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APPROXIMATION BY MULTIPLE REFINABLE FUNCTIONS

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ABSTRACT. We consider the shift-invariant space, $\mathbb{S}(\Phi)$, generated by a set $\Phi = \{\phi_1, \ldots, \phi_r\}$ of compactly supported distributions on $\mathbb{R}$ when the vector of distributions $\phi := (\phi_1, \ldots, \phi_r)^T$ satisfies a system of refinement equations expressed in matrix form as

$$\phi = \sum_{\alpha \in \mathbb{Z}} a(\alpha)\phi(2\,\cdot - \alpha)$$

where a is a finitely supported sequence of $r \times r$ matrices of complex numbers. Such *multiple refinable functions* occur naturally in the study of multiple wavelets.

The purpose of the present paper is to characterize the *accuracy* of Φ, the order of the polynomial space contained in $\mathbb{S}(\Phi)$, strictly in terms of the refinement mask a. The accuracy determines the L_p-approximation order of $\mathbb{S}(\Phi)$ when the functions in Φ belong to $L_p(\mathbb{R})$ (see Jia [10]). The characterization is achieved in terms of the eigenvalues and eigenvectors of the subdivision operator associated with the mask a. In particular, they extend and improve the results of Heil, Strang and Strela [7], and of Plonka [16]. In addition, a counterexample is given to the statement of Strang and Strela [20] that the eigenvalues of the subdivision operator determine the accuracy. The results do not require the linear independence of the shifts of ϕ.

1. **Introduction.** In this paper we investigate approximation by integer translates of multiple refinable functions. Multiple functions $\phi_1, \ldots, \phi_r$ on $\mathbb{R}$ are said to be refinable if they are linear combinations of the rescaled and translated functions $\phi_j(2\,\cdot - \alpha)$, $j = 1, \ldots, r$ and $\alpha \in \mathbb{Z}$. The coefficients in the linear combinations determine the refinement mask. It is desirable to characterize the approximation order provided by the multiple refinable functions in terms of the refinement mask. This study is important for our understanding of multiple wavelets.

Our study of multiple refinable functions is based on shift-invariant spaces. Let S be a linear space of distributions on $\mathbb{R}$. If $f \in S$ implies $f(\cdot - \alpha) \in S$ for all $\alpha \in \mathbb{Z}$, then S is said to be invariant under integer translates, or simply, S is *shift-invariant*.

Let ϕ be a compactly supported distribution on $\mathbb{R}$, and let $b\colon \mathbb{Z} \longrightarrow \mathbb{C}$ be a sequence. The *semi-convolution* of ϕ with b, denoted $\phi *' b$, is defined by

$$\phi *' b := \sum_{\alpha \in \mathbb{Z}} \phi(\cdot - \alpha) b(\alpha).$$

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Given a finite collection $\Phi = \{\phi_1, \ldots, \phi_r\}$ of compactly supported distributions on $\mathbb{R}$, we denote by $\mathbb{S}(\Phi)$ the linear space of all distributions of the form $\sum_{j=1}^{r} \phi_j *' b_j$, where $b_1, \ldots, b_r$ are sequences on $\mathbb{Z}$. Clearly, $\mathbb{S}(\Phi)$ is shift-invariant.

The linear space of all sequences from $\mathbb{Z}$ to $\mathbb{C}$ is denoted by $\ell(\mathbb{Z})$. The *support* of a sequence b on $\mathbb{Z}$ is defined by

$$\operatorname{supp} b := \{\alpha \in \mathbb{Z} : b(\alpha) \neq 0\}.$$

The sequence b is said to be *finitely supported* if $\operatorname{supp} b$ is a finite set. The *symbol* of b is the Laurent polynomial

$$\tilde{b}(z) := \sum_{\alpha \in \mathbb{Z}} b(\alpha) z^{\alpha}, \quad z \in \mathbb{C} \setminus \{0\}.$$

For an integer $k \geq 0$, Π_k will denote the set of all polynomials of degree at most k. We also agree that $\Pi_{-1} = \{0\}$. An element u of $\ell(\mathbb{Z})$ is called a *polynomial sequence* if there exists a polynomial p such that $u(\alpha) = p(\alpha)$ for all $\alpha \in \mathbb{Z}$. Such p is uniquely determined by u. The *degree* of u is the same as the degree of p.

