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Characterizations of Simple Isolated Line Singularities

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Abstract. A line singularity is a function germ $f\colon (\mathbf{C}^{n+1},0) \to \mathbf{C}$ with a smooth 1-dimensional critical set $\Sigma = \{(x,y) \in \mathbf{C} \times \mathbf{C}^n \mid y = 0\}$. An isolated line singularity is defined by the condition that for every $x \neq 0$, the germ of f at $(x,0)$ is equivalent to $y_1^2 + \cdots + y_n^2$. Simple isolated line singularities were classified by Dirk Siersma and are analogous of the famous $A - D - E$ singularities. We give two new characterizations of simple isolated line singularities.

1 Introduction

1.1

Let $\mathcal{O} := \{f\colon (\mathbf{C}^{n+1},0) \to \mathbf{C}\}$ be the ring of germs of holomorphic functions and let m be its maximal ideal. An important problem in Singularity Theory is the classification of holomorphic germs $f \in \mathcal{O}$ with respect to the coordinate changes in $(\mathbf{C}^{n+1},0)$. When we consider only germs f with an isolated singularity in the origin of $\mathbf{C}^{n+1}$, the list starts with the famous $A - D - E$ *simple isolated singularities*, see for instance [2]:

$$A_k : x^{k+1} + y_1^2 + \cdots + y_n^2, \quad k \geq 1$$

$$D_k : x^2 y_1 + y_1^{k-1} + y_2^2 + \cdots + y_n^2, \quad k \geq 4$$

$$E_6 : x^4 + y_1^3 + y_2^2 + \cdots + y_n^2$$

$$E_7 : x^3 y_1 + y_1^3 + y_2^2 + \cdots + y_n^2$$

$$E_8 : x^5 + y_1^3 + y_2^2 + \cdots + y_n^2.$$

Several characterizations of the $A - D - E$ singularities are well-known, see for instance Durfee's paper [3].

After isolated singularities, a next step would be to consider the case of function germs $f\colon (\mathbf{C}^{n+1},0) \to \mathbf{C}$ with a smooth 1-dimensional critical set. This approach was followed by Dirk Siersma, who introduced in [9] the class of germs of holomorphic functions with an *isolated line singularity*. Namely, if $(x,y) = (x, y_1, \ldots, y_n)$ denote the coordinates in $(\mathbf{C}^{n+1},0)$, consider the line $L := \{y_1 = \cdots = y_n = 0\}$, let $I := (y_1, \ldots, y_n) \subseteq \mathcal{O}$ be its ideal and let $\mathcal{D}_I$ denote the group of local analytic isomorphisms $\varphi\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C}^{n+1},0)$ for which $\varphi(L) = L$. Then $\mathcal{D}_I$ acts on I^2 and for $f \in I^2$, the *tangent space* of (the orbit of) f with respect to this action is the ideal defined by

$$\tau(f) := m\Big(\frac{\partial f}{\partial x}\Big) + I\Big(\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}\Big),$$

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while the *codimension* of (the orbit) of f is $c(f) := \dim_{\mathbf{C}} \frac{I^2}{\tau(f)}$.

A *line singularity* is a germ $f \in I^2$. An *isolated line singularity* (for short: ILS) is a line singularity f such that $c(f) < \infty$. Geometrically, $f \in I^2$ is an ILS if and only if the singular locus of f is L and for every $x \neq 0$, the germ of (a representative of) f at $(x, 0) \in L$ is equivalent to $y_1^2 + \cdots + y_n^2$. In Section 1 of [9], Siersma studied line singularities from the point of view of Thom-Mather theory. One of his results is the following theorem. (A topology on $\mathcal{O}$ is introduced as in [3, p. 145].)

