finally, let $X(T)\subset k[T]$ denote the weights of T [Jan03, Section I.2.4]. Recall that for a right $k[G]$-comodule M, the weight space with weight $\lambda\in X(T)$ is given by

$$M_\lambda = \{m\in M : \Delta_M(m) = m\otimes\lambda\}.$$

We may also define weight spaces for contramodules. Given a $k[G]$-contramodule B, we may view it as a $k[T]$-contramodule via restriction along the coalgebra map $k[G]\to k[T]$ corresponding to the inclusion $T\subset G$. Via a slight abuse of notation, we also write θ_B for the contramodule structure map when considering B as a $k[T]$-contramodule. Then, we define the weight space with weight $\lambda\in X(T)$ as

$$B_\lambda = \left\{b\in B : \text{ for all } \phi\in\mathrm{Hom}_k\left(k[T],\langle b\rangle\right) \text{ we have } \phi(\lambda) = \theta_B(\phi)\right\},$$

where $\langle b\rangle\subset B$ denotes the subspace spanned by b. It is a fact that $B=\prod_{\lambda\in X(T)} B_\lambda$ as a $k[T]$-contramodule, which follows readily from [Pos10, Lemma 2, A.2]. We now describe the weight spaces of $\mathrm{Hom}_k(M,B)$. Note that we use additive notation for the weights.

Lemma 1.3 *Let M be a right $k[G]$-comodule and B a left $k[G]$-contramodule. Then, we have*

$$\mathrm{Hom}_k(M,B)_\lambda = \prod_{\alpha+\beta=\lambda} \mathrm{Hom}_k(M_\alpha, B_\beta).$$

Proof Let $\alpha,\beta\in X(T)$ such that $\alpha+\beta=\lambda$. We calculate explicitly the image of $\mathrm{Hom}_k(M_\alpha,B_\beta)$ under the diagonal action.

Given $(h\mapsto\phi_h)\in\mathrm{Hom}_k(k[T],\mathrm{Hom}_k(M_\alpha,B_\beta))$ we have

$$\begin{aligned}\left(h\mapsto\phi_h\right) &\longmapsto \left(h\otimes h'\mapsto\phi_{hh'}\right)\longmapsto\left(m\otimes h\mapsto(h'\mapsto\phi_{hh'}(m))\right)\\ &\longmapsto\left(m\mapsto\theta_B(f\mapsto\phi_{\alpha f}(m))\right) = \phi_{\alpha+\beta}\end{aligned}$$

so indeed $\mathrm{Hom}_k(M_\alpha,B_\beta)\subset\mathrm{Hom}_k(M,B)_\lambda$. Equality follows from the fact that

$$\mathrm{Hom}_k(M,B) = \mathrm{Hom}_k\left(\bigoplus_\alpha M_\alpha, \prod_\beta B_\beta\right) = \prod_{\alpha,\beta}\mathrm{Hom}_k\left(M_\alpha,B_\beta\right). \qquad \blacksquare$$

The next lemma may be thought of as a contra-analog of the tensor identity for modules [Jan03, Proposition I.3.6]. We will dub this the “hom identity for contramodules”.

Lemma 1.4 (Hom identity) *Let M be a right $k[G]$-comodule. Let $H \subset G$ be a subgroup of G, and B a left $k[H]$-contramodule. Then,*

$$\mathrm{Hom}_k\left(M, Ind_{k[H]}^{k[G]} B\right) = Ind_{k[H]}^{k[G]}\Big(\mathrm{Hom}_k(M,B)\Big),$$

where the contramodule structure on $\mathrm{Hom}_k(-,-)$ is the diagonal action in both cases, and where $Ind_{k[H]}^{k[G]}(-)$ is defined using the map of coalgebras $k[G] \to k[H]$ corresponding to the inclusion $H \subset G$.

Proof For ease of notation throughout the proof, we assign the labels

$$L := \mathrm{Hom}_k\left(M, \mathrm{Ind}_{k[H]}^{k[G]} B\right) \quad R := \mathrm{Ind}_{k[H]}^{k[G]}\Big(\mathrm{Hom}_k(M,B)\Big).$$

Recalling the definition of Cohom from section 2, we see that as vector spaces both L and R are quotients of $\mathrm{Hom}_k(k[G]\otimes M, B)$. The steps of the proof will be as follows:

(1) Define linear maps $\alpha, \beta : \mathrm{Hom}_k(k[G]\otimes M, B) \to \mathrm{Hom}_k(k[G]\otimes M, B)$ with α and β inverse to one another (as maps of vector spaces).
(2) Show that α and β factor to give maps from L to R. (Note that this is a well-definedness check.)
(3) Show that in fact α, β are $k[G]$-contramodule homomorphisms.

