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SMALL ZEROS OF QUADRATIC L-FUNCTIONS

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We study the distribution of the imaginary parts of zeros near the real axis of quadratic L-functions. More precisely, let $K(s)$ be chosen so that $|K(1/2 \pm it)|$ is rapidly decreasing as t increases. We investigate the asymptotic behaviour of

$$F(\alpha, D) = \left(\frac{1}{2\zeta(2)} K\left(\frac{1}{2}\right) D\right)^{-1} \sum_{d \in \mathcal{F}(D)} \sum_{\rho(d)} K(\rho) D^{i\alpha\gamma}$$

as $D \to \infty$. Here $\sum_{\rho(d)}$ denotes the sum over the non-trivial zeros $\rho = 1/2 + i\gamma$ of the Dirichlet L-function $L(s, \chi_d)$, and $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol. The outer sum $\sum_{d \in \mathcal{F}(D)}$ is over all fundamental discriminants d that are in absolute value $\leqslant D$. Assuming the Generalized Riemann Hypothesis, we show that for

$$0 < |\alpha| < \frac{2}{3},\ F(\alpha, D) = -1 + o(1) \text{ as } D \to \infty.$$

1. INTRODUCTION

It is well known (see [3, 4, 8, 11]) that the zeros of Dirichlet L-functions $L(s, \chi)$ close to the real axis contain significant number-theoretic information. For example if χ is a quadratic character with $\chi(-1) = -1$, then zeros of $L(x, \chi)$ close to $s = 1/2$ have an effect on the class numbers of complex quadratic fields. In another direction, if χ is the non-principal character $(\bmod 4)$ then the "first" zero of $L(s, \chi)$ in the critical strip has a bearing on how primes are distributed in residue classes 1 and 3 $(\bmod 4)$, respectively, and in particular on a phenomenon first observed by Chebysev [5] concerning discrepancies in the distribution of primes into different residue classes.

Shanks [9] has given heuristic arguments for the predominance of primes in residue classes of non-quadratic type. He conjectured that if a_1 is a quadratic residue and a_2 is a quadratic non-residue $(\bmod q)$, then there are "more" primes congruent to a_2 than those congruent to $a_1 \bmod q$. Obviously the sense in which this predominance occurs needs to be specified. Bentz [4] and Bentz and Pintz [3] have made progress in

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this direction. Their work clearly displays the significance of "small" zeros of Dirichlet L-functions in comparative prime number theory. (See also [10].)

In this paper, we study under the assumption of the Generalized Riemann Hypothesis the distribution of "small" zeros of quadratic L-functions $L(x, \chi_d)$ for all fundamental discriminants d that are in absolute value less than or equal to a given constant D. More specifically, if $\sum_{d \in \mathcal{F}(D)}$ denotes a summation over all such d, we investigate the asymptotic properties of $\sum_{d \in \mathcal{F}(D)} \sum_{\rho(d)} K(\rho) D^{i\alpha\gamma}$ as a function of α as $D \to \infty$. Here K is a suitable kernel, $\sum_{\rho(d)}$ denotes the sum over the non-trivial zeros $\rho = 1/2 + i\gamma$ of $L(s, \chi_d)$ and $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol.

2. Preliminaries and results

Let x and D be positive real numbers and define

$$F_1(x, D) = \sum_{d \in \mathcal{F}(D)} \sum_{\rho(d)} K(\rho) x^{i\gamma}$$

where $\mathcal{F}(D)$ is the set of fundamental discriminants of quadratic number fields which are in absolute value less than or equal to D; $\sum_{\rho(d)}$ denotes the sum over the nontrivial zeros $\rho = 1/2 + i\gamma$ of the L-series $L(s, \chi_d)$ where $\chi_d = \left(\frac{d}{\cdot}\right)$, the Kronecker symbol. (Notice that we are assuming the Generalized Riemann Hypothesis.) Also we assume that $K(s)$ is analytic in the strip $-1 < \operatorname{Re}(s) < 2$ such that $\int_{c-\infty i}^{c+\infty i} K(s) x^{-s} ds$ is absolutely convergent for $-1 < c < 2$ and all $x > 2$, $K(1/2 + it) = K(1/2 - it)$, and where $a(x) = 1/(2\pi i) \int_{c-\infty i}^{c+\infty i} K(s) x^{-s} ds$ is real valued and of compact support on the interval $(0, \infty)$. As is well-known, we have