Theorem 1.2 *A germ $f \in I^2$ is $\mathcal{D}_I$-simple (i.e. $c(f) < \infty$ and f has a neighborhood in I^2 which intersects only a finite number of $\mathcal{D}_I$-orbits) if and only if f is $\mathcal{D}_I$-equivalent to one of the germs in the following table:*

Name	*Normal form*	*Conditions*	*Determined jet*
A_∞	$y_1^2 + \cdots + y_n^2$		2
D_∞	$xy_1^2 + y_2^2 + \cdots + y_n^2$		3
$J_{k,\infty}$	$x^k y_1^2 + y_1^3 + y_2^2 + \cdots + y_n^2$	$k \geq 2$	$k+2$
$T_{\infty,k,2}$	$x^2 y_1^2 + y_1^k + y_2^2 + \cdots + y_n^2$	$k \geq 4$	k
$Z_{k,\infty}$	$xy_1^3 + x^{k+2} y_1^2 + y_2^2 + \cdots + y_n^2$	$k \geq 1$	$k+4$
$W_{1,\infty}$	$x^3 y_1^2 + y_1^4 + y_2^2 + \cdots + y_n^2$		5
$T_{\infty,q,r}$	$xy_1 y_2 + y_1^q + y_2^r + y_3^2 + \cdots + y_n^2$	$q \geq r \geq 3$	q
$Q_{k,\infty}$	$x^k y_1^2 + y_1^3 + xy_2^2 + y_3^2 + \cdots + y_n^2$	$k \geq 2$	$k+2$
$S_{1,\infty}$	$x^2 y_1^2 + y_1^2 y_2 + xy_2^2 + y_3^2 + \cdots + y_n^2$		4

1.3

The singularities in Theorem 1.2 are analogous of the $A - D - E$ singularities and were considered also by V. V. Goryunov, see for instance [4]. Non-isolated singularities were studied, from different points of view, in many papers, *e.g.* [7], [6], [11], *etc.*

For convenience of notations, we consider also

$$J_{1,\infty} : xy_1^2 + y_1^3 + y_2^2 + \cdots + y_n^2, \quad \text{which is } \mathcal{D}_I\text{-equivalent to } D_\infty,$$

$$Z_{0,\infty} : xy_1^3 + x^2 y_1^2 + y_2^2 + \cdots + y_n^2, \quad \text{which is } \mathcal{D}_I\text{-equivalent to } T_{\infty,4,2},$$

$$A_0 : x + y_1^2 + \cdots + y_n^2, \quad \text{which is smooth (no singular points)},$$

$$D_3 : x^2 y_1 + y_1^2 + y_2^2 + \cdots + y_n^2, \quad \text{which is equivalent to } A_3.$$

Note that the normal forms of singularities in Theorem 1.2 are *quasihomogeneous* polynomials, *i.e.* there exist *weights* $w_0, w_1, \ldots, w_n \in \mathbf{N} \setminus \{0\}$ for the coordinates $x, y_1, \ldots, y_n$, and a natural number d, called the *weighted degree* of f, such that $d \geq 2w_j$ for all j and such that all monomials $x^{a_0} y_1^{a_1} \cdots y_n^{a_n}$ which are *contained* in f, *i.e.* which appear in f with a non-zero coefficient, satisfy

$$\operatorname{wdeg}(x^{a_0} y_1^{a_1} \cdots y_n^{a_n}) := a_0 w_0 + a_1 w_1 + \cdots + a_n w_n = d.$$

We denote by $\mathcal{O}_{\geq d}$ the ideal of $\mathcal{O}$ generated by all the monomials with weighted degree $\geq d$.

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1.4

The aim of this note is to give *new characterizations* of the simple isolated line singularities. We assume that $n = 2$ and we denote the coordinates (x, y_1, y_2) in $(\mathbf{C}^3, 0)$ by (x, y, z). Hence the line L has equations $y = z = 0$, $I = (y, z)$, the equation of the D_∞ singularity is $xy^2 + z^2$, *etc.*

In the next section we blow up an ILS, with center L, and we show that the singularities of the strict transform and of the exceptional curve characterize the simple isolated line singularities. In the last section we give a characterization of a simple ILS using its *inner modality*, as introduced in [10].

It would be interesting to have other characterizations for simple ILS, and also for other simple non-isolated singularities.

2 Blowing Up Line Singularities

2.1

Let $f\colon (\mathbf{C}^3, 0) \to \mathbf{C}$ be an ILS, $f \in (y, z)^2$. We fix a representative of f, defined on a small neighborhood of $0 \in \mathbf{C}^3$, and we continue to denote this representative by $f\colon (\mathbf{C}^3, 0) \to \mathbf{C}$.

