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Casting light on shadow Somos sequences

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Abstract
Recently Ovsienko and Tabachnikov considered extensions of Somos and Gale-Robinson sequences, defined over the algebra of dual numbers. Ovsienko used the same idea to construct so-called shadow sequences derived from other nonlinear recurrence relations exhibiting the Laurent phenomenon, with the original motivation being the hope that these examples should lead to an appropriate notion of a cluster superalgebra, incorporating Grassmann variables. Here, we present various explicit expressions for the shadow of Somos-4 sequences and describe the solution of a general Somos-4 recurrence defined over the $\mathbb{C}$-algebra of dual numbers from several different viewpoints: analytic formulae in terms of elliptic functions, linear difference equations, and Hankel determinants.

1. Introduction

The standard dual numbers, which were introduced by Clifford, take the form $x + y\varepsilon$, where x,y are a pair of real numbers and $\varepsilon^2 = 0$. Triples of such numbers on the dual unit sphere were employed by Study to describe the space of oriented lines in $\mathbb{R}^3$. This geometrical interpretation leads to contemporary applications of dual numbers in computer vision, while in an algebraic setting they can be used for automatic differentiation. Dual numbers also provide the simplest example of an algebra incorporating Grassmann variables, which encode fermionic fields in quantum theory.

Cluster algebras, introduced in [5], are a class of commutative algebras with a distinguished set of generators (cluster variables) that are defined recursively by a process called mutation and arise in a variety of different contexts, including Lie theory, Poisson geometry, Teichmüller theory, and integrable systems (see e.g. [9, 13, 15] and references therein). A notable feature of cluster algebras is that they exhibit the Laurent phenomenon: each cluster variable is a Laurent polynomial in the variables from an initial set of generators (called a seed), with integer coefficients. In tandem with the development of cluster algebras, based on the Caterpillar Lemma [6], Fomin and Zelevinsky found a systematic way to prove that the Laurent phenomenon holds for a wide variety of nonlinear recurrence relations, including many examples originally described by Gale [8]. One such example is the Somos-4 recurrence, namely

$$x_{n+4}x_n = \alpha\, x_{n+3}x_{n+1} + \beta\, x_{n+2}^2, \tag{1.1}$$

where α and β are coefficients. The Laurent property for (1.1) means that the iterates are Laurent polynomials in a set of four initial values x_0, x_1, x_2, x_3 with coefficients in $\mathbb{Z}[\alpha, \beta]$, that is,

$$x_n \in \mathbb{Z}\left[\alpha, \beta, x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}\right], \quad \forall n \in \mathbb{Z}. \tag{1.2}$$

In particular, this implies that the "classical" Somos-4 sequence, defined by taking coefficients $\alpha = \beta = 1$ and fixing all initial data to be 1, consists entirely of integers, beginning with

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots \tag{1.3}$$

(see [19]). It was subsequently shown by Fordy and Marsh that the Somos-4 recurrence is generated from a cluster algebra defined by a quiver that has periodicity under a specific sequence of mutations: the variables (x_0, x_1, x_2, x_3) can be taken as an initial seed, extended by a pair of frozen variables α, β that do not mutate [7].

Given the relevance of superalgebras and supermanifolds in both geometry and theoretical physics, and the myriad ways that cluster algebras interact with these different areas, it is currently of interest to find an appropriate notion of a cluster superalgebra, incorporating (anticommuting) Grassmann variables in addition to the usual (commuting) cluster variables. The first step in this direction was taken by Ovsienko [20], who introduced a type of extended quiver, given by a hypergraph obtained by adding extra odd vertices (associated with Grassmann variables) to the usual (even) vertices associated with the cluster variables in a cluster algebra defined by a quiver. This allowed the development of various examples, including superfriezes, proposed as superalgebra analogues of Coxeter's frieze patterns [17, 22], and also versions of Somos-4 and higher-order Somos-k (or Gale-Robinson) recurrences defined over the dual numbers [23]. In particular, in [20] and [23], the following dual number generalization of the original Somos-4 recurrence with $\alpha = \beta = 1$ was considered:

$$X_{n+4}X_n = X_{n+3}X_{n+1} + (1+\varepsilon)\, X_{n+2}^2; \tag{1.4}$$

while in [23], an alternative dual number version was also mentioned, namely

$$X_{n+4}X_n = (1+\varepsilon)\, X_{n+3}X_{n+1} + X_{n+2}^2. \tag{1.5}$$

Both of the latter examples define a sequence of dual numbers $X_n = x_n + y_n\varepsilon$, given a suitable set of four initial values X_0, X_1, X_2, X_3. It is proved in [20] that, within the setting of mutations of extended quivers considered there, the Laurent property holds in terms of both even and odd variables. As a consequence, for the recurrences (1.4) and (1.5), if the initial values are given by $x_0 = x_1 = x_2 = x_3 = 1$ and any four integers y_0, y_1, y_2, y_3, then the whole sequence (y_n) consists of integers.

More recently, Ovsienko has considered several other examples of nonlinear recurrence relations or birational transformations where each variable x is replaced by a dual number $X = x + y\varepsilon$, referring to the corresponding sequence of y values as the shadow sequence [21]. For instance, the Cassini relation for the Fibonacci sequence produces the convolution of the sequence with itself as a shadow, while certain shadow sequences of the Markov numbers appear to be new.

In this paper, we will take the complex numbers as the ambient field and work with the commutative $\mathbb{C}$-algebra of dual numbers $\mathbb{D} = \mathbb{D}(\mathbb{C})$ given by:

$$\mathbb{D} = \left\{ x + y\varepsilon \mid x, y \in \mathbb{C},\ \varepsilon^2 = 0 \right\},$$

which is isomorphic to the quotient $\mathbb{C}[t] \,/ < t^2 >$. Note that the units in $\mathbb{D}$ are the set of elements:

$$\mathbb{D}^* = \{ x + y\varepsilon \in \mathbb{D} \mid x \neq 0 \},$$

with

$$(x + y\varepsilon)^{-1} = x^{-1}(1 - x^{-1}y\varepsilon). \tag{1.6}$$

Then, we consider the general Somos-4 recurrence for dual numbers $X_n = x_n + y_n\varepsilon$, given by:

$$X_{n+4}X_n = \left(\alpha^{(0)} + \alpha^{(1)}\varepsilon\right)\, X_{n+3}X_{n+1} + \left(\beta^{(0)} + \beta^{(1)}\varepsilon\right)\, X_{n+2}^2, \tag{1.7}$$

which includes (1.4) and (1.5) as special cases. Iteration of (1.7) requires that $X_n \in \mathbb{D}^*$ at each step. If $x_n = 0$ at some stage, so that X_n is not a unit, then a priori it appears impossible to iterate further, but one can still consider sequences in $\mathbb{D}$ that satisfy (1.7). In fact, zero terms in Somos-4 sequences are generically isolated (see the discussion in [14]). Moreover, if one starts from initial data $X_0, X_1, X_2, X_3 \in \mathbb{D}^*$, then the orbit is defined for all $n \in \mathbb{Z}$, because the iterates can be obtained by evaluating Laurent polynomials at these four units, due to the Laurent phenomenon (see (1.10) below).