$$K(s) = \int_0^\infty a(t) t^s \frac{dt}{t}.$$

Since we are interested in zeros which are near the real axis we can choose $K(s)$ so that $|K(1/2 \pm it)|$ is rapidly decreasing as t increases. However for now we shall not specify any particular $K(s)$. As we derive properties of $F_1(x, D)$ we shall need to impose further restrictions on $a(x)$, and thus on $K(s)$, but we shall do so as we proceed.

We start by making use of the explicit formula

$$\sum_{\rho(d)} K(\rho) x^\rho = -\sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right) + a\left(\frac{1}{x}\right) \log \frac{|d|}{\pi} + O(1).$$

This can be derived as in [6].

Consequently

$$F_1(x, D) = A + B + O\left(x^{-1/2} D\right)$$

where

$$A = -x^{1/2} \sum_{d \in \mathcal{F}(D)} \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right)$$

and

$$B = x^{-1/2} a\left(\frac{1}{x}\right) \sum_{d \in \mathcal{F}(D)} \log \frac{|d|}{\pi}.$$

Now let $A = A_1 + A_2$ where

$$A_1 = -x^{-1/2} \sum_{d \in \mathcal{F}(D)} \sum_{\substack{n=1 \\ n=\square}}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right)$$

and

$$A_2 = -x^{-1/2} \sum_{d \in \mathcal{F}(D)} \sum_{\substack{n=1 \\ n\neq\square}}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \left(\frac{d}{n}\right);$$

here $\sum_{n=\square}$ denotes the sun over those integers which are perfect squares.

Consider first A_1. Since $\left(\frac{d}{n}\right) = 1$ if n and d are relatively prime and $\left(\frac{d}{n}\right) = 0$ if not, we see that

$$A_1 = -x^{-1/2} \sum_{d \in \mathcal{F}(D)} \sum_{\substack{n=1 \\ n=\square \\ (d,n)=1}} a\left(\frac{n}{x}\right) \Lambda(n).$$

We now write $A_1 = A_{11} + A_{12}$ where

$$A_{11} = -x^{-1/2} \sum_{d \in \mathcal{F}(D)} \sum_{\substack{n=1 \\ n=\square}}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n)$$

and

$$A_{12} = x^{-1/2} \sum_{d \in \mathcal{F}(D)} \sum_{\substack{n=1 \\ n=\square \\ (d,n)>1}}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n).$$

But notice

$$A_{11} = -x^{-1/2} \left|\mathcal{F}(D)\right| \sum_{\substack{n=1 \\ n=\square}}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n)$$

where $|\mathcal{F}(D)|$ denotes the cardinality of $\mathcal{F}(D)$. We now seek an asymptotic expansion of A_{11}. To this end we have

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Lemma 1. *If the Riemann Hypothesis (R.H.) holds and*

$$\int_0^\infty v^{1/4}\log^2 v\,|a'(v)|\,dv \text{ exists and is finite,}$$

then
$$\sum_{\substack{n=1\\ n=\square}}^{\infty} a\Big(\frac{n}{x}\Big)\Lambda(n)=\frac{1}{2}K\Big(\frac{1}{2}\Big)x^{1/2}+O\Big(x^{1/4}\log^2 x\Big)\quad (x\to\infty).$$

Proof: We use Riemann-Stieltjes integration to write

$$\sum_{n=\square} a\Big(\frac{n}{x}\Big)\Lambda(n)=\int_0^\infty a\Big(\frac{u}{x}\Big)d\psi(\sqrt{u})$$

where $\psi(u)=\sum_{n\leqslant u}\Lambda(n)$. Then under R.H. $\psi(u)=u+E(u)$ with $E(u)\ll u^{1/2}\log^2 u$. Consequently

$$\int_0^\infty a\Big(\frac{u}{x}\Big)d\psi(\sqrt{u})=\int_0^\infty a\Big(\frac{u}{x}\Big)d\sqrt{u}+\int_0^\infty a\Big(\frac{u}{x}\Big)dE(\sqrt{u}).$$