Let us put $V := f^{-1}(0) \subseteq (\mathbf{C}^3, 0)$ and let M be the blowing up of $\mathbf{C}^3$ with center L, *i.e.* M is the subset of $\mathbf{C}^3 \times \mathbf{P}^1$ described by $M := \{\big((x, y, z), [u : v]\big) \mid yv = zu\}$. There are two coordinate charts on M, namely $\mathcal{U}_1 := M \cap \{u \neq 0\}$, with coordinates (x, y, v), and $\mathcal{U}_2 := M \cap \{v \neq 0\}$, with coordinates (x, z, u). Let $\sigma\colon M \to \mathbf{C}^3$ be the projection map, let X denote the strict transform of V, let $H := \sigma^{-1}(L)$ be the exceptional divisor of M and let $Y := X \cap H$ be the exceptional curve of X. More precisely, X is the closure in $\mathbf{C}^3 \times \mathbf{P}^1$ of the set $\{\big((x, y, z), [y : z]\big) \mid f(x, y, z) = 0, (y, z) \neq (0, 0)\}$, the equations of X are

$$\text{in } \mathcal{U}_1 : y^{-2} \cdot f(x, y, vy) = 0; \quad \text{in } \mathcal{U}_2 : z^{-2} \cdot f(x, uz, z) = 0,$$

and the equations of Y are

$$\text{in } \mathcal{U}_1 : y = 0,\ y^{-2} \cdot f(x, y, vy) = 0; \quad \text{in } \mathcal{U}_2 : z = 0,\ z^{-2} \cdot f(x, uz, z) = 0.$$

By a direct computation one can show the following

Proposition 2.2 *If f is a simple isolated line singularity, then the singularities of X and of Y are described in the following table:*

Name of f	(X, Y) in $\mathcal{U}_1$	(X, Y) in $\mathcal{U}_2$
A_∞	(smooth, smooth)	(smooth, smooth)
D_∞	(smooth, smooth)	(smooth, smooth)
$J_{k,\infty}$	(smooth, one A_{k-1})	(smooth, smooth)
$T_{\infty,k,2}$	(one A_{k-3}, one A_1)	(smooth, smooth)
$Z_{k,\infty}$	(one A_1, one A_{k+1})	(smooth, smooth)
$W_{1,\infty}$	(one A_2, one A_2)	(smooth, smooth)
$T_{\infty,q,r}$	(one A_{q-3}, one A_1)	(one A_{r-3}, one A_1)
$Q_{k,\infty}$	(smooth, one D_{k+1})	(smooth, smooth)
$S_{1,\infty}$	(one A_1, one A_3)	(smooth, smooth)

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2.3

Note that all the singularities of X and Y are in the origin of the coordinates charts $\mathcal{U}_1$ and $\mathcal{U}_2$ and that they are not "too complicated". We prove that the converse is also true. Before stating our results, let us recall a definition.

Let $f \in (y,z)^2$ be a line singularity and write it as $f = y^2\psi_1 + 2yz\psi_2 + z^2\psi_3$, for some germs $\psi_1, \psi_2, \psi_3 \in \mathcal{O}$. These germs are not uniquely determined, but the *corank* of f, *i.e.* the corank of the Hessian matrix

$$H_f(0) = \begin{pmatrix} \psi_1(0) & \psi_2(0) \\ \psi_2(0) & \psi_3(0) \end{pmatrix}$$

is well defined. It is clear that the corank of f is equal to 0 if and only if f is $\mathcal{D}_I$-equivalent to A_∞. For $g \in \mathcal{O}$, the k-jet of g will be denoted by $j^k(g)$.

Theorem 2.4 **(Case: corank is one)** *Let $f\colon (\mathbf{C}^3, 0) \to \mathbf{C}$ be an isolated line singularity with singular locus $L = \{y = z = 0\}$, let $V = f^{-1}(0)$ and let X denote the strict transform of V after blowing up the line L in $\mathbf{C}^3$. Let Y be the exceptional curve of X and let us suppose that the corank of f is equal to* 1. *Then we have:*

(i) If X is smooth, then f is $\mathcal{D}_I$-equivalent to $J_{k,\infty}$ for some $k \geq 1$.

(ii) If X has an A_1 singularity, then f is $\mathcal{D}_I$-equivalent to $Z_{k,\infty}$ for some $k \geq 0$.