For a positive integer r, $\mathbb{C}^r$ denotes the linear space of $r \times 1$ vectors of complex numbers. By $\ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ we denote the linear space of all sequences of $r \times 1$ vectors. As usual, the transpose of a matrix A will be denoted by A^T.

The Fourier transform of an integrable function f on $\mathbb{R}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx, \quad \xi \in \mathbb{R}.$$

The Fourier transform has a natural extension to compactly supported distributions.

We will consider approximation in the space $L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$ with the p-norm of a function f in $L_p(\mathbb{R})$ denoted by $\|f\|_p$. The distance between two functions $f, g \in L_p(\mathbb{R})$ is $\operatorname{dist}_p(f, g) := \|f - g\|_p$, while the distance from f to a subset G of $L_p(\mathbb{R})$ is

$$\operatorname{dist}_p(f, G) := \inf_{g \in G} \|f - g\|_p.$$

For any p, $1 \leq p \leq \infty$, and a finite collection, Φ, of compactly supported functions in $L_p(\mathbb{R})$, set $S := \mathbb{S}(\Phi) \cap L_p(\mathbb{R})$, and $S^h := \{g(\cdot/h) : g \in S\}$ for $h > 0$. Given a real number $s \geq 0$, $\mathbb{S}(\Phi)$ is said to provide *L_p-approximation order s* if, for each sufficiently smooth function $f \in L_p(\mathbb{R})$,

$$\operatorname{dist}_p(f, S^h) \leq C h^s,$$

where C is a positive constant independent of h (C may depend on f).

In [8] Jia characterized the L_∞-approximation order provided by $\mathbb{S}(\Phi)$ as follows: *$\mathbb{S}(\Phi)$ provides L_∞-approximation order k if and only if there exists a compactly supported function $\psi \in \mathbb{S}(\Phi)$ such that*

$$\sum_{\alpha \in \mathbb{Z}} \psi(\cdot - \alpha) q(\alpha) = q \quad \forall q \in \Pi_{k-1}. \tag{1.1}$$

It follows from the Poisson summation formula that (1.1) is equivalent to the following conditions:

$$\hat{\psi}^{(j)}(2\pi\beta) = \delta_{j0}\delta_{\beta 0} \quad \text{for } j = 0, 1, \ldots, k-1 \text{ and } \beta \in \mathbb{Z}, \tag{1.2}$$

where $\hat{\psi}^{(j)}$ denotes the j-th derivative of the Fourier transform of ψ. This equivalence was observed by Schoenberg in his celebrated paper [18]. The conditions in (1.2) are now referred to as the Strang-Fix conditions (see [19]). When Φ consists of a single generator ϕ, Ron [17] proved that $\mathbb{S}(\phi)$ provides L_∞-approximation order k if and only if $\mathbb{S}(\phi)$ contains Π_{k-1}. In [10] Jia proved that, for $1 \le p \le \infty$, $\mathbb{S}(\Phi)$ provides L_p-approximation order k if and only if $\mathbb{S}(\Phi)$ contains Π_{k-1}. We caution the reader that this result is no longer true for shift-invariant spaces on $\mathbb{R}^d$, $d > 1$. See the counterexamples given in [2] and [3].

Following [7], we say that Φ has *accuracy* k if $\Pi_{k-1} \subseteq \mathbb{S}(\Phi)$. Thus, $\mathbb{S}(\Phi)$ provides L_p-approximation order k for any p, $1 \le p \le \infty$, if and only if Φ is a subset of $L_p(\mathbb{R})$ and has accuracy k. However, the concept of accuracy does not require the members in Φ to belong to any $L_p(\mathbb{R})$.

Thus, from now on we allow Φ to be a finite collection of compactly supported distributions $\phi_1, \ldots, \phi_r$ on $\mathbb{R}$. For simplicity, we write ϕ for the (column) vector $(\phi_1, \ldots, \phi_r)^T$, and write $\mathbb{S}(\phi)$ for $\mathbb{S}(\{\phi_1, \ldots, \phi_r\})$. We say that ϕ has accuracy k if $\{\phi_1, \ldots, \phi_r\}$ does.