But
$$\int_0^\infty a\Big(\frac{u}{x}\Big)d\sqrt{u}=\frac{1}{2}\int_0^\infty a\Big(\frac{u}{x}\Big)u^{-1/2}du$$

and changing variable $v=u/x$,

$$\begin{aligned}\frac{1}{2}\int_0^\infty a\Big(\frac{u}{x}\Big)u^{-1/2}du&=\frac{1}{2}\int_0^\infty a(v)x^{-1/2}v^{-1/2}x\,dv\\ &=\frac{1}{2}x^{1/2}\int_0^\infty a(v)v^{-1/2}dv=\frac{1}{2}K\Big(\frac{1}{2}\Big)x^{1/2}.\end{aligned}$$

On the other hand, by using integration by parts we get

$$\begin{aligned}\int_0^\infty a\Big(\frac{u}{x}\Big)dE(\sqrt{u})&=a\Big(\frac{u}{x}\Big)E(\sqrt{u})\Big|_0^\infty-\int_0^\infty E(\sqrt{u})\,d\,a\Big(\frac{u}{x}\Big)\\ &=-\int_0^\infty E(\sqrt{u})\,d\,a\Big(\frac{u}{x}\Big)\end{aligned}$$

since $a(x)$ has compact support in $(0,\infty)$. By using $E(\sqrt{u})\ll u^{1/4}\log^2 u$ we have

$$\int_0^\infty E(\sqrt{u})\,d\,a\Big(\frac{u}{x}\Big)\ll\int_0^\infty u^{1/4}\log^2 u\,d\,a\Big(\frac{u}{x}\Big)=x^{-1}\int_0^\infty u^{1/4}\log^2 u\,a'\Big(\frac{u}{x}\Big)du.$$

Changing variable $v=u/x$ leads to

$$\begin{aligned}x^{-1}\int_0^\infty u^{1/4}\log^2 u\,a'\Big(\frac{u}{x}\Big)du&=x^{-1}\int_0^\infty x^{1/4}v^{1/4}\log^2(xv)a'(v)x\,dv\\ &=x^{1/4}\int_0^\infty v^{1/4}(\log^2 x+2\log x\log v+\log^2 v)a'(v)dv\ll x^{1/4}\log^2 x\end{aligned}$$

by the hypothesis. This establishes the lemma. $\square$

LEMMA 2. $|\mathcal{F}(D)| = \frac{1}{\zeta(2)} D + O\left(\sqrt{D}\right) \quad (D \to \infty).$

PROOF: As is well-known, see for example, Davenport [6], a fundamental discriminant is a product of relatively prime factors of the form $-4,\ 8,\ -8,\ (-1)^{(p-1)/2}p$ (p any odd prime). Then we have

$$\mathcal{F}(D) = \mathcal{F}_1(D) \dot{\cup} \mathcal{F}_4(D) \dot{\cup} \mathcal{F}_8(D)$$

where

$$\mathcal{F}_1(D) = \{d \colon d \text{ is an odd fund. disc. and } |d| \leqslant D\}$$

$$\mathcal{F}_4(D) = \{d \colon d \text{ is a fund. disc., } d \equiv 4(8),\ |d| \leqslant D\}$$

$$\mathcal{F}_8(D) = \{d \colon d \text{ is a fund. disc., } d \equiv 0(8),\ |d| \leqslant D\}.$$

Notice that in all cases the odd part of d is square-free. Also notice that if $m, \neq \pm 1$, is odd or $\equiv 4(8)$ and that the odd part of m is square-free, then precisely one of m or $-m$ is a fundamental discriminant. On the other hand, if $m = 8m_0$ where m_0 is odd and square-free, then both m and $-m$ are fundamental discriminants. Thus we see

$$|\mathcal{F}_1(D)| = \sum_{\substack{1 < m \leqslant D \\ m \text{ odd}}} \mu^2(m),$$

$$|\mathcal{F}_4(D)| = \sum_{\substack{1 \leqslant m \leqslant \frac{D}{4} \\ m \text{ odd}}} \mu^2(m),$$

and

$$|\mathcal{F}_8(D)| = 2 \sum_{\substack{1 \leqslant m \leqslant \frac{D}{8} \\ m \text{ odd}}} \mu^2(m).$$

We now use the asymptotic formula,

$$M_2(x) \colon = \sum_{n \leqslant x} \mu^2(n) = \frac{1}{\zeta(2)} x + O\left(x^{1/2}\right)$$

see for example, Ellison [7].