(iii) If X has an A_{k-3} singularity, for some $k \geq 5$, and Y has an A_1 singularity, then f is $\mathcal{D}_I$-equivalent to $T_{\infty,k,2}$.

(iv) If X has an A_2 singularity and Y has an A_2 singularity, then f is $\mathcal{D}_I$-equivalent to $W_{1,\infty}$.

Proof Since f is an ILS with corank 1, one can find suitable coordinates in $(\mathbf{C}^3, 0)$ such that $f(x,y,z) = y^2 g(x,y) + z^2$. Moreover, $g(x,y)$ has an isolated singularity in $(0,0) \in \mathbf{C}^2$ and $g(x,0)$ has an isolated singularity in $0 \in \mathbf{C}$. Thus, X and Y are smooth in $\mathcal{U}_2$ and only the origin of $\mathcal{U}_1$ could be a singular point of X or of Y. Note that the equation of X in $\mathcal{U}_1$ is $g(x,y) + v^2 = 0$ and the equations of Y in $\mathcal{U}_1$ are $y = g(x,0) + v^2 = 0$.

If X is smooth, then $g(x,y) = \alpha x + \beta y + \cdots$ for some $(\alpha, \beta) \in \mathbf{C}^2 \setminus \{(0,0)\}$. If $\alpha \neq 0$, then f is $\mathcal{D}_I$-equivalent to $D_\infty = J_{1,\infty}$. When $\alpha = 0$ and $\beta \neq 0$, then f is $\mathcal{D}_I$-equivalent to $J_{k,\infty}$, for some $k \geq 2$. Thus, point (i) is proved.

If X is not smooth, then $j^2(g)$ is $\mathcal{D}_I$-equivalent to one of the following: xy, x^2, y^2 or 0. If $j^2(g) = 0$, then X has a singularity which is not of type A_s, for any s, contradicting the hypothesis. It remains that $j^2(g) \neq 0$.

Note that X has an A_1 singularity if and only if $j^2(g)$ is $\mathcal{D}_I$-equivalent to xy; and in this situation it is easy to see that f is $\mathcal{D}_I$-equivalent to $Z_{k,\infty}$ for some $k \geq 0$.

If Y has an A_1 singularity, then $j^2\big(g(x,0)\big) = x^2$. If, moreover, X has an A_{k-3} singularity, for some $k \geq 5$, then $j^2(g) = x^2$ and it is easy to see that f is $\mathcal{D}_I$-equivalent to $T_{\infty,k,2}$.

Suppose now that X and Y have singularities of type A_2. By the above remarks it follows that $j^2(g) = y^2$. Thus, g is $\mathcal{D}_I$-equivalent to $y^2 + yh(x,y) + a(x)$, for suitable germs $h \in m^2$ and $a \in m^3 \setminus m^4$. And now it is easy to see that f is $\mathcal{D}_I$-equivalent to $W_{1,\infty}$. ■

Theorem 2.5 **(Case: corank is two)** *Let $f\colon (\mathbf{C}^3, 0) \to \mathbf{C}$ be an isolated line singularity with singular locus $L = \{y = z = 0\}$, let $V = f^{-1}(0)$ and let X denote the strict transform of V*

after blowing up the line L in $\mathbf{C}^3$. Let Y be the exceptional curve of X and let us suppose that the corank of f is equal to 2. Then we have:

(i) If X is smooth and Y has an isolated singularity, not of type A_1, then f is $\mathcal{D}_I$-equivalent to $Q_{k,\infty}$, for some $k \geq 2$.

(ii) If X has an A_1 singularity and Y has an isolated singularity, not of type A_1, then f is $\mathcal{D}_I$-equivalent to $S_{1,\infty}$.

(iii) If Y has only singularities of type A_1, then f is $\mathcal{D}_I$-equivalent to $T_{\infty,q,r}$, for some $q \geq r \geq 3$.