Let $K(\phi)$ be the linear space defined by

$$K(\phi) := \Big\{ b \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r) : \textstyle\sum_{\alpha\in\mathbb{Z}} b(\alpha)^T \phi(\cdot - \alpha) = 0 \Big\}. \tag{1.3}$$

Since $K(\phi)$ clearly represents linear dependency relations among the shifts of $\phi_1, \ldots, \phi_r$, we say that the shifts of $\phi_1, \ldots, \phi_r$ are *linearly independent* when $K(\phi) = \{0\}$.

Now assume that $\phi = (\phi_1, \ldots, \phi_r)^T$ satisfies the following refinement equation:

$$\phi = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2 \cdot -\alpha), \tag{1.4}$$

where each $a(\alpha)$ is an $r \times r$ matrix of complex numbers and $a(\alpha) = 0$ except for finitely many α. Thus, a is a finitely supported sequence of $r \times r$ matrices. We call a the *refinement mask*.

The *subdivision operator* S_a associated with a is the linear operator on $\ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ defined by

$$S_a u(\alpha) := \sum_{\beta\in\mathbb{Z}} a(\alpha - 2\beta)^T u(\beta), \quad \alpha \in \mathbb{Z},$$

where $u \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$. For the scalar case ($r = 1$), the subdivision operator was studied by Cavaretta, Dahmen, and Micchelli in [4].

When $\phi_1, \ldots, \phi_r$ are integrable functions on $\mathbb{R}$ and the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent, Strang and Strela [20] proved: *ϕ has accuracy k implies that the subdivision operator S_a has eigenvalues* $1, 1/2, \ldots, 1/2^{k-1}$. In [16] Plonka obtained a similar result. However, they did not give any criterion to check linear independence of the shifts of $\phi_1, \ldots, \phi_r$ in terms of the refinement mask.

In Section 2 we will establish the following theorem without any condition imposed on the finitely supported mask.

THEOREM 1.1. *Suppose ϕ is a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation in (1.4) with mask a. If ϕ has accuracy k, then $1, 1/2, \ldots, (1/2)^{k-1}$ are eigenvalues of the subdivision operator S_a. Moreover, if a is supported in $[N_1, N_2]$, where N_1 and N_2 are integers, then a nonzero complex number σ is an eigenvalue of S_a if and only if σ is an eigenvalue of the block matrix*

$$\left(a(-\alpha + 2\beta)^T\right)_{N_1 \le \alpha, \beta \le N_2}.$$

In [7], Heil, Strang, and Strela raised the question about whether the existence of the eigenvalues $1, 1/2, \ldots, 1/2^{k-1}$ for S_a is sufficient to ensure that ϕ has accuracy k. They conjectured that this would be true for the scalar case ($r = 1$) if the shifts of ϕ are linearly independent. The following counterexample, which will be verified in Section 2, gives a negative answer to their conjecture and disproves the statement of Strang and Strela [20] that the eigenvalues of the subdivision operator determine the accuracy.

COUNTEREXAMPLE. Let a be the sequence on $\mathbb{Z}$ given by

$$a(0) = 1/2,\ a(1) = 3/4,\ a(2) = 1/2,\ a(3) = 1/4, \text{ and}$$
$$a(\alpha) = 0 \text{ for } \alpha \in \mathbb{Z} \setminus \{0, 1, 2, 3\}.$$

Then the subdivision operator S_a has eigenvalues $1, 1/2, 1/4$. Let ϕ be the solution of the refinement equation $\phi = \sum_{\alpha \in \mathbb{Z}} a(\alpha)\phi(2 \cdot -\alpha)$ subject to $\hat{\phi}(0) = 1$. Then ϕ is a compactly supported continuous function with linearly independent shifts. But ϕ does not have accuracy 2.