From this we show that

$$M_0(x) \colon = \sum_{\substack{n \leqslant x \\ n \text{ odd}}} \mu^2(n) = \frac{2}{3\zeta(2)} x + O\left(x^{1/2}\right).$$

For notice that

$$M_2(x) = M_0(x) + M_0\left(\frac{x}{2}\right)$$
$$M_2\left(\frac{x}{2}\right) = M_0\left(\frac{x}{2}\right) + M_0\left(\frac{x}{4}\right)$$
$$\vdots$$
$$M_2\left(\frac{x}{2^\nu}\right) = M_0\left(\frac{x}{2^\nu}\right) + M_0\left(\frac{x}{2^{\nu+1}}\right)$$

and so $$M_0(x) = \sum_{0\leqslant\nu\leqslant N} (-1)^\nu M_2\left(\frac{x}{2^\nu}\right) \quad \text{where} \quad N = \left[\frac{\log x}{\log 2}\right].$$

Then
$$\begin{aligned} M_0(x) &= \frac{1}{\zeta(2)}\sum_{\nu=0}^{N}\left(\frac{(-1)^\nu x}{2^\nu} + O\left(\left(\frac{x}{2^\nu}\right)^{1/2}\right)\right) \\ &= \frac{x}{\zeta(2)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2^\nu} - \frac{x}{\zeta(2)}\sum_{\nu>N}\frac{(-1)^\nu}{2^\nu} + O\left(x^{1/2}\right) \\ &= \frac{2}{3\zeta(2)}x - \frac{x}{\zeta(2)}O\left(\frac{1}{x}\right) + O\left(x^{1/2}\right) = \frac{2}{3\zeta(2)}x + O\left(x^{1/2}\right). \end{aligned}$$

But then
$$\begin{aligned} |\mathcal{F}(D)| &= |\mathcal{F}_1(D)| + |\mathcal{F}_4(D)| + |\mathcal{F}_8(D)| \\ &= M_0(D) - 1 + M_0\left(\frac{D}{4}\right) + 2M_0\left(\frac{D}{8}\right) \\ &= \frac{2}{3\zeta(2)}\left(D + \frac{D}{4} + \frac{2D}{8}\right) + O\left(\sqrt{D}\right) = \frac{1}{\zeta(2)}D + O\left(\sqrt{D}\right) \end{aligned}$$

as desired. □

Combining Lemmas 1 and 2, we have proved

Proposition 1.

$$A_{11} = -\frac{1}{2\zeta(2)}K\left(\frac{1}{2}\right)D + O\left(\sqrt{D}\right) + O\left(Dx^{-1/4}\log^2 x\right) \quad \begin{pmatrix} x\to\infty \\ D\to\infty\end{pmatrix}.$$

We next consider A_{12}.

Proposition 2. *Under the assumptions of Lemma 1,*

$$A_{12} \ll x^{-1/2}(\log x)D\log\log D \quad \begin{pmatrix} x\to\infty \\ D\to\infty\end{pmatrix}.$$

Proof:

$$A_{12} = x^{-1/2}\sum_{d\in\mathcal{F}(D)}\sum_{\substack{n=\square \\ (d,n)>1}} a\left(\frac{n}{x}\right)\Lambda(n) = x^{-1/2}\sum_{d\in\mathcal{F}(D)}\sum_{\substack{p,m \\ p\text{ prime} \\ m\geqslant 1 \\ p\mid d}} a\left(\frac{p^{2m}}{x}\right)\log p.$$