Proof Since $f \in I^2$ has corank two, we can write

$$f = x\big(y^2a(x) + yzb(x) + z^2c(x)\big) + g(y,z) + xh(x,y,z)$$

for suitable germs $a, b, c \in \mathcal{O}$ and $g, h \in I^3$. The equations of X are:

$$\text{in } \mathcal{U}_1 : xa(x) + xvb(x) + xv^2c(x) + y^{-2}\cdot g(y,vy) + xy^{-2}\cdot h(x,y,vy) = 0,$$

$$\text{in } \mathcal{U}_2 : xu^2a(x) + xub(x) + xc(x) + z^{-2}\cdot g(uz,z) + xz^{-2}\cdot h(x,uz,z) = 0.$$

Since Y has only isolated singularities, we have: $\min\{\operatorname{ord}(a), \operatorname{ord}(b), \operatorname{ord}(c)\} = 0$.

Consider now the quadratic form $Q(y,z) := y^2a(0) + yzb(0) + z^2c(0)$. After a suitable linear coordinate change $\varphi \in \mathcal{D}_I$, we will have either $Q = z^2$, or $Q = yz$.

If $Q = z^2$, then $c(0) = 1$ and $a(0) = b(0) = 0$. Thus, the origin $0 \in \mathcal{U}_1$ is a singular point of Y, but not of type A_1.

If $Q = yz$, then $b(0) = 1$ and $a(0) = c(0) = 0$. Using the standard classification methods, one can easily show that f is $\mathcal{D}_I$-equivalent to a $T_{\infty,q,r}$ singularity, for suitable $q \geq r \geq 3$. Thus, point (iii) is proved.

Suppose that Y has at least one isolated singularity which is not of type A_1. Then $Q = z^2$, $c(0) = 1$ and $a(0) = b(0) = 0$, hence X can be singular only in the origin $0 \in \mathcal{U}_1$.

Assume moreover that X is smooth. Then $j^1\big(y^{-2}\cdot g(y,vy)\big) \neq 0$. After a suitable coordinate change $\varphi \in \mathcal{D}_I$, we can obtain $j^3(f) = xz^2 + y^3$. Using the standard classification methods, one can show that f is $\mathcal{D}_I$-equivalent to a $Q_{k,\infty}$ singularity, for a suitable $k \geq 2$.

If Y has at least one isolated singularity which is not of type A_1 and X has an A_1 singularity, we write

$$a(x) = a_1x + x^2\gamma(x), \quad h(x,y,z) = zH_1(x,y,z) + h_1y^3 + y^3H_2(x,y)$$

and

$$g(y,z) = g_1y^3 + g_2y^2z + g_3y^4 + z^2G_1(y,z) + y^3zG_2(y) + y^5G_3(y)$$

for suitable coefficients $a_1, g_1, g_2, g_3, h_1 \in \mathbf{C}$ and functions $\gamma(x)$, $G_2(y)$, $G_3(y) \in \mathcal{O}$, $G_1(y,z) \in I$, $H_1(x,y,z) \in I^2$ and $H_2(x,y) \in m$. Since X has an A_1 singularity in the origin $0 \in \mathcal{U}_1$, it follows that $g_1 = 0$, $a_1 \neq 0$ and $g_2 \neq 0$. After a coordinate change $\varphi \in \mathcal{D}_I$, we can assume that $a(x) = x$ and $g_2 = 1$. Thus, for suitable $\beta_j \in \mathbf{C}$ and homogeneous polynomials $\alpha_1(y,z) \in I^4$, $\alpha_2(y,z) \in I^3$, we have:

$$j^4(f) = xz^2 + \beta_1x^2yz + x^2y^2 + y^2z + \beta_2yz^2 + \beta_3z^3 + \alpha_1(y,z) + x\alpha_2(y,z).$$

And now, the usual classification methods give us that f is $\mathcal{D}_I$-equivalent to $S_{1,\infty}$. ■

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2.6

Combining Proposition 2.2 with Theorems 2.4 and 2.5, we obtain the following

Corollary *A simple isolated line singularity $f \in (y,z)^2$ can be characterized by the corank of f and by the singularities of X and Y.*

Remark 2.7 In [5], G. Jiang extended the above results to the case of line singularities on an A_1 surface. However, these results can not be generalized to any class of non-isolated singularities, as the next example shows us.