If we separate the above equation into even/odd components (i.e. the two components of the dual number on each side of the equation), then we obtain the triangular system:

$$x_{n+4}x_n = \alpha^{(0)}\, x_{n+3}x_{n+1} + \beta^{(0)}\, x_{n+2}^2, \tag{1.8}$$

$$x_n y_{n+4} - \alpha^{(0)}x_{n+1}y_{n+3} - 2\beta^{(0)}x_{n+2}y_{n+2} - \alpha^{(0)}x_{n+3}y_{n+1} + x_{n+4}y_n = \alpha^{(1)}x_{n+1}x_{n+3} + \beta^{(1)}x_{n+2}^2. \tag{1.9}$$

The even equation (1.8) is just the ordinary Somos-4 recurrence for x_n, with coefficients $\alpha^{(0)}$, $\beta^{(0)}$, while the odd equation (1.9) is an inhomogeneous linear equation for y_n, where the coefficients and the inhomogeneity (i.e. the source term on the right-hand side) are given in terms of x_n. In the special case that $\alpha^{(1)} = \beta^{(1)} = 0$ and the right-hand side of (1.9) vanishes, the resulting homogeneous equation is just the linearization of (1.8), corresponding to shadow sequences in the sense of [21].

The original methods for proving the Laurent property for (1.1), such as the one described by Gale in [8], or the approach taken by Fomin and Zelevinsky in [6], treat the initial data as formal variables but do not rely on the auxiliary structure of a cluster algebra. Since $\mathbb{D}$ is a commutative ring, these methods carry over directly to (1.7), immediately yielding the assertion that

$$X_n \in \mathbb{Z}\left[\alpha, \beta, X_0^{\pm 1}, X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}\right] \tag{1.10}$$

for all n, where $\alpha = \alpha^{(0)} + \alpha^{(1)}\varepsilon$, $\beta = \beta^{(0)} + \beta^{(1)}\varepsilon$. Thus, taking even/odd components and using (1.6), this allows us to state the Laurent property for the system (1.8), (1.9) in the following form.

Lemma 1.1. *The system (1.8), (1.9) has the Laurent property, in the sense that*

$$x_n \in \mathbb{Z}\left[\alpha^{(0)}, \beta^{(0)}, x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}\right],$$

and

$$y_n \in \mathbb{Z}\left[\alpha^{(0)}, \alpha^{(1)}, \beta^{(0)}, \beta^{(1)}, x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, y_0, y_1, y_2, y_3\right],$$

for all $n \in \mathbb{Z}$.

The rest of the paper is devoted to providing various representations for explicit solutions of (1.7), in three different formats: analytic formulae, based on the results of [10]; an elementary algebraic description, based on the theory of linear difference equations; and Hankel determinant expressions, using more recent results on Jacobi continued fractions in [12]. Each of the next three sections will deal with one of these representations, before we end with some conclusions.

2. Analytic formulae

To begin with, we paraphrase the main result of [10] and briefly describe it in a way that incorporates certain later observations made in [11] and [14].

Theorem 2.1. *The general solution of the initial value problem for (1.1) over $\mathbb{C}$ is*

$$x_n = A\, B^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}}, \tag{2.1}$$

where $\sigma(z) = \sigma(z; g_2, g_3)$ is the Weierstrass sigma function for an associated elliptic curve:

$$y^2 = 4x^3 - g_2 x - g_3, \tag{2.2}$$

where $A, B \in \mathbb{C}^*$ and $z, z_0, g_2, g_3 \in \mathbb{C}$ are explicitly determined by four nonzero initial values and coefficients $x_0, x_1, x_2, x_3, \alpha, \beta$.

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More precisely, the solution of the initial value problem for (1.1) is achieved by considering the sequence of ratios:

$$d_n = \frac{x_{n+1}x_{n-1}}{x_n^2}, \tag{2.3}$$

which satisfy a recurrence of second order, namely

$$d_{n+1}d_{n-1} = \frac{\alpha d_n + \beta}{d_n^2}. \tag{2.4}$$

The latter recurrence is equivalent to a birational map of the plane, being an example of a symmetric QRT map [24], and it has a rational conserved quantity:

$$J = d_n d_{n-1} + \alpha\left(\frac{1}{d_n} + \frac{1}{d_{n-1}}\right) + \frac{\beta}{d_n d_{n-1}} \tag{2.5}$$

that defines a pencil of biquadratic plane curves, each of which is generically of genus 1 (except for a finite number of singular fibres, for certain special values of J [3]) and hence isomorphic to a cubic curve of the form (2.2). Explicit computations with Weierstrass functions then show that the formula (2.1) does indeed provide a solution of the Somos-4 recurrence (1.1), with the coefficients being given by:

$$\alpha = \frac{\sigma(2z)^2}{\sigma(z)^8}, \qquad \beta = -\frac{\sigma(3z)}{\sigma(z)^9}, \tag{2.6}$$

allowing g_2, g_3 to be obtained in terms of α, β, J via the expressions:

$$g_2 = 12\lambda^2 - 2J, \quad g_3 = 4\lambda^3 - g_2\lambda - \alpha, \quad \lambda = \frac{1}{3\alpha}\left(\frac{J^2}{4} - \beta\right), \tag{2.7}$$

and then (up to fixing an overall sign) z and z_0 are determined modulo periods by the elliptic integrals:

$$z = \int_\infty^\lambda \frac{dx}{y}, \qquad z_0 = \int_\infty^{\lambda - d_0} \frac{dx}{y}, \tag{2.8}$$

so that once these are given the values of A,B can be found from the initial data. (Note that when $g_2^3 - 27g_3^2 = 0$, corresponding to a singular fibre, the formula (2.1) is still valid in terms of suitable trigonometric/rational limits of the sigma function.)

For the purposes of what follows, it is important to note that from (2.3) the expression (2.5) is equivalent to a rational conserved quantity for the Somos-4 recurrence (1.1) itself, that is,

$$J = \frac{x_n^2 x_{n+3}^2 + \alpha\left(x_{n+1}^3 x_{n+3} + x_n x_{n+2}^3\right) + \beta\, x_{n+1}^2 x_{n+2}^2}{x_n x_{n+1} x_{n+2} x_{n+3}}. \tag{2.9}$$

An elementary observation, which is the basis for the application of dual numbers to automatic differentiation, is that for any differentiable function Φ and $X = x + y\varepsilon \in \mathbb{D}$, the value of Φ at X is

$$\Phi(X) = \Phi(x) + \Phi'(x)y\varepsilon.$$

We can use this result to describe solutions of (1.7) in analytic form.