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We first show that for a fixed prime p,

$$\sum_{m=1}^{\infty} a\left(\frac{p^{2m}}{x}\right)\log p = O(\log x).$$

Since $a(x)$ has compact support in $(0,\infty)$, there exist positive constants $c_1 < c_2$ such that $a(x) = 0$ if $x \notin [c_1, c_2]$, and suppose $M = \max$ of $|a(x)|$. Then $a(p^{2m}/x) \neq 0$ implies that $c_1 \leqslant p^{2m}/x \leqslant c_2$ or equivalently that $(\log(c_1 x))/(2\log p) \leqslant m \leqslant (\log(c_2 x))/(2\log p)$. But then $\sum_{m=1}^{\infty} a(p^{2m}/x)\log p \ll M(\log x)/(\log p)\cdot\log p \ll \log x$. Notice that the implied constant is independent of p. Now

$$\begin{aligned} A_{12} = x^{-1/2}\sum_{d\in\mathcal{F}(D)}\sum_{\substack{p,m\\ p \text{ prime}\\ m\geqslant 1\\ p\mid d}} a\left(\frac{p^{2m}}{x}\right)\log p \ll x^{-1/2}\sum_{d\in\mathcal{F}(D)}\sum_{p\mid d}\log x \\ \ll x^{-1/2}\log x\sum_{d\leqslant D}\sum_{p\mid d} 1. \end{aligned}$$

But, as is known, see [2], $\sum_{d\leqslant D}\sum_{p\mid d} 1 \sim D\log\log D$. This establishes Proposition 2. □

Combining the two propositions yields

PROPOSITION 3. *Under the assumptions in Lemma 1, as $x\to\infty$, $D\to\infty$*

$$\begin{aligned} A_1 = -\frac{1}{\zeta(2)}\cdot\frac{1}{2}\cdot K\left(\frac{1}{2}\right)D + O\left(D^{1/2}\right) + O\left(Dx^{-1/4}\log^2 x\right) \\ + O\left(x^{-1/2}(\log x)D\log\log D\right). \end{aligned}$$

Now consider A_2.

PROPOSITION 4. *If the Riemann Hypothesis holds and*

$$\int_0^{\infty}\left|v^{3/2}\log^{5/2} v\, a'(v)\right| dv < \infty, \text{ then } A_2 \ll D^{1/2}x^{3/4}\log^{1/2} x.$$

PROOF: By the arguments in Ayoub [1], we have for n not a square, that $\sum_{d\in\mathcal{F}(D)}(d/n) = O\left(D^{1/2}n^{1/4}\log^{1/2} n\right)$ where the implied constant is independent of

n. Hence

$$A_2 \ll x^{-1/2} \sum_{n=1}^{\infty} a\Big(\frac{n}{x}\Big) \Lambda(n) D^{1/2} n^{1/4} \log^{1/2} n.$$

Now we consider

$$\begin{aligned}\sum_{n=1}^{\infty} a\Big(\frac{n}{x}\Big)\Lambda(n) n^{1/4}\log^{1/2} n &= \int_0^\infty a\Big(\frac{u}{x}\Big) u^{1/4}\log^{1/2} u\, d\psi(u)\\ &= \int_0^\infty a\Big(\frac{u}{x}\Big) u^{1/4}\log^{1/2} u\, du + \int_0^\infty a\Big(\frac{u}{x}\Big) u^{1/4}\log^{1/2} u\, dE(u)\end{aligned}$$

where $\psi(u) = u + E(u)$ and again by R.H. $E(u) \ll u^{1/2}\log^2 u$. We consider the first integral, change variable $v = u/x$, use $\sqrt{x+y} \leqslant \sqrt{x} + \sqrt{y}$ and obtain

$$\begin{aligned}\int_0^\infty a\Big(\frac{u}{x}\Big) u^{1/4}\log^{1/2} u\, du &= \int_0^\infty a(v) x^{1/4} v^{1/4} \log^{1/2}(xv) x\, dv\\ &= x^{5/4}\int_0^\infty a(v) v^{1/4} (\log x + \log v)^{1/2} dv\\ &< x^{5/4}\int_0^\infty a(v) v^{1/4}\Big(\log^{1/2} x + \log^{1/2} v\Big) dv \ll x^{5/4}\log^{1/2} x.\end{aligned}$$