To begin, we define α and β as follows:

$$\begin{aligned}\alpha : \mathrm{Hom}_k(k[G]\otimes M, B) &\longrightarrow \mathrm{Hom}_k(k[G]\otimes M, B)\\ \phi &\longmapsto \phi \circ \mu^T \circ (\mathrm{Id}_{k[G]\otimes M} \otimes S) \circ (\mathrm{Id}_{k[G]} \otimes \Delta_M)\end{aligned}$$

$$\begin{aligned}\beta : \mathrm{Hom}_k(k[G]\otimes M, B) &\longrightarrow \mathrm{Hom}_k(k[G]\otimes M, B)\\ \psi &\longrightarrow \psi \circ \mu^T \circ (\mathrm{Id}_{k[G]} \otimes \Delta_M),\end{aligned}$$

where $\mu^T : k[G]\otimes M \otimes k[G] \longrightarrow k[G]\otimes M$ denotes a certain twisted multiplication, given by $\mu^T(f\otimes m\otimes g) = gf \otimes m$. Concretely, we have for $f\otimes m \in k[G]\otimes M$

$$\alpha(\phi)(f\otimes m) = \phi\big(S(m_{(1)})f \otimes m_{(0)}\big)$$

$$\beta(\psi)(f\otimes m) = \psi\big(m_{(1)}f \otimes m_{(0)}\big).$$

We now check that these maps are inverse to one another. We will only check that $\beta\circ\alpha \equiv \mathrm{Id}_{\mathrm{Hom}_k(k[G]\otimes M,B)}$, as checking that $\alpha\circ\beta \equiv \mathrm{Id}$ is similar.

Let $\phi \in \mathrm{Hom}_k(k[G]\otimes M, B)$. Then, we have

$$\begin{aligned}(\beta\circ\alpha)(\phi) &= \beta\Big(f\otimes m \longmapsto \phi\big(S(m_{(1)})f\otimes m_{(0)}\big)\Big)\\ &= \Big(f\otimes m \longmapsto \phi\big(S(m_{(1)})m_{(2)}f\otimes m_{(0)}\big)\Big)\\ &= \Big(f\otimes m \longmapsto \phi\big(\varepsilon(m_{(1)})f\otimes m_{(0)}\big)\Big) = \Big(f\otimes m \longmapsto \phi(f\otimes m)\Big)\end{aligned}$$

and, therefore, $(\beta\circ\alpha)(\phi) = \phi \in \mathrm{Hom}_k(k[G]\otimes M, B)$.

Now, as vector spaces, we have (with implicit applications of the tensor hom adjunction):

$$L = \operatorname{coeq}\left(\operatorname{Hom}_k(M \otimes k[H] \otimes k[G], B) \underset{\operatorname{Hom}_k\left(M \otimes k[G], \theta_B\right)}{\overset{\operatorname{Hom}_k\left(M \otimes \Delta_{k[G]}, B\right)}{\rightrightarrows}} \operatorname{Hom}_k(M \otimes k[G], B) \right).$$

$$R = \operatorname{coeq}\left(\operatorname{Hom}_k(M \otimes k[H] \otimes k[G], B) \underset{\operatorname{Hom}_k\left(k[G], \theta_{\operatorname{Hom}_k(M,B)}\right)}{\overset{\operatorname{Hom}_k\left(M \otimes \Delta_{k[G]}, B\right)}{\rightrightarrows}} \operatorname{Hom}_k(M \otimes k[G], B) \right).$$

To assist in easing notation, we will write $f_L = \operatorname{Hom}_k\left(M \otimes \Delta_{k[G]}, B\right)$, and $g_L = \operatorname{Hom}_k\left(M \otimes k[G], \theta_B\right)$ for the two maps defining L. Similarly, we will write $f_R = \operatorname{Hom}_k\left(M \otimes \Delta_{k[G]}, B\right)$ and $g_R = \operatorname{Hom}_k\left(k[G], \theta_{\operatorname{Hom}_k(M,B)}\right)$ for the two maps defining R.

Now, α composed with the natural quotient map from $\operatorname{Hom}_k(M \otimes k[G], B)$ to R gives us a map from $\operatorname{Hom}_k(M \otimes k[G], B)$ to R which we also denote α. We now want to check that this α factors through L. In other words, we wish to check that $\operatorname{Im}\left(\alpha \circ (f_L - g_L)\right) \subset \operatorname{Im}\left(f_R - g_R\right)$.

This is equivalent to finding a linear endomorphism $T \in \operatorname{End}\Big(\operatorname{Hom}_k(k[H] \otimes k[G] \otimes M, B)\Big)$ with $\alpha \circ (f_L - g_L) = (f_R - g_R) \circ T$. One checks that $T := \operatorname{Hom}_k \left(\mu_{4,2}^{5,1} \circ \left(\operatorname{Id}^{\otimes 2} \otimes \left((\operatorname{Id} \otimes S^{\otimes 2}) \circ \Delta_M^2\right)\right), B\right)$ satisfies this, where both μ_{ij} and μ^{ij} denotes multiplication given by taking the element in the i^{th} factor of the tensor and the j^{th} factor of the tensor, multiplying them together, and letting the resulting product replace the factor taken from the j^{th} position. Concretely, we have, for an algebra A, say,

$$\begin{aligned} \mu_{ij} : A^{\otimes n} &\longrightarrow A^{\otimes n-1} \\ a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_j \otimes \cdots \otimes a_n &\longmapsto a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes \cdots \otimes a_i a_j \otimes \cdots \otimes a_n. \end{aligned}$$

Similarly, to show that β gives a well defined map from R to L, we must find a linear endomorphism U such that $(f_L - g_L) \circ U = \beta \circ (f_R - g_R)$. Once again, one readily checks that $U := \operatorname{Hom}_k\left(\mu_{4,1}^{5,2} \circ \left(\operatorname{Id}^{\otimes 2} \otimes \Delta_M^2\right), B\right)$ satisfies this condition.