Proposition 2.2. *Given parameters* $A = A^{(0)} + A^{(1)}\varepsilon$, $B = B^{(0)} + B^{(1)}\varepsilon$, $G_2 = g_2 + \tilde{g}_2\varepsilon$, $G_3 = g_3 + \tilde{g}_3\varepsilon$, $Z_0 = z_0 + \tilde{z}_0\varepsilon$, $Z = z + \tilde{z}\varepsilon \in \mathbb{D}$, *the expression*

$$X_n = A\,B^n \frac{\sigma(Z_0 + nZ)}{\sigma(Z)^{n^2}} \tag{2.10}$$

satisfies the dual Somos-4 recurrence (1.7) with coefficients:

$$\alpha = \alpha^{(0)} + \alpha^{(1)}\varepsilon = \frac{\sigma(2Z)^2}{\sigma(Z)^8}, \quad \beta = \beta^{(0)} + \beta^{(1)}\varepsilon = -\frac{\sigma(3Z)}{\sigma(Z)^9},$$

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provided that $A, B \in \mathbb{D}^*$ and $\sigma(Z) = \sigma(Z; G_2, G_3) \in \mathbb{D}^*$. In terms of even/odd components, this can be written as:

$$X_n = x_n + x_n \left(\frac{A^{(1)}}{A^{(0)}} + \frac{B^{(1)}}{B^{(0)}} n + \tilde{z}_0 \zeta(z_0 + nz) + \left(\tilde{z}\partial_z + \tilde{g}_2 \partial_{g_2} + \tilde{g}_3 \partial_{g_3} \right) \log x_n \right) \varepsilon, \tag{2.11}$$

where x_n denotes the right-hand side of (2.1) with the replacement $A \to A^{(0)}$, $B \to B^{(0)}$, and the other parameters z, z_0, g_2, g_3 being the same, $\zeta(z) = \zeta(z; g_2, g_3)$ is the Weierstrass zeta function, and ∂ denotes a partial derivative.

Proof. As pointed out in [11], the fact that the analytic expression (2.1) satisfies a Somos-4 relation with coefficients given by (2.6) is a direct consequence of the three-term identity for the sigma function, which can be viewed as an algebraic identity between power series. The sigma function $\sigma(z; g_2, g_3)$ is a holomorphic function of the argument z and is also holomorphic in the two invariants g_2, g_3. Therefore, the same proof carries over to the commutative algebra $\mathbb{D}$, with the requirement that $\sigma(Z) \in \mathbb{D}^*$ so that one can divide by suitable powers of this quantity, and $A, B \in \mathbb{D}^*$ so that the even part $x_n \neq 0$. The solution depends analytically on six dual parameters, so it can be expanded as $X_n = x_n + y_n \varepsilon$ by differentiating with respect to each of these parameters, giving

$$y_n = (A^{(1)} \partial_{A^{(0)}} + B^{(1)} \partial_{B^{(0)}} + + \tilde{z}_0 \partial_{z_0} + \tilde{z}\partial_z + \tilde{g}_2 \partial_{g_2} + \tilde{g}_3 \partial_{g_3}) \, x_n.$$

It is convenient to rewrite this in terms of logarithmic derivatives of x_n, so that y_n is written as x_n times a sum of such derivatives, and the first three derivatives are straightforward to calculate explicitly, as in (2.11). □

We believe that the analytic formula (2.10) represents the general solution of the dual Somos-4 recurrence (1.7). However, a complete proof would require showing that, given $\alpha, \beta \in \mathbb{D}$, one can solve the initial value problem for (1.7) by determining the four dual parameters G_2, G_3, Z_0, Z. The difficulty lies in constructing a suitable theory of elliptic functions and uniformization of elliptic curves over the dual numbers, which in particular would provide analogues of the elliptic integrals (2.8). An abstract theory of super Riemann surfaces is available [28], but to the best of our knowledge the genus 1 case has not been developed explicitly in terms of Weierstrass functions with odd components. Rather than trying to pursue this further here, we will take the formula (2.11) as the starting point for a more straightforward algebraic approach in the next section.

3. Solution of linear difference equation

Since the system consisting of (1.8) and (1.9) is triangular, and we already have the general solution of (1.8), given by Theorem 2.1, the main outstanding problem is to understand the solutions of (1.9). We beginning by considering the special case $\alpha^{(1)} = \beta^{(1)} = 0$, which is just a homogeneous linear equation for y_n, namely

$$x_n y_{n+4} - \alpha^{(0)} x_{n+1} y_{n+3} - 2\beta^{(0)} x_{n+2} y_{n+2} - \alpha^{(0)} x_{n+3} y_{n+1} + x_{n+4} y_n = 0. \tag{3.1}$$

The solutions of the latter equation are the shadow Somos-4 sequences, in the sense of [21].

However, the equation (3.1) is just the linearization of the Somos-4 recurrence (1.8), and we have the explicit solution of this, which (for fixed coefficients $\alpha^{(0)}, \beta^{(0)}$) depends on four arbitrary parameters; these can be taken to be $A^{(0)}, B^{(0)}, z_0$, and $J^{(0)}$, where the last is the value of the first integral (2.9). Thus, by adapting a technique that goes back to Lie's work on ordinary differential equations, we can observe that the derivative of the solution x_n with respect to each of these parameters provides a solution of the linearized equation, and thus we obtain four linearly independent solutions, which span the whole vector space of solutions of (3.1).

Lemma 3.1. *The Somos-4 shadow equation (1.9) has four linearly independent solutions, given by $y_n^{(i)} = x_n$, $y_n^{(ii)} = nx_n$, $y_n^{(iii)} = x_n\zeta(z_0+nz)$ and*

$$y_n^{(iv)} = x_n \partial_{J^{(0)}} \log x_n = x_n \left(\frac{dz}{dJ^{(0)}}\partial_z + \frac{dg_2}{dJ^{(0)}}\partial_{g_2} + + \frac{dg_3}{dJ^{(0)}}\partial_{g_3} \right) \log x_n. \tag{3.2}$$

Proof. The derivative of the formula (2.1) with respect to the scaling parameter A is just proportional to x_n, while the derivative with respect to B is proportional to nx_n. Differentiating the numerator in (2.1) with respect to z_0 produces x_n times the Weierstrass zeta function, but the dependence on $J^{(0)}$ is more complicated, as both z and the invariants g_2 and g_3 vary with this parameter. □

Remark 3.2. *The fact that $y_n = x_n$ is a solution of (1.9) was already noted in [23].*

The parameters A,B in (2.1) correspond to the action of a two-parameter scaling group, $x_n \to AB^n\, x_n$ for $(A,B) \in (\mathbb{C}^*)^2$, which leaves the Somos-4 recurrence invariant. The freedom to vary z_0 corresponds to an arbitrary choice of initial point on the elliptic curve (2.2), viewed as a complex torus $\mathbb{C}/\Lambda$ where Λ is the period lattice. The first three terms in the odd part of (2.11) are clearly proportional to $y_n^{(i)}$, $y_n^{(ii)}$, and $y_n^{(iii)}$. We will shortly present an algebraic method to calculate the third independent shadow solution $y_n^{(iii)}$.