Next we consider the second integral and integrate by parts, and changing variable as usual:

$$\begin{aligned}&\int_0^\infty a\Big(\frac{u}{x}\Big) u^{1/4}\log^{1/2} u\, dE(u) = -\int_0^\infty E(u)\, d\Big(a\Big(\frac{u}{x}\Big) u^{1/4}\log^{1/2} u\Big)\\ &\ll \int_0^\infty u^{1/2}(\log^2 u) u^{1/4}\log^{1/2} u\, da\Big(\frac{u}{x}\Big) + \int_0^\infty u^{1/2}(\log^2 u)\, a\Big(\frac{u}{x}\Big) d\Big(u^{1/4}\log^{1/2} u\Big)\\ &\ll \int_0^\infty u^{3/4}\Big(\log^{5/2} u\Big) a'\Big(\frac{u}{x}\Big)\frac{1}{x} du\\ &\qquad + \int_0^\infty u^{1/2}\log^2 u\, a\Big(\frac{u}{x}\Big)\Big(u^{-3/4}\log^{1/2} u + u^{-3/4}\log^{-1/2} u\Big) du\\ &\ll \int_0^\infty x^{3/4} v^{3/4}\log^{5/2}(xv) a'(v)\, dv + \int_0^\infty u^{-1/4}\log^{5/2} u\, a\Big(\frac{u}{x}\Big) du\\ &\ll x^{3/4}\log^{5/2} x\int_0^\infty v^{3/4}\log^{5/2} v\, a'(v)\, dv\\ &\qquad + x^{-5/4}\log^{5/2} x\int_0^\infty v^{-1/4}\log^{5/2} v\, a(v)\, dv\\ &\ll x^{3/4}\log^{5/2} x.\end{aligned}$$

Combining the results establishes the proposition. □

From Proposition 3 and 4 we obtain

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Proposition 5. *Under the assumptions of Proposition 4,*

$$A = -\frac{1}{\zeta(2)}\cdot\frac{1}{2}\cdot K\Big(\frac{1}{2}\Big)D + O\Big(Dx^{-1/4}\log^2 x\Big) + O\Big(x^{-1/2}(\log x)D\log\log D\Big) + O\Big(D^{1/2}x^{3/4}\log^{1/2} x\Big).$$

Now we consider B.

Proposition 6.

$$B = \frac{1}{\zeta(2)}x^{-1/2}a\Big(\frac{1}{x}\Big)D\log D + O\Big(x^{-1/2}a\Big(\frac{1}{x}\Big)D\Big).$$

Proof: We have

$$\begin{aligned} B &= x^{-1/2}a\Big(\frac{1}{x}\Big)\sum_{d\in\mathcal{F}(D)}\log\frac{|d|}{\pi} \\ &= x^{-1/2}a\Big(\frac{1}{x}\Big)\sum_{d\in\mathcal{F}(D)}\log|d| - x^{-1/2}a\Big(\frac{1}{x}\Big)\log\pi\,|\mathcal{F}(D)|. \end{aligned}$$

First consider

$$\sum_{d\in\mathcal{F}(D)}\log|d| = \int_1^D \log u\, d\mathcal{F}(u) = \int_1^D \log u\, d\Big(\frac{1}{\zeta(2)}u + E_1(u)\Big)$$

where $\mathcal{F}(u) = \zeta(2) + E_1(u)$ and $E_1(u) \ll u^{1/2}$ by Lemma 2. Now $(1/\zeta(2))\int_1^D \log u\,du = (1/\zeta(2))D\log D + O(D)$ by evaluation. On the other hand,

$$\begin{aligned} \int_1^D \log u\, dE_1(u) &= E_1(u)\log u\,\Big|_1^D - \int_1^D E_1(u)d\log u \\ &\ll \sqrt{D}\log D + \int_1^D u^{-1/2}du \ll \sqrt{D}\log D. \end{aligned}$$

Thus
$$B = \frac{1}{\zeta(2)}x^{-1/2}a\Big(\frac{1}{x}\Big)(D\log D + O(D)),$$

as desired. □

Combining Propositions 5 and 6, we have

Theorem 1. *Under the assumptions of Proposition 4,*

$$\begin{aligned} F_1(x, D) = &-\frac{1}{\zeta(2)}\cdot\frac{1}{2}\cdot K\Big(\frac{1}{2}\Big)D + \frac{1}{\zeta(2)}x^{-1/2}a\Big(\frac{1}{x}\Big)D\log D \\ &+ O\Big(x^{-1/2}(\log x)D\log\log D\Big) + O\Big(D^{1/2}x^{3/4}\log^{1/2} x\Big). \end{aligned}$$