So far, we have that $\alpha : L \to R$ and $\beta : R \to L$ are isomorphisms of vector spaces which are inverse to one another. To conclude, we show that, in fact, α is a map of contramodules. It will be sufficient to check that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}_k\left(k[G], \operatorname{Hom}_k\left(M, \operatorname{Hom}_k(k[G], B)\right)\right) & \xrightarrow{\theta_{\text{diag}}} & \operatorname{Hom}_k\left(M, \operatorname{Hom}_k(k[G], B)\right) \\ \Big\downarrow {\scriptstyle \operatorname{Hom}_k(k[G], \alpha)} & & \Big\downarrow {\scriptstyle \alpha} \\ \operatorname{Hom}_k\left(k[G], \operatorname{Hom}_k\left(M, \operatorname{Hom}_k(k[G], B)\right)\right) & \xrightarrow{\theta_{\text{free}}} & \operatorname{Hom}_k\left(M, \operatorname{Hom}_k(k[G], B)\right). \end{array}$$

Here, θ_{diag} denotes the diagonal contra-action on $\mathrm{Hom}_k\left(M, \mathrm{Hom}_k(k[G], B)\right)$, and θ_{free} denotes the free contra-action on $\mathrm{Hom}_k\Big(M, \mathrm{Hom}_k(k[G], B)\Big) \cong \mathrm{Hom}_k\Big(k[G], \mathrm{Hom}_k(M, B)\Big)$.

Let $\varphi \in \mathrm{Hom}_k\Big(k[G], \mathrm{Hom}_k\left(M, \mathrm{Hom}_k(k[G], B)\right)\Big)$ be denoted as the map $f \mapsto \Big(m \mapsto \left(g \mapsto b(f, m, g)\right)\Big)$, where $b(-,-,-) : k[G] \otimes M \otimes k[G] \longrightarrow B$. Travelling vertically then horizontally yields

$$\Big(\theta_{\mathrm{free}}(\alpha \circ \varphi)\Big)(m) = \Bigg(m \longmapsto \Big(f \longmapsto b\big(f_{(2)}, m_{(0)}, S(m_{(1)})f_{(1)}\big)\Big)\Bigg).$$

On the other hand, travelling horizontally then vertically yields

$$\Big(\alpha \circ \theta_{\mathrm{diag}}(\varphi)\Big)(m) = \Bigg(m \longmapsto \Big(f \longmapsto b\big(m_{(1)}S(m_{(2)})f_{(2)}, m_{(0)}, S(m_{(3)})f_{(1)}\big)\Big)\Bigg)$$

which after observing that $m_{(0)} \otimes m_{(1)}S(m_{(2)}) \otimes m_{(3)} = m_{(0)} \otimes m_{(1)} \otimes 1$ and using that $b(-,-,-)$ is tensorial gives equality.

Finally, since $\alpha : L \to R$ is a $k[G]$-contramodule isomorphism with linear inverse $\beta : R \to L$, we deduce that β is also a $k[G]$-contramodule isomorphism and the proof is complete. ■

We immediately obtain the following corollary.

Corollary 1.5 *Let P be a projective $k[G]$-contramodule. Then, for any right $k[G]$-comodule M we have that $\mathrm{Hom}_k(M, P)$, with diagonal action, is a projective $k[G]$-contramodule.*

Proof Since P is projective, the contra-action map $\theta : \mathrm{Hom}_k(k[G], P) \longrightarrow P$ splits. Now, consider the additive functor $\mathrm{Hom}_k(M, -) : k[G]\text{-Contra} \longrightarrow k[G]\text{-Contra}$ which equips the resulting contramodule with the diagonal action. Applying this to θ above we have

$$\mathrm{Hom}_k\left(M, \mathrm{Hom}_k(k[G], P)\right) \xrightarrow{\mathrm{Hom}_k(M,\theta)} \mathrm{Hom}_k(M, P)$$

which splits. Thus, $\mathrm{Hom}_k(M, P)$ is a direct summand of $\mathrm{Hom}_k\left(M, \mathrm{Hom}_k(k[G], P)\right)$. However, by the previous lemma, we have

$$\mathrm{Hom}_k\left(M, \mathrm{Hom}_k(k[G], P)\right) \cong \mathrm{Hom}_k\left(k[G], \mathrm{Hom}_k(M, P)\right),$$

where the latter is the free contramodule on the vector space $\mathrm{Hom}_k(M, P)$. Therefore, $\mathrm{Hom}_k(M, P)$ is a direct summand of a free contramodule and is, therefore, projective. ■

To conclude the section, we give a final contra-analog of a construction well-known for modules over a group. Namely, let $G = N \rtimes K$, and M be a (right) K-module. Then, M has a (left) $k[K]$-comodule structure and $\mathrm{ind}_K^G M = k[G] \boxtimes_{k[K]} M \cong k[N] \otimes_k M$, where $K < G$ acts on $k[N]$ via conjugation, and $\boxtimes$ denotes the cotensor product. For contramodules, the obvious analog to this holds. We have the following.

Lemma 1.6 *Let $G = N \rtimes K$ with associated coordinate rings $k[G], k[N]$ and $k[K]$. Then, for any $k[K]$-contramodule (M, θ_M), we have*

$$\mathit{Ind}_{k[K]}^{k[G]}(M) \cong \mathrm{Hom}_k(k[N], M),$$

where the contramodule structure on the right hand side is the diagonal action and the right $k[K]$-comodule structure on $k[N]$ is induced from the conjugation action of K on $k[N]$.