As for the fourth solution $y_n^{(iv)}$, it is much more complicated, because it involves the modular derivatives ∂_{g_2} and ∂_{g_3}. Indeed, if we fix the coefficients $\alpha^{(0)}$ and $\beta^{(0)}$, then we can consider the following system of three equations in terms of the Weierstrass $\wp$ function and its derivatives:

$$\alpha^{(0)} = \wp'(z)^2, \qquad \frac{\beta^{(0)}}{\alpha^{(0)}} = \wp(2z) - \wp(z), \qquad J^{(0)} = \wp''(z); \tag{3.3}$$

the first two equations are just equivalent to (2.6), corresponding to identities between elliptic functions (see [10, 11]). Differentiating each of these equations with respect to $J^{(0)}$ and applying the chain rule gives a system of three linear equations for the derivatives of z, g_2, g_3 with respect to $J^{(0)}$, appearing in (3.2). As for the modular derivatives of the sigma function, note that it is a weighted homogeneous function of z, g_2, g_3, so it satisfies the linear partial differential equation (PDE):

$$\left(4g_2\partial_{g_2} + 6g_3\partial_{g_3} - z\partial_z + 1\right)\sigma\,(z; g_2, g_3) = 0.$$

As was shown by Weierstrass [27], it satisfies another linear PDE which is first order in g_2, g_3 and second order in z: this is equivalent to the well-known heat equation for the Jacobi theta function. (It also satisfies a fourth-order nonlinear ODE in z that can be written in Hirota bilinear form [4], which can be viewed as the continuous analogue of Somos-4.) In principle, one could combine all these facts to rewrite (3.2) in terms of z derivatives only. However, we do not need to calculate these analytic expressions in more detail, because in due course we will present an algebraic way to characterize a fourth independent shadow solution, which is much more straightforward.

In order to derive the third shadow solution in a more algebraic way, we consider the following coupled pair of recurrence relations with parameters u,f, which was considered in [12] as the simplest example of a map arising from Jacobi continued fractions in hyperelliptic function fields, based on a construction introduced by van der Poorten [26]:

$$\begin{aligned} v_n &= -v_{n-1} + u/d_n, \\ d_{n+1} &= -d_n - v_n^2 - f. \end{aligned} \tag{3.4}$$

The above system defines an integrable map, and here we will present its analytic solution.

Proposition 3.3. *The map $(v_{n-1}, d_n) \mapsto (v_n, d_{n+1})$ defined by (3.4) is integrable: it preserves the symplectic form $dv_{n-1} \wedge dd_n$, and has the first integral:*

$$H = d_n\left(v_{n-1}^2 + d_n + f\right) - uv_{n-1}.$$

The general solution of the map can be written in terms of elliptic functions, up to an overall choice of sign as:

$$d_n = \wp(z) - \wp(z_0 + nz), \qquad v_n = \pm\big(\zeta(z_0 + (n+1)z) - \zeta(z_0 + nz) - \zeta(z)\big), \tag{3.5}$$

with the parameters given by:

$$u = \pm\wp'(z), \qquad f = -3\wp(z),$$

on the level set $H = -\frac{1}{2}\wp''(z)$.

Proof. It is straightforward to verify from the map (3.4) that $\omega = dv_{n-1} \wedge dd_n = dv_n \wedge dd_{n+1}$, so it is a symplectic map, and a short calculation also shows that H is independent of n, so this is a discrete integrable system with one degree of freedom. By Proposition 5.1 of [12], on each level set $H = \text{const}$, the sequence of values of d_n coincides with an orbit of (2.4), if the coefficients and the value of the first integral (2.5) are identified as:

$$\alpha = u^2, \qquad \beta = u^2(v^2 + f), \qquad J = 2uv = -2H. \tag{3.6}$$

As for the analytic solution of the map, it was shown in [10] that when x_n is given by (2.1), the corresponding solution of (2.4), associated with it via (2.3), is given in terms of the Weierstrass $\wp$ function by the first formula in (3.5), and when this expression for d_n is substituted into the second component of (3.4), the explicit formula for v_n is found from taking the square root in the left-hand side of the elliptic function identity:

$$\big(\zeta(a) + \zeta(b) + \zeta(c)\big)^2 = \wp(a) + \wp(b) + \wp(c), \quad \text{for} \quad a + b + c \equiv 0 \bmod \Lambda.$$

Then given the two formulae in (3.5), another identity between elliptic functions verifies that the first component of the system (3.4) is satisfied for all $n \in \mathbb{Z}$, with the parameter $u = \pm\sqrt{\alpha} = \pm\wp'(z)$, making the same choice of sign as for v_n. □

Remark 3.4. *Given the canonical Poisson bracket* $\{ d_n, v_{n-1} \} = 1$ *associated with* ω*, the first integral* H *defines the Hamiltonian vector field* $\{ \cdot, H \}$*, whose flow commutes with the shift* $n \to n + 1$ *corresponding to the map. A direct calculation shows that, for a fixed choice of scale, this flow implies that the sequence of* d_n *and* v_n *satisfy the set of differential equations:*

$$\begin{aligned} \dot{v}_n &= d_n - d_{n+1}, \\ \dot{d}_n &= d_n(v_{n-1} - v_n), \end{aligned} \tag{3.7}$$

for all $n \in \mathbb{Z}$*, which (up to squaring one of the variables) is just the infinite Toda lattice written in Flaschka coordinates. Thus, the simultaneous solutions of the map and the Hamiltonian flow provide genus 1 solutions of the Toda lattice, and for genus* $g > 1$ *an analogous statement holds for the other maps constructed in [12]; further details will be presented elsewhere.*

Corollary 3.5. *Up to subracting off multiples of* $y_n^{(i)} = x_n$ *and* $y_n^{(ii)} = nx_n$ *and overall scale, the third independent shadow solution of Somos-4 can be obtained from the associated solution of the map (3.4), in the form:*

$$y_n^{(iii)} = -x_n \sum_{j=0}^{n-1} v_j, \qquad n \geq 0 \tag{3.8}$$

(where the case $n = 0$ *corresponds to an empty sum).*

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Proof. If we take the plus sign in the formula for v_n in (3.5), then we have a telescopic sum:

$$-x_n \sum_{j=0}^{n-1} v_j = x_n\big(n\zeta(z) + \zeta(z_0) - \zeta(z_0 + nz)\big)$$

for $n \geq 0$, and after subtracting off $\zeta(z_0)y_n^{(i)}$ and $\zeta(z)y_n^{(ii)}$ this is just the third independent solution $y_n^{(iii)}$ in Lemma 3.1, up to an overall sign. □

We now present an observation that dramatically simplifies the solution of the dual Somos-4 equation and allows the order of (1.9) to be reduced from 4 to 3. The point is that, since $\mathbb{D}$ is a commutative algebra, the calculation which shows that (2.9) is a conserved quantity for (1.1) carries over directly to the dual numbers, so the same expression with the replacement $x_n \to X_n$ provides a first integral $J = J^{(0)} + J^{(1)}\varepsilon \in \mathbb{D}$ for (1.7). This implies that the system of even/odd equations has a pair of rational first integrals, namely $J^{(0)}$, which is just the first integral for the even part (1.8) given by the original expression (2.9) in terms of x_n alone, but with parameters $\alpha \to \alpha^{(0)}$, $\beta \to \beta^{(0)}$, and $J^{(1)}$, which depends on both sets of variables x_n and y_n, and is linear in y_n. The most efficient way to calculate $J^{(1)}$ is to clear the denominator in the formula for $J \in \mathbb{D}$, multiplying both sides of the relation by $X_nX_{n+1}X_{n+2}X_{n+3}$, to obtain a polynomial relation, and then the leading order (even) part of the resulting expression just returns the usual formula for $J^{(0)}$, while the $O(\varepsilon)$ (odd) part gives a linear equation in $J^{(1)}$, but also contains $J^{(0)}$. Upon eliminating $J^{(0)}$ from the latter expression, the desired formula for $J^{(1)}$ is obtained.