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We now normalise $F_1(x, D)$ by taking $x = D^\alpha$ and dividing by

$$\frac{1}{\zeta(2)} D \cdot \frac{1}{2} \cdot K\left(\frac{1}{2}\right).$$

Hence let
$$F(\alpha, D) = \left(\frac{1}{\zeta(2)} \cdot \frac{1}{2} \cdot K\left(\frac{1}{2}\right) D\right)^{-1} F_1(D^\alpha, D).$$

Then we have

Theorem 2. *Under the assumptions of Proposition 4, as $D \to \infty$*

$$F(\alpha, D) = -1 + \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} a(D^\alpha) D^{-\alpha/2} \log D + O\left(D^{-\alpha/2} \log D^\alpha \log\log D\right) + O\left(D^{(3/4)\alpha - 1/2} \log^{1/2} D^\alpha\right).$$

In particular if $0 < |\alpha| < 2/3$, then

$$F(\alpha, D) = -1 + o(1).$$

We are now in a position of using Theorem 2 to investigate the distribution of the zeros of these L-functions.

Theorem 3. *Assume the hypotheses of Theorem 2. Suppose that $r(\alpha)$ is an even function defined on $(-\infty, \infty)$ with $\widehat{r}(\alpha)$ existing and such that $\widehat{r}(\alpha)$ is supported in $[-2/3, 2/3]$. Moreover suppose $\int_{-\infty}^{\infty} \alpha r(\alpha) d\alpha$ converges. Then*

$$\left(\frac{D}{\zeta(2)}\right)^{-1} \sum_{d \in \mathcal{F}(D)} \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} \sum_{\rho(d)} K(\rho) r\left(\frac{\gamma \log D}{2\pi}\right) = 2\int_{-\infty}^{\infty} \left(1 - \frac{\sin 2\pi\alpha}{2\pi\alpha}\right) r(\alpha) d\alpha + o(1),$$

where the implied constant depends only on the kernel K.

Proof: Consider $\int_{-\infty}^{\infty} F(\alpha, D) \widehat{r}(\alpha) d\alpha$ which by Theorem 2 is equal to

$$\int_{-1}^{1} \left(-1 + \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} a(D^{-\alpha}) D^{-\alpha/2} \log D\right) \widehat{r}(\alpha) d\alpha + o(1)$$
$$= \int_{-\infty}^{\infty} \left(-\chi_{[-1,1]}(\alpha) + \left(\left(\frac{1}{2}\right) K\left(\frac{1}{2}\right)\right)^{-1} a(D^\alpha) D^{-\alpha/2} \log D\right) \widehat{r}(\alpha) d\alpha + o(1)$$

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where $\chi_{[-1,1]}$ is the characteristic function of $[-1, 1]$. But

$$\int_{-\infty}^{\infty} \chi_{[-1,1]}(\alpha)\widehat{r}(\alpha)d\alpha = \int_{-\infty}^{\infty} \widehat{\chi}_{[-1,1]}(\alpha)r(\alpha)d\alpha = 2\int_{-\infty}^{\infty} \frac{\sin(2\pi\alpha)}{2\pi\alpha}r(\alpha)d\alpha.$$

On the other hand,

$$\int_{-\infty}^{\infty} D^{-\alpha/2}a(D^{-\alpha})\widehat{r}(\alpha)d\alpha = \int_{-\infty}^{\infty} \widehat{D^{-\alpha/2}a(D^{-\alpha})}r(\alpha)d\alpha.$$

But
$$\widehat{D^{-\alpha/2}a(D^{-\alpha})} = \int_{-\infty}^{\infty} D^{-\beta/2}a(D^{-\beta})e^{-2\pi i\alpha\beta}d\beta$$

and by the change of variable $t = D^{-\beta}$, this integral equals

$$\frac{1}{\log D}\int_0^{\infty} a(t)t^{1/2+2\pi i\alpha/\log D}\frac{dt}{t}$$
$$= \frac{\alpha}{\log D}\int_0^{\infty} a(t)t^{1/2}\frac{dt}{t} + \frac{1}{\log D}\int_0^{\infty} a(t)t^{1/2}\left(t^{2\pi i\alpha/\log D} - 1\right)\frac{dt}{t}.$$