Proof To begin, it will serve us well to establish some notation. Let $\iota : N \to G$ denote the natural inclusion and $\iota^* : k[G] \to k[N]$ be the corresponding map of coordinate rings. Let $\Delta_R : k[G] \to k[G] \otimes k[K]$ denote the right $k[K]$-comodule structure on $k[G]$ corresponding to left multiplication $K \times G \to G$, similarly define $\Delta_L : k[G] \to k[K] \otimes k[G]$ corresponding to right multiplication of K on G. Finally, let $\Delta_{\mathrm{cong}} : k[N] \to k[N] \otimes k[K]$ denote the $k[K]$-comodule structure on $k[N]$ induced from conjugation $K \times N \to N; (k, n) \mapsto knk^{-1}$.

Recall that as vector spaces we have $\mathrm{Ind}_{k[K]}^{k[G]}(M) \cong \mathrm{Cohom}_{k[K]}(k[G], M)$, and one equips this space with contramodule structure by realizing it as a quotient of the free contramodule on M. Now, consider the following map:

$$\begin{aligned} \mathrm{Hom}_k(k[N], M) &\longrightarrow \mathrm{Ind}_{k[K]}^{k[G]}(M) \\ \phi &\longmapsto [\phi \circ \iota^*], \end{aligned}$$

where $[\cdot]$ denotes the equivalence class. This is clearly an isomorphism of vector spaces. So all that remains is to check that it preserves the $k[G]$-contramodule structure. We also observe that the crux of the proof lies in checking that the $k[K]$-contramodule structure (given by restriction) is preserved. Thus, consider the following diagram, which we wish to show commutes:

$$\begin{array}{ccc} \mathrm{Hom}_k\left(k[K], \mathrm{Hom}_k(k[N], M)\right) & \longrightarrow & \mathrm{Hom}_k\left(k[K], \mathrm{Ind}(M)\right) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_k(k[N], M) & \longrightarrow & \mathrm{Ind}(M). \end{array}$$

Let $\left(k \mapsto (n \mapsto m(k, n))\right) \in \mathrm{Hom}_k\left(k[K], \mathrm{Hom}_k(k[N], M)\right)$. Travelling horizontally and then vertically gives $\left[g \longmapsto m\Big(\Delta_R(g)_{(1)}, \iota^*\big(\Delta_R(g)_{(0)}\big)\Big)\right] \in \mathrm{Ind}(M)$.

On the other hand, travelling vertically and then horizontally gives

$$\begin{aligned} &\left[g \longmapsto \theta_M\Big(k \longmapsto m\Big(\Delta_{\mathrm{cong}}(g)_{(1)} \cdot k, \iota^*(\Delta_{\mathrm{cong}}(g)_{(0)})\Big)\Big)\right] \\ \equiv &\left[g \longmapsto m\Big(\Delta_{\mathrm{cong}}\big(\Delta_L(g)_{(0)}\big)_{(1)} \cdot \Delta_L(g)_{(-1)}, \iota^*\Big(\Delta_{\mathrm{cong}}\big(\Delta_L(g)_{(0)}\big)_{(0)}\Big)\Big)\right] \in \mathrm{Ind}(M). \end{aligned}$$

Observe that is it now sufficient to show that

$$\Delta_R(g) = \Delta_{\text{cong}}\Big(\Delta_L(g)_{(0)}\Big)_{(0)} \otimes \Delta_{\text{cong}}\Big(\Delta_L(g)_{(0)}\Big)_{(1)} \cdot \Delta_L(g)_{(-1)}$$

or more concisely that $\Delta_R \equiv (\mathrm{id} \otimes \mu) \circ \omega \circ (\mathrm{id} \otimes \Delta_{\text{cong}}) \circ \Delta_L$, where $\omega(x \otimes y \otimes z) = (y \otimes z \otimes x)$ and $(\mathrm{id} \otimes \mu)(x \otimes y \otimes z) = x \otimes yz$.

However, recall that Δ_R is the map of coordinate rings associated with $K \times G \to G; (k, g) \mapsto kg$ and the right hand side is the map corresponding to the composition

$$\begin{aligned} K \times G \longrightarrow K \times K \times G \longrightarrow K \times G \times K \longrightarrow G \times K \to G \\ (k, g) \longmapsto (k, k, g) \longmapsto (k, g, k) \longmapsto (kgk^{-1}, k) \longmapsto kg \end{aligned}$$

and so they are equal, as required. ∎

3 Conditions on $k[G]$ for the existence of proper mock projective contramodules

Let G be an affine group scheme over an algebraically closed field k. Given a subgroup scheme H of G, we say that H is contra-exact in G if the induction functor (of contramodules), $\mathrm{Ind}_{k[H]}^{k[G]}(-)$, is exact. Analogously, one says that H is exact in G if the induction functor of modules is exact. It turns out that these are equivalent [Joh25]. Therefore, we may drop the prefix "contra" and just say that H is exact in G. As an example, all finite subgroup schemes of G are exact.

Before giving our first result on conditions on G for mock projective contramodules to exist we give a useful lemma, showing that restriction takes projective contramodules to projective contramodules, provided the subscheme is exact.

Lemma 2.1 *Let H be exact in G. Then, the restriction functor $\mathrm{Res}_{k[H]}^{k[G]}$ takes projective contramodules to projective contramodules.*

Proof Let $B \in k[G]$-Contra be projective. Then, B is a direct summand of $\mathrm{Hom}_k(k[G], B)$, a free contramodule. Since restriction is an additive functor, B is also a direct summand of $\mathrm{Hom}_k(k[G], B)$ as $k[H]$-contramodules. As H is exact in G, $k[G]$ is an injective $k[H]$-contramodule and so $k[G]$ is a direct summand of $k[G] \otimes k[H]$.