Let $k \geq 4$ and let $g\colon (\mathbf{C}^{k+2}, 0) \to \mathbf{C}$ be defined by $g(y_1, y_2, x_1, \dots, x_k) = x_1 y_1^2 + x_2 y_2^2 + y_1 y_2 h(x_3, \dots, x_k)$, where $h(x_3, \dots, x_k)$ is an isolated singularity. Then the singular locus of g is $\{y_1 = y_2 = 0\}$ and under the blowing up of $\mathbf{C}^{k+2}$ with center $\{y_1 = y_2 = 0\}$, the strict transform of $g^{-1}(0)$ is smooth and intersects transversally the exceptional divisor. On the other hand, it follows from [12] that if h is not an $A - D - E$ singularity, then g is not a simple non-isolated singularity.

3 Inner Modality

3.1

Let $w_0, w_1, w_2 \in \mathbf{N} \setminus \{0\}$ be the weights of x, y, z and let $d \in \mathbf{N}$. We assume that

(1) $$w_1 \leq w_2 \text{ and } d \geq 2w_j > 0 \quad \text{for all } j.$$

Let $f \in \mathbf{C}[x,y,z]$ be a quasihomogeneous polynomial of degree d and assume that $f \in I^2$ is an ILS. Following [10, p. 286], we define the *inner modality* of f by

$$m_0(f) = \dim_{\mathbf{C}} \frac{I \cap \mathcal{O}_{\geq d}}{J(f) \cap \mathcal{O}_{\geq d}}, \quad \text{where } J(f) := \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big),$$

and we say that f is *i-simple* if $m_0(f) = 0$. By [10], we have:

(2) $$f \text{ is } i\text{-simple} \iff 2d < 2w_0 + 3w_1 + 2w_2.$$

In this section we prove the following

Theorem 3.2 *Let $f \in \mathbf{C}[x,y,z]$ be a quasihomogeneous polynomial such that $f \in I^2$ is an isolated line singularity. Then f is i-simple if and only if f is $\mathcal{D}_I$-equivalent to one of the normal forms listed in Theorem 1.2.*

Remark 3.3 This theorem is similar to results obtained, for the $A - D - E$ singularities, by V. I. Arnold [1] and K. Saito [8].

Proof In [10, p. 289], it is already shown that the normal forms listed in Theorem 1.2 are i-simple. Moreover, in the same place it is remarked that the converse is true for all i-simple f, if the corank is equal to 1. Hence we have to prove that f is $\mathcal{D}_I$-simple only when f is i-simple and has the corank equal to 2. This fact follows from the next proposition. ■

Proposition 3.4 *Let $f \in \mathbf{C}[x, y, z]$ be a quasihomogeneous polynomial such that $f \in I^2$ is an isolated line singularity of corank 2. Then we have:*

(i) If f is i-simple, then $j^3(f)$ contains at least one monomial from the following list:

$$xy^2, \quad xyz, \quad xz^2. \tag{3}$$

(ii) If $j^3(f)$ contains at least two monomials from the list (3), then f is $\mathcal{D}_I$-equivalent to a germ $T_{\infty,r,r}$ for a suitable $r \geq 3$.

(iii) There is no such f for which $j^3(f)$ contains only xy^2 from the list (3).

(iv) If $j^3(f)$ contains only xyz from the list (3), then f is $\mathcal{D}_I$-equivalent to a germ $T_{\infty,q,r}$ for suitable $q \geq r \geq 3$.

(v) If f is i-simple and $j^3(f)$ contains only xz^2 from the list (3), then f is $\mathcal{D}_I$-equivalent either to $S_{1,\infty}$, or to a germ $Q_{k,\infty}$ for a suitable $k \geq 2$.

Proof To prove (ii), note that $w_1 = w_2$, hence $f(0, y, z)$ is a homogeneous polynomial. If r denotes the (usual) degree of $f(0, y, z)$, then using the classification methods one can easily show that f is $\mathcal{D}_I$-equivalent to $T_{\infty,r,r}$.

To prove (iii), assume the contrary. The condition $c(f) < \infty$ implies that f contains at least one monomial of the form $x^k yz$ or $x^k z^2$, for some $k \geq 2$. A contradiction is given by the inequalities: $\operatorname{wdeg}(x^k z^2) \geq \operatorname{wdeg}(x^k yz) \geq \operatorname{wdeg}(x^k y^2) > w_0 + 2w_1 = d$.

The point (iv) can be proved using the classification methods.