Lemma 3.6. *In addition to the conserved quantity $J^{(0)}$, given by replacing the parameters $\alpha \to \alpha^{(0)}$, $\beta \to \beta^{(0)}$ in the right-hand side of (2.9), the system consisting of (1.8) and (1.9) has a second rational first integral, namely*

$$J^{(1)} = \frac{D_n - \sum_{j=0}^{3} C_n^{(j)} x_{n+j}^{-1} y_{n+j}}{x_n x_{n+1} x_{n+2} x_{n+3}}, \tag{3.9}$$

where

$$\begin{aligned}
C_n^{(0)} &= \alpha^{(0)} x_{n+1}^3 x_{n+3} + \beta^{(0)} x_{n+1}^2 x_{n+2}^2 - x_n^2 x_{n+3}^2, \\
C_n^{(1)} &= \alpha^{(0)} x_n x_{n+2}^3 - 2\alpha^{(0)} x_{n+1}^3 x_{n+3} - \beta^{(0)} x_{n+1}^2 x_{n+2}^2 + x_n^2 x_{n+3}^2, \\
C_n^{(2)} &= -2\alpha^{(0)} x_n x_{n+2}^3 + \alpha^{(0)} x_{n+1}^3 x_{n+3} - \beta^{(0)} x_{n+1}^2 x_{n+2}^2 + x_n^2 x_{n+3}^2, \\
C_n^{(3)} &= \alpha^{(0)} x_n x_{n+2}^3 + \beta^{(0)} x_{n+1}^2 x_{n+2}^2 - x_n^2 x_{n+3}^2, \\
D_n &= \alpha^{(1)} x_n x_{n+2}^3 + \alpha^{(1)} x_{n+1}^3 x_{n+3} + \beta^{(1)} x_{n+1}^2 x_{n+2}^2.
\end{aligned}$$

For fixed $J^{(1)}$, the equation (3.9) can be rearranged as a linear inhomogeneous difference equation of third order for y_n, that is,

$$L_n(y_n) = F_n, \tag{3.10}$$

where L_n is the linear difference operator:

$$L_n = C_n^{(3)} x_{n+3}^{-1} \mathcal{S}^3 + C_n^{(2)} x_{n+2}^{-1} \mathcal{S}^2 + C_n^{(1)} x_{n+1}^{-1} \mathcal{S} + C_n^{(0)} x_n^{-1}, \tag{3.11}$$

given in terms of the shift operator $\mathcal{S}$ that sends $n \to n+1$, and

$$F_n = D_n - J^{(1)} x_n x_{n+1} x_{n+2} x_{n+3}. \tag{3.12}$$

Thus, we have succeeded in reducing the order of (1.9) by 1, as claimed. The corresponding third-order homogeneous equation which arises when $\alpha^{(1)} = \beta^{(1)} = J^{(1)} = 0$, namely

$$L_n(y_n) = 0, \tag{3.13}$$

is nothing other than the linearization of the equation defining $J^{(0)}$, and three linearly independent solutions are obtained by varying the analytic solution (2.1) with respect to the three parameters that do

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not depend on this modular quantity: in other words, up to rescaling and taking linear combinations, they are given by the first three shadow solutions $y_n^{(i)}, y_n^{(ii)}, y_n^{(iii)}$ in Lemma 3.1. Then, a fourth independent Somos-4 shadow solution is found by taking $\alpha^{(1)} = \beta^{(1)} = 0$ but $J^{(1)} \neq 0$.

The form of the solutions of the homogeneous equation (3.13) is made more transparent by setting $y_n = x_n Y_n$. This gives

$$L_n(y_n) = x_n x_{n+1} x_{n+2} x_{n+3} \tilde{L}_n(Y_n) = 0,$$

where

$$\tilde{L}_n = \left(\left(\frac{\beta^{(0)}}{d_{n+1}d_{n+2}} - d_{n+1}d_{n+2} \right) (\mathcal{S}+1) + \frac{\alpha^{(0)}}{d_{n+2}} \mathcal{S} + \frac{\alpha^{(0)}}{d_{n+1}} \right) (\mathcal{S}-1)^2.$$

So clearly, $Y_n = 1$ and $Y_n = n$ lie in the kernel of the latter operator, corresponding to $y_n^{(i)}$ and $y_n^{(ii)}$.

Finally, we can construct the general solution of (1.9) by applying the discrete analogue of the method of variation of parameters to find an arbitrary element in the affine space of solutions of (3.10). In other words, we can write the solution as the sum of a particular integral plus an arbitrary linear combination of solutions of the homogeneous equation (3.13). For variation of parameters, the initial ansatz is to write the solution of (3.10) in the form:

$$y_n = \sum_j f_n^{(j)} y_n^{(j)}, \tag{3.14}$$

where the index j ranges over the three lower-case Roman numerals i,ii,iii, and then impose the constraints that

$$\sum_j \left(f_{n+1}^{(j)} - f_n^{(j)}\right) y_{n+1}^{(j)} = 0 = \sum_j \left(f_{n+1}^{(j)} - f_n^{(j)}\right) y_{n+2}^{(j)}, \tag{3.15}$$

which together imply that $y_{n+1} = \sum_j f_n^{(j)} y_{n+1}^{(j)}$, $y_{n+2} = \sum_j f_n^{(j)} y_{n+2}^{(j)}$. Putting all this into (3.10) gives

$$L_n(y_n) = C_n^{(3)} x_{n+3}^{-1} \sum_j \left(f_{n+1}^{(j)} - f_n^{(j)}\right) y_{n+3}^{(j)} + \sum_j f_n^{(j)} L_n\left(y_n^{(j)}\right) = C_n^{(3)} x_{n+3}^{-1} \sum_j \left(f_{n+1}^{(j)} - f_n^{(j)}\right) y_{n+3}^{(j)},$$

which must equal F_n. Combining the latter equality with the two constraints (3.15) gives a linear system for the three differences $f_{n+1}^{(j)} - f_n^{(j)}$, which is solved to yield