Now, the functor $\mathrm{Hom}_k(-, B) : \mathrm{Comod} - k[H] \longrightarrow k[H]$-Contra is additive and thus we have that $\mathrm{Hom}_k(k[G], B)$ is a direct summand of $\mathrm{Hom}_k(k[G] \otimes k[H], B)$ as $k[H]$- contramodules. Combining both direct summand inclusions, we have that B is a direct summand of $\mathrm{Hom}_k(k[G] \otimes k[H], B) \cong \mathrm{Hom}_k\big(k[H], \mathrm{Hom}_k(k[G], B)\big)$, with the latter free. Thus, B is projective, as it is a direct summand of a free contramodule. ∎

Proposition 2.2 *Let H be a finite subgroup scheme in G with coordinate rings $k[H]$ and $k[G]$ respectively. Then:*

a) *$\mathrm{Ind}_{k[H]}^{k[G]} k$ is a projective $k[G]$-contramodule if and only if k is a projective $k[H]$-contramodule.*

b) *If the Frobenius map $F : G \longrightarrow G$ restricts to an automorphism of H, then $\mathrm{Ind}_{k[H]}^{k[G]} k$ is projective over G_r for all $r > 0$.*

Before giving the proof, let us remark that when H is finite, then there is an equivalence of categories between $k[H]$-Contra and $k[H]$-Comod (or equivalently Mod-H, the category of right H-modules) [Joh25].[2] Explicitly, if $M \in k[H]$-Comod, then M can be equipped with a H-contramodule structure via the following composition:

$$\mathrm{Hom}_k(H, M) \cong H^* \otimes M \xrightarrow{H^* \otimes \Delta_M} H^* \otimes H \otimes M \xrightarrow{\mathrm{eval} \otimes M} k \otimes M \cong M.$$

In the other direction, if $B \in H$-Contra, then we have a (right) H^*-module structure (equivalently, a left H-comodule structure) on B via $\mathrm{Hom}_k(H, B) \cong B \otimes H^*$.

Proof We first prove part a). For the if direction, simply observe that since restriction takes epimorphisms to epimorphisms, induction takes projective contramodules to projective contramodules. To prove the only if direction we wish for some sort of "generalised Frobenius reciprocity" for contramodules; we develop this via the Grothendieck spectral sequence [Jan03, Section I.4.1] [Lan12, Theorem XX.9.6].

Observe that for any $k[G]$-contramodule V, the adjunction between induction and restriction may be viewed as an isomorphism of functors

$$\mathrm{Hom}^{k[G]}(-, V) \circ \mathrm{Ind}_{k[H]}^{k[G]\ \mathrm{op}}(-) \ \cong\ \mathrm{Hom}^{k[H]}\left(-, V|_{k[H]}\right),$$

where notably we have $\mathrm{Ind}_{k[H]}^{k[G]\ \mathrm{op}} : k[H]\text{-Contra}^{\mathrm{op}} \longrightarrow k[G]\text{-Contra}^{\mathrm{op}}$. Since contramodule categories have enough projectives, opposite contramodule categories have enough injectives. Furthermore, $\mathrm{Ind}_{k[H]}^{k[G]\ \mathrm{op}}$ is exact since H is a finite subgroup scheme of G. One checks that all other requirements to apply Grothendieck's spectral sequence (specifically special case (2) of the Proposition in [Jan03, Section I.4.1]) are satisfied, and so we have an isomorphism

$$\mathrm{Ext}^n_{k[G]\text{-Contra}}\left(\mathrm{Ind}_{k[H]}^{k[G]} W, V\right) \ \cong\ \mathrm{Ext}^n_{k[H]\text{-Contra}}\left(W, V\right)$$

for each $V \in k[G]$-Contra, $W \in k[H]$-Contra. In particular for $V = W = k$ and $n > 0$, we have

$$\mathrm{Ext}^n_{k[H]\text{-Contra}}\left(k, k\right) = 0 \text{ for all } n > 0$$

since $\mathrm{Ind}_{k[H]}^{k[G]} k$ is projective by assumption. Finally, since H is finite, we have an equivalence of categories between $k[H]$-Contra and Mod-H, the category of right H modules. Now, by the theory of cohomological support varieties, we have that k is a projective H-Mod, and thus a projective $k[H]$-contramodule [FP05, Theorem 5.6 (5)].

For part b), let I be an injective $k[G]$-comodule. Since H is exact in G the restriction of I to $k[H]$ is an injective $k[H]$-comodule [CPS77, Proposition 2.1]. If the Frobenius

[2] The equivalence holds for arbitrary H if we restrict ourselves to only finite dimensional objects. The key is to have an isomorphism $\mathrm{Hom}_k(H, -) \cong - \otimes H^*$.

morphism F restricts to an automorphism of H, then $I^{(r)}$ is also an injective $k[H]$-comodule for any $r > 0$. Therefore, $\mathrm{Hom}_k\left(I^{(r)}, k\right)$ is a projective $k[H]$-contramodule by Lemma 1.1. Furthermore, by Lemma 1.4, we have

$$\mathrm{Hom}_k\left(I^{(r)}, \mathrm{Ind}_{k[H]}^{k[G]} k\right) \cong \mathrm{Ind}_{k[H]}^{k[G]}\Big(\mathrm{Hom}_k\left(I^{(r)}, k\right)\Big).$$