To prove (v), note that if $j^3\big(f(0, y, z)\big) \neq 0$, then the usual classification methods give us that f is $\mathcal{D}_I$-equivalent either to $S_{1,\infty}$, or to $Q_{k,\infty}$, for some $k \geq 2$.

Assume now that $j^3(f)$ contains only xz^2 from list (3) and that $j^3\big(f(0, y, z)\big) = 0$. Using (2), we will show that f is not i-simple. The condition $c(f) < \infty$ implies that f contains at least one monomial of type y^a, xy^b, zy^c, for some $a \geq 3$, $b \geq 3$, $c \geq 2$, and at least one monomial of type $x^\ell yz$ or $x^\ell y^2$, for some $\ell \geq 2$. Since $x^{2\ell} y^2 z^2 = xz^2 \cdot x^{2\ell-1} y^2$, it follows that there exists some $k \geq 2$ such that $\operatorname{wdeg}(x^k y^2) = d$.

If f contains y^a for some $a \geq 4$, then $4d = \operatorname{wdeg}(y^a \cdot x^2 z^4 \cdot x^k y^2) = (2 + k)w_0 + (a + 2)w_1 + 4w_2 \geq 4w_0 + 6w_1 + 4w_2$.

If f contains xy^b for some $b \geq 3$, then $2d = \operatorname{wdeg}(xz^2 \cdot xy^b) = 2w_0 + bw_1 + 2w_2 \geq 2w_0 + 3w_1 + 2w_2$.

If f contains zy^c for some $c \geq 3$, then $3d = \operatorname{wdeg}(x^k y^2 \cdot zy^c \cdot xz^2) = (k + 1)w_0 + (c + 2)w_1 + 3w_2 \geq 3w_0 + 5w_1 + 3w_2$.

The point (i) is a consequence of the following Lemma. ■

Lemma 3.5 *Let $f \in \mathbf{C}[x, y, z]$ be a quasihomogeneous polynomial such that $f \in I^2$ is an isolated line singularity of corank 2 and such that $j^3(f)$ contains no monomials from the list (3). Then f is not i-simple.*

Proof We list seven cases and we show that $2d \geq 2w_0 + 3w_1 + 2w_2$ in each of them. Thus, by (2), f is not i-simple. We leave almost all the details of the proof to the reader.

(i) f contains xz^b for some $b \geq 3$.
(ii) $j^3(f)$ contains at least two monomials from the set $\{y^3, y^2z, yz^2, z^3\}$.
(iii) $j^3(f) = \alpha y^3$, with $\alpha \neq 0$.

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It follows that $w_1 = \frac{d}{3}$ and that f contains at least one monomial of the form yz^a, with $a \geq 3$, or xz^b, with $b \geq 3$, or z^c, with $c \geq 4$. But $w_2 \geq w_1 = \frac{d}{3}$, hence f does not contain z^c, with $c \geq 4$. Also, by case (i), if f contains xz^b, with $b \geq 3$, then f is not i-simple. It remains to consider the situation when f contains yz^a, with $a \geq 3$. It follows that $w_2 = \frac{2d}{3a} < w_1 = \frac{d}{3}$, in contradiction with our assumption (1).

(iv) $j^3(f) = \alpha y^2 z$, with $\alpha \neq 0$.
(v) $j^3(f) = \alpha y z^2$, with $\alpha \neq 0$.
(vi) $j^3(f) = \alpha z^3$, with $\alpha \neq 0$.
(vii) $j^3(f) = 0$.

Then f contains at least one monomial from each of the following three lists:

$$x^k y^2,\ x^k yz,\ x^k z^2 \quad \text{for some } k \geq 2;$$

$$y^{a+4},\ xy^{b+3},\ zy^{b+3} \quad \text{for some } a, b \geq 0; \quad z^{u+4},\ xz^{v+3},\ yz^{v+3} \quad \text{for some } u, v \geq 0.$$

The last two lists show that $d > 3w_1$ and $d > 3w_2$. If f contains $x^k y^2$, then $2d > \operatorname{wdeg}(x^k y^2) + 3w_2 = kw_0 + 2w_1 + 3w_2 \geq 2w_0 + 3w_1 + 2w_2$. A similar argument works also when f contains $x^k z^2$ or when f contains $x^k yz$. ■

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