Theorem 3.7. *The general solution of (3.10) can be written in the form (3.14), where*

$$f_n^{(j)} = f_0^{(j)} + \sum_{k=0}^{n-1} v_k^{(j)}, \qquad j = i, ii, iii, \qquad \text{for } n \geq 0, \tag{3.16}$$

with

$$\begin{pmatrix} v_n^{(i)} \\ v_n^{(ii)} \\ v_n^{(iii)} \end{pmatrix} = \frac{x_{n+3}F_n}{C_n^{(3)}} \begin{vmatrix} y_{n+1}^{(i)} & y_{n+1}^{(ii)} & y_{n+1}^{(iii)} \\ y_{n+2}^{(i)} & y_{n+2}^{(ii)} & y_{n+2}^{(iii)} \\ y_{n+3}^{(i)} & y_{n+3}^{(ii)} & y_{n+3}^{(iii)} \end{vmatrix}^{-1} \begin{pmatrix} y_{n+1}^{(ii)}y_{n+2}^{(iii)} - y_{n+1}^{(iii)}y_{n+2}^{(ii)} \\ y_{n+1}^{(iii)}y_{n+2}^{(i)} - y_{n+1}^{(i)}y_{n+2}^{(iii)} \\ y_{n+1}^{(i)}y_{n+2}^{(ii)} - y_{n+1}^{(ii)}y_{n+2}^{(i)} \end{pmatrix}. \tag{3.17}$$

The three parameters $f_0^{(j)}$ for $j = i, ii, iii$ in (3.16) are arbitrary, and together with the freedom to choose $J^{(1)}$ arbitrarily, this in turn provides the general solution of (1.9).

Example 3.8. *As a particular example of solving (3.10), let us consider the original Somos-4 sequence (1.3) and obtain four independent shadow sequences. For ease of comparison with Example 4.2 in [12], it is convenient to index the original sequence so that $x_{-1} = x_0 = x_1 = x_2 = 1$, and then the first shadow sequence $y_n^{(i)}$ is just given by the same sequence, starting with index $n = -1$, extending to a sequence of positive terms for all $n \in \mathbb{Z}$ (as due to a symmetry it repeats the same values when run in reverse), while a second independent sequence is $y_n^{(ii)} = nx_n$, starting with $y_{-1}^{(ii)} = -1$, $y_0^{(ii)} = 0$, $y_1^{(ii)} = 1$, $y_2^{(ii)} = 2$, which is*

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Table 1. *Four independent shadows of the original Somos-4 sequence (1.3).*

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
$y_n^{(i)}$	1	1	1	1	2	3	7	23	59	314	1529	8209	83,313	620,297
$y_n^{(ii)}$	-1	0	1	2	6	12	35	138	413	2512	13,761	82,090	916,443	7,443,564
$y_n^{(iii)}$	0	0	1	1	3	7	15	70	202	1107	6906	36,386	420,371	3,594,979
$y_n^{(iv)}$	0	0	0	1	1	3	10	22	108	472	2174	17,792	120,536	1,161,627

positive for all $n \geq 1$. As for a third independent sequence, it is found by applying Corollary 3.5, using the system (3.4) with parameters $u = -1$, $f = -3$ to generate a sequence of pairs (v_n, d_{n+1}) starting from $v_0 = -1$, $d_1 = 1$ (and note that we also have $d_0 = 1$ and $H = -J/2 = -2$ in this case). Then, from (3.8), we find $y_0^{(iii)} = 0$, $y_1^{(iii)} = 1$, $y_2^{(iii)} = 1$, and going one step back with the homogeneous equation (3.13) shows that $y_{-1}^{(iii)} = 0$. From the first equation in (3.4) with $u = -1$, it follows that $-(v_n + v_{n-1}) = 1/d_n > 0$ for all n, and so from (3.8) and the initial value $v_0 = -1$ this implies that this third shadow sequence is positive whenever $n \geq 1$. Finally, for a fourth independent shadow sequence, we must solve (3.10) with $\alpha^{(1)} = \beta^{(1)} = 0$ and a nonzero value of $J^{(1)}$, so we fix $J^{(1)} = -1$ and take $y_n^{(iv)} = 0$ for $n = -1, 0, 1$, giving $y_2^{(iv)} = 1$. Empirical evidence suggests the conjecture that this fourth sequence should be positive for all $n \geq 2$, but we do not have a proof. Table 1 presents the first few values in these shadow sequences.

4. Hankel determinant formulae

It was conjectured by Barry and proved by various authors that certain Somos-4 sequences could be expressed as Hankel determinants [1, 2, 29]. In [12], we showed that these results can be unified and further generalized by applying van der Poorten's work on Jacobi continued fractions (J-fractions) in hyperelliptic function fields [26]. In the genus 1 case, one expands a certain function G on a quartic curve as a J-fraction, that is,

$$G = \cfrac{s_0}{X + v_1 - \cfrac{d_2}{X + v_2 - \cfrac{d_3}{X + v_3 - \cdots}}}, \tag{4.1}$$

where X^{-1} is a local parameter around one of the points at infinity, and the recursion relation for the continued fraction leads to the map (3.4) for (v_{n-1}, d_n). The numerators and denominators of the convergents provide associated orthogonal polynomials, and standard results imply that the power series expansion of the generating function near $X = \infty$, that is $G = \sum_{j\geq 1} s_{j-1} X^{-j}$, provides a sequence of moments (s_j) such that d_n and v_n can be written in terms of ratios of the corresponding determinants Δ_n or Δ_n^* of Hankel/bordered Hankel type, respectively. In particular, the solution of (2.4) is given by $d_n = \Delta_{n-2}\Delta_n/\Delta_{n-1}^2$, and in [12] we further showed how this result extends to the solution of an integrable symplectic map associated with the J-fraction expansion of a function on a hyperelliptic curve of any genus $g > 1$.

Since $\mathbb{D}$ is a commutative algebra, identities for Hankel determinants carry over directly to suitable sequences of moments $s_j \in \mathbb{D}$ and allow the solution of (1.7) to be expressed in the same form as for Somos-4 sequences over $\mathbb{C}$. Thus, simply by writing a dual number version of the statement of Theorem 5.2 in [12] (also making use of the formulae in Theorem 4.1 therein for the particular case of genus 1), and taking care that certain parameters should be units, we arrive at the following result.