Since induction takes projective objects to projective objects, the right hand side (and, therefore, the left hand side) is a projective $k[G]$-contramodule. Furthermore, since G_r is exact in G, restriction takes projective objects to projective objects, and so the restriction of the left hand side is a projective $k[G_r]$-contramodule. But now, as a $k[G_r]$-contramodule we have

$$\mathrm{Hom}_k\left(I^{(r)}, \mathrm{Ind}_{k[H]}^{k[G]} k\right) \cong \mathrm{Hom}_k\left(\bigoplus_{i=1}^{\dim(I)} k, \mathrm{Ind}_{k[H]}^{k[G]} k\right) \cong \prod_{i=1}^{\dim(I)} \mathrm{Ind}_{k[H]}^{k[G]} k.$$

Thus, as $k[G_r]$-contramodules, $\mathrm{Ind}_{k[H]}^{k[G]} k$ is a direct summand of a projective contramodule, and so is itself projective. ∎

Proposition 2.3 *Let H be a finite subgroup scheme of G for which every simple $k[H]$-contramodule is the restriction of a $k[G]$-contramodule. Then, for any right $k[G]$-comodule M,* $\mathrm{Hom}_k\left(M, Ind_{k[H]}^{k[G]} k\right)$ *is a projective $k[G]$-contramodule if and only if* $\mathrm{Hom}_k(M, k)$ *is a projective $k[H]$- contramodule.*

Proof Using the Grothendieck spectral sequence, as seen in the proof of Proposition 2.2, we have that for any $k[G]$-contramodule B:

$$\mathrm{Ext}^n_{k[G]\text{-Contra}}\left(\mathrm{Ind}_{k[H]}^{k[G]}\big(\mathrm{Hom}_k(M, k)\big), B\right) \cong \mathrm{Ext}^n_{k[H]\text{-Contra}}\big(\mathrm{Hom}_k(M, k), B\big).$$

As every simple $k[H]$-contramodule comes from a $k[G]$-contramodule, by assumption, we immediately conclude that $\mathrm{Hom}_k(M, k)$ is projective if and only if $\mathrm{Ind}_{k[H]}^{k[G]}\big(\mathrm{Hom}_k(M, k)\big)$ is. Moreover, by Lemma 1.4, we have that $\mathrm{Ind}_{k[H]}^{k[G]}\big(\mathrm{Hom}_k(M, k)\big) \cong \mathrm{Hom}_k\left(M, \mathrm{Ind}_{k[H]}^{k[G]} k\right)$. ∎

We may now describe conditions on an algebraic group scheme for it to have proper mock projective contramodules. In order to assist with the proof, we first give the following lemma.

Lemma 2.4 [Pos10, Lemma 2, Appendix A.2] *Let C be a coalgebra which is the direct sum of a family of coalgebras C_α. Then, any left contramodule B over C is the product of a uniquely defined family of left contramodules B_α over C_α.*

In particular, if C is cosemisimple, then any contramodule over C is the direct product of simple contramodules. With this fact in hand, we may now state and prove our theorem.

Theorem 2.5 *Let G be an affine algebraic group scheme over a field k which is defined and split over a finite subfield $\mathbb{F}_q \subset k$. Let G^0 denote the connected component of G. Then, the following are equivalent:*

i) *$k[G]$ has proper mock projective contramodules;*
ii) *G has proper mock injective modules;*
iii) *either G^0 is not a torus or G/G^0 has order divisible by p.*

Proof The analogous result of Hardesty, Nakano, and Sobaje gives $ii) \iff iii)$ [HNS17, Theorem 2.2.1]. To show $i) \iff iii)$ we use a similar proof technique. If p does not divide the order of $G(\mathbb{F}_q)$, then G^0 is a torus and G/G^0 is a finite group of order not divisible by p. Thus, every element of G is semisimple ([Nag61, Theorem 2]) and so $k[G]$ is cosemisimple. Therefore, by Lemma 2.4, every contramodule is a direct product of simple contramodules. Thus, every $k[G]$-contramodule is projective since all maps of contramodules are given as products of maps between the simple constituents, and so there cannot be any proper mock projective contramodules. On the other hand, if p divides the order of $G(\mathbb{F}_q)$ then k is a non-projective $k[G(\mathbb{F}_q)]$-contramodule. Thus, by Proposition 2.2, $\mathrm{Ind}_{k[G(\mathbb{F}_q)]}^{k[G]} k$ is a non-projective $k[G]$-contramodule whilst being projective as a contramodule over $k[G_r]$ for all $r > 0$. ∎

We now produce a family of non-projective $k[G]$-contramodules which are projective with respect to the fixed point subgroups of powers of the Frobenius map.

Proposition 2.6 *Let G be an affine algebraic group defined over $\mathbb{F}_p$. Let P be a projective $k[G]$-contramodule. Then, $P^{(r)}$, $r > 0$, is projective as a $k[G(\mathbb{F}_q)]$-contramodule, $q = p^s$ for all large enough s, but is not projective as a $k[G]$-contramodule.*

Proof As $G(\mathbb{F}_q)$ is finite, it is exact in G and therefore $\mathrm{Res}_{k[G(\mathbb{F}_q)]}^{k[G]} P$ is a projective contramodule. For $r < s$, the r^{th} power of the Frobenius map is an automorphism of $G(\mathbb{F}_q)$, and so $P^{(r)}$ is also projective over $k[G(\mathbb{F}_q)]$. Finally, as a $k[G_r]$-contramodule, it is trivial. Thus, it is not projective over $k[G_r]$ and so cannot be projective over $k[G]$. ∎

4 Mock projectives with cofinite radicals

In this section, G is a connected reductive algebraic group scheme over a field k. We wish to investigate mock projective contramodules which have a finite head, that is, contramodules whose largest semisimple quotient is finite dimensional. We begin our investigation by looking at contramodules over the coordinate ring of a unipotent group.