Notice that

$$t^{2\pi i\alpha/\log D} - 1 = e^{2\pi i(\log t/\log D)\alpha} - 1 = 2\pi i\frac{\log t}{\log D}\alpha e^{2\pi i(\log t/\log D)\theta_\alpha}$$

for some θ_α between 0 and α, whence

$$t^{2\pi i\alpha/\log D} - 1 \ll \frac{\log t}{\log D}\alpha$$

and
$$\frac{1}{\log D}\int_0^{\infty} a(t)t^{1/2}\left(t^{2\pi i\alpha/\log D} - 1\right)\frac{dt}{t} \ll \frac{\alpha}{\log^2 D}$$

where the constant is independent of α and D. Consequently,

$$\int_{-\infty}^{\infty} \widehat{D^{-\alpha/2}a(D^{-\alpha})}r(\alpha)d\alpha = \frac{1}{\log D}K\left(\frac{1}{2}\right)\int_{-\infty}^{\infty} r(\alpha)d\alpha + O\left(\frac{1}{\log^2 D}\int_{-\infty}^{\infty}\alpha r(\alpha)d\alpha\right)$$
$$= \frac{K(\frac{1}{2})}{\log D}\int_{-\infty}^{\infty} r(\alpha)d\alpha + O\left(\frac{1}{\log^2 D}\right).$$

Thus
$$\int_{-\infty}^{\infty}\left(-\chi_{[-1,1]}(\alpha) + \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1}a(D^{-\alpha})D^{-\alpha/2}\log D\right)d\alpha$$
$$= -2\int_{-\infty}^{\infty}\frac{\sin 2\pi\alpha}{2\pi\alpha}r(\alpha)d\alpha + 2\int_{-\infty}^{\infty} r(\alpha)d\alpha + O\left(\frac{1}{\log D}\right)$$
$$= 2\int_{-\infty}^{\infty}\left(1 - \frac{\sin 2\pi\alpha}{2\pi\alpha}\right)r(\alpha)d\alpha + O\left(\frac{1}{\log D}\right).$$

Therefore
$$\int_{-\infty}^{\infty} F(\alpha, D)\widehat{r}(\alpha)d\alpha = 2\int_{-\infty}^{\infty}\left(1 - \frac{\sin 2\pi\alpha}{2\pi\alpha}\right)r(\alpha)d\alpha + o(1).$$

On the other hand, by the definition of $F(\alpha, D)$ we have

$$\int_{-\infty}^{\infty} F(\alpha, D)\widehat{r}(\alpha)d\alpha$$
$$= \left(\frac{D}{\zeta(2)}\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} \sum_{d\in\mathcal{F}(D)} \sum_{\rho(d)} K(\rho) \int_{-\infty}^{\infty} e^{i\gamma\alpha\log D}\widehat{r}(\alpha)d\alpha$$
$$= \left(\frac{D}{\zeta(2)}\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} \sum_{d\in\mathcal{F}(D)} \sum_{\rho(d)} K(\rho) \int_{-\infty}^{\infty} \widehat{r}(\alpha)e^{\frac{2\pi i\alpha\gamma\log D}{2\pi}}d\alpha.$$

But
$$\int_{-\infty}^{\infty} \widehat{r}(\alpha)e^{\frac{2\pi i\alpha\gamma\log D}{2\pi}}d\alpha = \widehat{\widehat{r}}\left(-\frac{\gamma\log D}{2\pi}\right) = r\left(\frac{\gamma\log D}{2\pi}\right).$$

Thus
$$\left(\frac{D}{\zeta(2)}\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} \sum_{d\in\mathcal{F}(D)} \sum_{\rho(d)} K(\rho)r\left(\frac{\gamma\log D}{2\pi}\right)$$
$$= 2\int_{-\infty}^{\infty}\left(1-\frac{\sin 2\pi\alpha}{2\pi\alpha}\right)r(\alpha)d\alpha + o(1).$$

□

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