Theorem 4.1. *Given arbitrary $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, s_0 \in \mathbb{D}^*$ and $s_1 \in \mathbb{D}$, define a sequence of dual numbers $(s_j)_{j\geq 0}$ by the recursion:*

$$s_j = \hat{\alpha} s_{j-2} + \hat{\beta} \sum_{i=0}^{j-2} s_i s_{j-2-i} + \hat{\gamma} \sum_{i=0}^{j-3} s_i s_{j-3-i}, \qquad j \geq 2, \tag{4.2}$$

and form the associated sequence of Hankel determinants:

$$\Delta_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & & & \vdots \\ \vdots & \iddots & & \vdots \\ s_{n-1} & \cdots & \cdots & s_{2n-2} \end{vmatrix} = \det\,(s_{i+j-2})_{i,j=1,\ldots,n} \tag{4.3}$$

for $n \geq 1$*, with the usual convention that* $\Delta_0 = 1$*. Then,*

$$X_n = \Delta_{n-1} \qquad \text{for } n \geq 1$$

is a solution of the dual Somos-4 recurrence (1.7) with coefficients given by:

$$\alpha = U^2, \qquad \beta = \alpha F + \frac{1}{4}J^2,$$

where

$$U = -s_0\hat{\gamma} - s_1\hat{\beta}, \qquad F = -\hat{\alpha} - 2s_0\hat{\beta},$$

and the value of the first integral is fixed to be

$$J = 2\left(s_0\hat{\alpha}\hat{\beta} + s_0^2\hat{\beta}^2 + s_1\hat{\gamma}\right).$$

Remark 4.2. *The Hankel determinant expression above only depends on five dual number parameters, which is one less than is required to produce the general solution of (1.7): there are four initial values and two coefficients in the initial value problem for Somos-4. However, the missing parameter can be recovered with the scaling symmetry* $X_n \to AX_n$ *for* $A \in \mathbb{D}^*$*. The other scaling symmetry* $X_n \to B^n X_n$ *for* $B \in \mathbb{D}^*$ *just rescales the value of* s_0*, which is an overall multiplier in the moment sequence. As discussed in [12], there is another moment sequence which provides Hankel determinant formulae for negative indices n, and in general it is necessary to apply these two scaling symmetries in order to glue the two sequences of Hankel determinants together into a valid Somos-4 sequence for all* $n \in \mathbb{Z}$*.*

Example 4.3. *To obtain Hankel determinant formulae for the fourth shadow sequence, as in Example 3.8, we take dual numbers that are* $O(\varepsilon)$ *perturbations of the corresponding quantities in Example 4.2 of [12]. Setting*

$$\hat{\alpha} = 1 - 4\varepsilon, \quad \hat{\beta} = 1, \quad \hat{\gamma} = 1 - \frac{3}{2}\varepsilon, \quad s_0 = 1 + \varepsilon, \quad s_1 = \frac{1}{2}\varepsilon \tag{4.4}$$

gives $U = -1$*,* $F = -3 + 2\varepsilon$*, and hence*

$$\alpha = 1, \qquad \beta = 1, \qquad J = 4 - \varepsilon,$$

and this corresponds to $\alpha^{(0)} = \beta^{(0)} = 1$*,* $J^{(0)} = 4$ *and* $\alpha^{(1)} = \beta^{(1)} = 0$*,* $J^{(1)} = -1$ *as required. Then upon iterating the recursion (4.2) with parameters and initial values as in (4.4), the sequence* (s_j) *is found to be*

$$1 + \varepsilon, \frac{1}{2}\varepsilon, 2 - \varepsilon, 1 + 2\varepsilon, 6 - 6\varepsilon, 7 + 2\varepsilon, 24 - 28\varepsilon, 41 - 23\varepsilon, 115 - 154\varepsilon, 236 - \frac{527}{2}\varepsilon, \ldots,$$

and the corresponding sequence of Hankel determinants begins with $\Delta_0 = 1$*,* $\Delta_1 = 1 + \varepsilon$*,*

$$\Delta_2 = \begin{vmatrix} 1+\varepsilon & \frac{1}{2}\varepsilon \\ \frac{1}{2}\varepsilon & 2-\varepsilon \end{vmatrix} = 2 + \varepsilon, \quad \Delta_3 = \begin{vmatrix} 1+\varepsilon & \frac{1}{2}\varepsilon & 2-\varepsilon \\ \frac{1}{2}\varepsilon & 2-\varepsilon & 1+2\varepsilon \\ 2-\varepsilon & 1+2\varepsilon & 6-6\varepsilon \end{vmatrix} = 3 + 3\varepsilon,$$

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$$\Delta_4 = \begin{vmatrix} 1+\varepsilon & \frac{1}{2}\varepsilon & 2-\varepsilon & 1+2\varepsilon \\ \frac{1}{2}\varepsilon & 2-\varepsilon & 1+2\varepsilon & 6-6\varepsilon \\ 2-\varepsilon & 1+2\varepsilon & 6-6\varepsilon & 7+2\varepsilon \\ 1+2\varepsilon & 6-6\varepsilon & 7+2\varepsilon & 24-28\varepsilon \end{vmatrix} = 7+10\varepsilon, \ldots,$$

which (after shifting the index) gives the correct values of $X_1, X_2, X_3, X_4, X_5, \ldots$ for the combination of the original Somos-4 sequence (1.3) together with its odd part, namely the shadow sequence $y_n^{(iv)}$ as in the fourth row of Table 1.

Example 4.4. *For the recurrence (1.4), the sequence with initial data given by four 1s with zero odd part was considered in [23], which corresponds to $X_{-1} = X_0 = X_1 = X_2 = 1$ and $\alpha^{(0)} = \beta^{(0)} = \beta^{(1)} = 1$, $\alpha^{(1)} = 0$ with our notation and indexing conventions. In this case, we take*

$$\hat{\alpha} = 1+\varepsilon, \quad \hat{\beta} = 1, \quad \hat{\gamma} = 1+\frac{1}{2}\varepsilon, \quad s_0 = 1, \quad s_1 = -\frac{1}{2}\varepsilon,$$

giving $U = -1$, $F = -3-\varepsilon$, so that

$$\alpha = 1, \qquad \beta = 1+\varepsilon, \qquad J = 4+\varepsilon.$$

Then, the recursion (4.2) yields the sequence (s_j) as:

$$1, -\frac{1}{2}\varepsilon, 2+\varepsilon, 1-\varepsilon, 6+4\varepsilon, 7, 24+18\varepsilon, 41+18\varepsilon, 115+98\varepsilon, 236+\frac{345}{2}\varepsilon, \ldots,$$

and the corresponding sequence of Hankel determinants begins with $\Delta_0 = 1$, $\Delta_1 = 1$,

$$\Delta_2 = \begin{vmatrix} 1 & -\frac{1}{2}\varepsilon \\ -\frac{1}{2}\varepsilon & 2+\varepsilon \end{vmatrix} = 2+\varepsilon, \quad \Delta_3 = \begin{vmatrix} 1 & -\frac{1}{2}\varepsilon & 2+\varepsilon \\ -\frac{1}{2}\varepsilon & 2+\varepsilon & 1-\varepsilon \\ 2+\varepsilon & 1-\varepsilon & 6+4\varepsilon \end{vmatrix} = 3+2\varepsilon,$$

$$\Delta_4 = \begin{vmatrix} 1 & -\frac{1}{2}\varepsilon & 2+\varepsilon & 1-\varepsilon \\ -\frac{1}{2}\varepsilon & 2+\varepsilon & 1-\varepsilon & 6+4\varepsilon \\ 2+\varepsilon & 1-\varepsilon & 6+4\varepsilon & 7 \\ 1-\varepsilon & 6+4\varepsilon & 7 & 24+18\varepsilon \end{vmatrix} = 7+10\varepsilon, \ldots,$$

of which the odd parts for index $n \geq 2$ give the (conjecturally) positive sequence of integers 1, 2, 10, 48, 160, 1273, 7346, 51, 394, 645, 078, ... as found in [23].