Let U be a connected unipotent group over k which is defined over $\mathbb{F}_p$ and let $k[U]$ be its coordinate ring. We first want to classify all simple $k[U]$-contramodules. It turns out that, just as in the case of $k[U]$-comodules, there is only one.

Lemma 3.1 *Let U be unipotent. Then, k is the only simple $k[U]$-contramodule.*

Proof Let $\varepsilon : k[U] \longrightarrow k$ denote the counit map. Then $k[U] = k \oplus \ker(\varepsilon)$, where $\ker(\varepsilon)$ is a conilpotent coalgebra. Let (S, θ) be a simple $k[U]$-contramodule. Let $S' = \mathrm{Im}\left(\theta|_{\mathrm{Hom}_k(\ker(\varepsilon),S)}\right)$ denote the image of θ under the restriction to $\mathrm{Hom}_k\left(\ker(\varepsilon), S\right)$. Then, we have $S' \subsetneq S$ [Pos10, Appendix A.2, Lemma 1]. Now, consider the restriction of θ to $\mathrm{Hom}_k(k, S') \subset \mathrm{Hom}_k(k[U], S)$. We have by counity that $\theta|_{\mathrm{Hom}_k(k,S')} : (1 \mapsto s') \longmapsto s'$ and so S' is a $k[U]$-subcontramodule of S properly contained in S and, therefore, $S' = 0$. Thus, S must be simple as a k-contramodule, which implies $S = k$. ■

We now produce a proper mock projective contramodule over a unipotent group with cofinite radical.

Proposition 3.2 *Let $r \geq 1$ and $q = p^r$. Then, the proper mock projective $k[U]$-contramodule $D_r = Ind_{U(\mathbb{F}_q)}^{U} k$ satisfies $D_r/rad(D_r) \cong k$.*

Proof Indeed, D_r is a mock projective contramodule, as seen in the proof of Theorem 2.5. By the previous lemma, we know that k is the only simple $k[U]$-contramodule. Furthermore, by the adjunction between induction and restriction we have

$$\mathrm{Hom}^{U}(D_r, k) \cong \mathrm{Hom}^{U(\mathbb{F}_q)}(k, k) \cong k.$$

It follows that $D_r/\mathrm{rad}(D_r) \cong k$. ■

4.1 Parabolic subgroups of G

We now turn our attention to contramodules associated with parabolic and Levi subgroups of algebraic groups. Let G be a connected reductive algebraic group; fix a maximal torus T and a Borel subgroup B containing T. Then, we have maps of coordinate rings $k[G] \to k[B] \to k[T]$ induced from the inclusions $T \subset B \subset G$. Let Φ denote the resulting root system. Choose simple roots Δ such that the root subgroups contained in B correspond to negative roots. Note that our choice of simple roots Δ determines the set of dominant weights, which we denote $X(T)_+$.

For $J \subset \Delta$, let P_J denote the corresponding parabolic subgroup of G containing B, with unipotent radical U_J and Levi factor L_J. Let $Z_J := Z(L_J)$ denote the center of the Levi factor; it can be verified that the central characters are given by $X(Z_J) = X(T)/\mathbb{Z}J$. Letting $\pi : X(T) \longrightarrow X(Z_J)$ denote the canonical quotient map, we see that $\pi(\mathbb{Z}\Phi) = \mathbb{Z}I$, where $I = \Delta/J$. It follows immediately that any $k[L_J]$ contramodule (D, θ_D) has a central character decomposition of the form

$$D = \prod_{\chi \in \mathbb{Z}I} D_\chi,$$

where $D_\chi = \left\{d \in D : \phi(\chi) = \theta_D(\phi) \text{ for all } \phi \in \mathrm{Hom}_k(k[L_J], \langle d \rangle)\right\}$ with the notation $\langle d \rangle$ denoting the one-dimensional subspace of D spanned by $d \in D$. Now, $\mathrm{Hom}_k(k[U_J], k)$ has a natural contramodule structure induced from the $k[L_J]$-comodule structure on $k[U_J]$. The preceding discussion, along with [HNS17, Lemma 3.3.2], gives the following result.

Lemma 3.3 *The $k[L_J]$-contramodule $\mathrm{Hom}_k(k[U_J],k)$ has a central character decomposition*

$$\mathrm{Hom}_k(k[U_J],k) = \prod_{\chi\in\mathbb{N}I} \mathrm{Hom}_k(k[U_J]_\chi,k),$$

where $\dim\big(\mathrm{Hom}_k(k[U_J]_\chi,k)\big) < \infty$ *for all* $\chi \in \mathbb{N}I$.

Let $\lambda \in X(T)_+$ be a dominant weight. Denote the simple module of highest weight λ by $L(\lambda)$. Recall (from the discussion after the statement of Proposition 2.2) that $L(\lambda)$ may be viewed as a $k[G]$-contramodule. Let $P(\lambda) \in k[G]$-Contra denote the projective cover of $L(\lambda)$ [Joh25].