Example 4.5. *The analogous sequence for the recurrence (1.5) was also presented in [23], corresponding to $X_{-1} = X_0 = X_1 = X_2 = 1$ and $\alpha^{(0)} = \beta^{(0)} = \alpha^{(1)} = 1$, $\beta^{(1)} = 0$. For this example, we take*

$$\hat{\alpha} = 1+\varepsilon, \quad \hat{\beta} = 1, \quad \hat{\gamma} = 1+\frac{1}{2}\varepsilon, \quad s_0 = 1, \quad s_1 = 0,$$

giving $U = -1-\frac{1}{2}\varepsilon$, $F = -3-\varepsilon$, so that

$$\alpha = 1+\varepsilon, \qquad \beta = 1, \qquad J = 4+2\varepsilon.$$

Thus, from the recursion (4.2), the sequence of dual number moments (s_j) is found to be

$$1, 0, 2+\varepsilon, 1+\frac{1}{2}\varepsilon, 6+5\varepsilon, 7+\frac{13}{2}\varepsilon, 24+27\varepsilon, 41+\frac{105}{2}\varepsilon, 115+164\varepsilon, 236+378\varepsilon, \ldots,$$

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hence, the corresponding sequence of Hankel determinants begins with $\Delta_0 = 1$, $\Delta_1 = 1$,

$$\Delta_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2+\varepsilon \end{vmatrix} = 2+\varepsilon, \quad \Delta_3 = \begin{vmatrix} 1 & 0 & 2+\varepsilon \\ 0 & 2+\varepsilon & 1+\frac{1}{2}\varepsilon \\ 2+\varepsilon & 1+\frac{1}{2}\varepsilon & 6+5\varepsilon \end{vmatrix} = 3+3\varepsilon,$$

$$\Delta_4 = \begin{vmatrix} 1 & 0 & 2+\varepsilon & 1+\frac{1}{2}\varepsilon \\ 0 & 2+\varepsilon & 1+\frac{1}{2}\varepsilon & 6+5\varepsilon \\ 2+\varepsilon & 1+\frac{1}{2}\varepsilon & 6+5\varepsilon & 7+\frac{13}{2}\varepsilon \\ 1+\frac{1}{2}\varepsilon & 6+5\varepsilon & 7+\frac{13}{2}\varepsilon & 24+27\varepsilon \end{vmatrix} = 7+10\varepsilon, \ldots,$$

so the integer sequence beginning with 1, 3, 10, 59, 198, 1387, 9389, 57, 983, 752, 301, ..., *as in [23], is obtained from the odd parts for index* $n \geq 2$, *and this is also conjectured to be positive.*

There is one more use for the s_j, obtained from the moment generating function G in (4.1), that is relevant to shadow sequences, but now it is the classical case of $\mathbb{C}$-valued moments that concerns us. In that setting, the quantities v_n appearing in the continued fraction (4.1) can be written in the form:

$$v_n = \frac{\Delta^*_{n-1}}{\Delta_{n-1}} - \frac{\Delta^*_n}{\Delta_n} \quad \text{for } n \geq 1, \tag{4.5}$$

where $\Delta^*_0 = 0$ and

$$\Delta^*_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-2} & s_n \\ s_1 & & \iddots & \vdots & \vdots \\ \vdots & \iddots & & \vdots & \vdots \\ s_{n-2} & \cdots & \cdots & s_{2n-4} & s_{2n-2} \\ s_{n-1} & \cdots & \cdots & s_{2n-3} & s_{2n-1} \end{vmatrix} \tag{4.6}$$

is a bordered Hankel determinant for $n \geq 1$. Thus, as pointed out in [12], the solution of the system (3.4) is expressed in terms of ratios of these Hankel/bordered Hankel determinants, whenever the moment sequence (s_j) is generated by a recursion of the form (4.2) over $\mathbb{C}$. This yields a bordered Hankel formula for the third shadow Somos-4 sequence.

Proposition 4.6. *Let* $(s_j)_{j\geq 0}$ *be a moment sequence satisfying a recursion (4.2) over* $\mathbb{C}$, *corresponding to a particular Somos-4 sequence* (x_n), *so that (up to rescaling) it is given in terms of the associated Hankel determinants by* $x_n = \Delta_{n-1}$ *for* $n \geq 1$. *Then, up to removing multiples of* $y^{(i)}_n = x_n$, *the third independent shadow Somos-4 sequence is given in terms of the corresponding bordered Hankel determinants (4.6) by* $y^{(iii)}_n = \Delta^*_{n-1}$ *for* $n \geq 1$.

Proof. Substitution of (4.5) into the formula (3.8) gives a telescopic sum, which simplifies to

$$y^{(iii)}_n = -x_n \left(v_0 - \frac{\Delta^*_{n-1}}{\Delta_{n-1}} \right),$$

and then using $x_n = \Delta_{n-1}$ this gives the required result after removing the multiple $-v_0 x_n$ from the front. □

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5. Conclusions

We have given a very detailed decription of the solution of the initial value problem for the general dual Somos-4 recurrence (1.7), from several different perspectives. The fact that this is possible relies heavily on the commutativity of the algebra of dual numbers but is also a reflection of the fact that there is an underlying discrete integrable system: either the QRT map (2.4), which preserves the log-canonical symplectic form $d \log d_{n-1} \wedge d \log d_n$, or the map (3.4). (The latter map preserves a different symplectic structure, but the orbits of the two maps can be identified via a correspondence between their parameters and initial data.) We have not explicitly described the analogs of these maps and their solutions over $\mathbb{D}$, but such a description is a straightforward consequence of our results on the dual Somos-4.

Continuous integrable systems with Grassmann variables have been studied for several decades, with one of the most recent results being a symmetry classification of $N = 1$ supersymmetric scalar homogeneous evolutionary PDEs [25]. Many of our considerations here extend naturally to the dual number analogs of other discrete integrable systems, such as the family of maps in [13], which are connected with Gale-Robinson sequences and cluster algebras, or the higher genus analogs of (3.4) in [12]. There are already versions of Yang–Baxter maps that include Grassmann variables, together with associated integrable lattice equations that satisfy the multidimensional consistency property [16]. Hopefully, some of these techniques could also shed more light on superfriezes and cluster superalgebras, as in [17, 22], which are relevant to Ptolemy relations in super-Teichmüller theory [18].

Finally, it would potentially be interesting to develop the analytic theory of elliptic functions over the dual numbers, since the ubiquity of the derivatives $\partial_{g_2}, \partial_{g_3}$ suggests that modular identities might appear very naturally in this setting.

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