Lemma 3.4 *Let M be a finite dimensional right $k[G]$-comodule with linear dual M^*, and let $\lambda,\mu \in X(T)_+$. Then:*

$$\dim\Big(\mathrm{Hom}^{k[G]}\big(\mathrm{Hom}(M,P(\lambda)),L(\mu)\big)\Big) = \big[\mathrm{Hom}\big(M^*,L(\mu)\big) : L(\lambda)\big],$$

where the right-hand side denotes the multiplicity of $L(\lambda)$ as a composition factor of $\mathrm{Hom}\big(M^*,L(\mu)\big)$.

Proof Let M have basis $\{m_i\}$ and M^* have dual basis $\{m_i^*\}$. Then, one checks that we have the following isomorphism:

$$\mathrm{Hom}^{k[G]}\big(\mathrm{Hom}\big(M,P(\lambda)\big),L(\mu)\big) \cong \mathrm{Hom}^{k[G]}\big(P(\lambda),\mathrm{Hom}\big(M^*,L(\mu)\big)\big)$$
$$\Big(f \longmapsto \sum_i \big((\phi\circ f)(m_i)\big)(m_i^*)\Big) \longleftarrow \phi$$
$$\psi \longrightarrow \Big(p \longmapsto \big(\alpha \longmapsto \phi\big(m \longmapsto \alpha(m)p\big)\big)\Big),$$

where both M and M^* are viewed as right comodules and $\mathrm{Hom}(-,-)$ is a contramodule via the diagonal action. Since $P(\lambda)$ is projective, $\mathrm{Hom}(P(\lambda),-)$ is exact and so by induction on the composition length, one may show that $\dim\Big(\mathrm{Hom}^{k[G]}\big(\mathrm{Hom}(M,P(\lambda)),L(\mu)\big)\Big)$ is exactly the number of times $L(\lambda)$ appears as a composition factor in $\mathrm{Hom}\big(M^*,L(\mu)\big)$, as required. ∎

One may inflate a $k[L_J]$ contramodule M to a $k[P_J]$ contramodule via the composition

$$\mathrm{Hom}_k(k[P_J],M) \longrightarrow \mathrm{Hom}_k(k[L_J],M) \longrightarrow M.$$

Given a weight $\lambda \in X(T)$ which is dominant for L_J, let $M = L_J(\lambda)$ denote the simple L_J module with highest weight λ. We denote the inflation by $L_{P_J}(\lambda)$. The projective cover $P_{P_J}(\lambda)$ of $L_{P_J}(\lambda)$ is given by

$$P_{P_J}(\lambda) \cong \mathrm{Ind}_{k[L_J]}^{k[P_J]}(P_{L_J}(\lambda)) \cong \mathrm{Hom}_k\big(k[U_J],P_{L_J}(\lambda)\big),$$

where the isomorphism is a consequence of Lemma 1.6.

Lemma 3.5 *Let P be a projective $k[P_J]$ contramodule with cofinite dimensional radical. Then, we have*

$$P|_{k[L_J]} = \prod_{\lambda \in X(T)} P_{L_J}(\lambda)^{n_\lambda},$$

where $n_\lambda < \infty$ for all weights $\lambda \in X(T)$.

Much as in the analogous result for injective modules of P_J with finite dimensional socle, [HNS17, Proposition 3.4.3], one must turn to using central characters and the fact that homomorphisms preserve weight spaces for the proof. All required results needed to produce an analog of this proof have been proven for contramodules. We leave the necessary modifications to the reader.

Let $F : P_J \longrightarrow P_J$ be the Frobenius morphism and let $(P_J)_r L_J = (F^r)^{-1}(L_J)$. We have the following result.

Proposition 3.6 *Let D be a $k[P_J]$ contramodule with cofinite dimensional radical which is projective as a $k[(P_J)_r L_J]$ contramodule for all $r \geq 1$. Then, D is projective as a $k[P_J]$ contramodule.*

Proof Since $D/\mathrm{rad}(D)$ is finite dimensional, it follows that the projective cover of D in the category $k[P_J]$-Contra is of the form $\mathrm{Hom}_k(k[U_J], P)$ for some projective $k[L_J]$ contramodule P with cofinite dimensional radical. This gives us a projection $\mathrm{Hom}_k(k[U_J], P) \longrightarrow D$. Since by assumption $D|_{k[(P_J)_r L_J]}$ is projective for all $r > 0$ we have projections of the form $D \longrightarrow \mathrm{Hom}_k\left((k[U_J])_r, P\right)$ $\big($of $k[(P_J)_r L_J]$ contramodules$\big)$. It suffices for us to show that we have

$$\mathrm{Hom}_k(k[U_J], P) = \bigcup_r \mathrm{Hom}_k\left((k[U_J])_r, P\right),$$

but this follows from the fact that the coordinate ring of an algebraic group is the projective limit of the coordinate rings of its Frobenius kernels.

Since colimits commute with colimits, and in particular, unions commute with cokernels, we have a projection $D \longrightarrow \mathrm{Hom}_k(k[U_J], P)$ and so $D = \mathrm{Hom}_k(k[U_J], P)$. Thus, D is projective as a $k[P_J]$ contramodule, as required. ■

Corollary 3.7 *Let P be a mock projective $k[B]$ contramodule which cofinite dimensional radical, then P is a projective $k[B]$ contramodule.*

Proof Let $J = \varnothing$. Then, $P_\varnothing = B$ and $L_\varnothing = T$. The result follows from the previous proposition, along with the fact that a contramodule is projective as a $k[B_r T]$ contramodule if and only if it is projective as a $k[B_r]$ contramodule [Jan03, Lemma II.9.4]. ■

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