This example shows that the mere existence of the eigenvalues $1, 1/2, \ldots, 1/2^{k-1}$ for S_a does not guarantee that ϕ has accuracy k. In order to characterize the accuracy of ϕ in terms of the subdivision operator S_a, we will need to know information about the corresponding eigenvectors of S_a.

In Section 3 we will prove the following theorem.

THEOREM 1.2. *Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation (1.4) with mask a. Then ϕ has accuracy k, provided that there exist polynomial sequences $u_1, \ldots, u_r$ of degree at most $k - 1$ satisfying the following two conditions:*

(a) $S_a u = (1/2)^{k-1} u$, where u is given by $u(\alpha) = \left(u_1(\alpha), \ldots, u_r(\alpha)\right)^T$, $\alpha \in \mathbb{Z}$, and

(b) $\sum_{j=1}^r \hat{\phi}_j(0) u_j$ has degree $k - 1$.

Under the conditions on linear independence or stability of the shifts of the functions $\phi_1, \ldots, \phi_r$, Heil, Strang, and Strela in [7], Plonka in [16], and Lian in [14] gave methods to check the accuracy of ϕ. In contrast to their methods, Theorem 1.2 only requires information about eigenvectors of the subdivision operator S_a corresponding to *one* eigenvalue.

Theorem 1.2 provides a lower bound for the accuracy of a vector of multiple refinable functions. In some cases, however, it fails to give the optimal accuracy. For example, let

ϕ be the characteristic function of the interval $[0, 2)$. Then ϕ satisfies the refinement equation

$$\phi(x) = \phi(2x) + \phi(2x - 2), \quad x \in \mathbb{R},$$

with the mask a given by $a(0) = a(2) = 1$ and $a(\alpha) = 0$ for all $\alpha \in \mathbb{Z} \setminus \{0, 2\}$. Let u be a sequence on $\mathbb{Z}$. Then the subdivision operator S_a has the property that $S_a u(2j+1) = 0$ for all $j \in \mathbb{Z}$. If u is a polynomial sequence such that $S_a u = \sigma u$ for some nonzero complex number σ. Then u vanishes at every odd integer; hence u is identically 0. This shows that there is no polynomial sequence that is an eigenvector of S_a corresponding to a nonzero eigenvalue. But ϕ has accuracy 1. Theorem 1.2 fails to give the optimal accuracy of ϕ, so do the methods discussed in [7], [14], and [16].

To fill this gap, we will establish in Section 4 the following result which gives a characterization for the accuracy of a vector of multiple refinable functions in terms of the refinement mask.

THEOREM 1.3. *Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation (1.4). Then ϕ has accuracy k if and only if there exist polynomial sequences $u_1, \ldots, u_r$ on $\mathbb{Z}$ such that the element $u \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ given by $u(\alpha) = \big(u_1(\alpha), \ldots, u_r(\alpha)\big)^T$, $\alpha \in \mathbb{Z}$, satisfies*

$$u \notin K(\phi) \quad \text{and} \quad S_a u - (1/2)^{k-1} u \in K(\phi).$$

Our theory will be applied to an analysis of the accuracy of a class of double refinable functions. Suppose $\phi = (\phi_1, \phi_2)^T$ satisfies the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}} a(\alpha) \phi(2 \cdot - \alpha),$$

where the mask is supported on $[0, 2]$. If we require that ϕ_1 be symmetric about $x = 1$ and ϕ_2 anti-symmetric about $x = 1$, then it is natural (see [12]) to assume that the mask a has the following form: $a(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus \{0, 1, 2\}$ and

$$a(0) = \begin{bmatrix} 1/2 & s/2 \\ t & \lambda \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad a(2) = \begin{bmatrix} 1/2 & -s/2 \\ -t & \lambda \end{bmatrix},$$

where s, t, λ, μ are real numbers. If $|2\lambda + \mu| < 2$, then by a result of Heil and Colella [6], the above refinement equation has a unique distributional solution $\phi = (\phi_1, \phi_2)^T$ subject to the condition $\hat{\phi}_1(0) = 1$ and $\hat{\phi}_2(0) = 0$. Such a solution is said to be the normalized solution. In Sections 3 and 4, we will give a detailed analysis of the accuracy of ϕ. In particular, we will show that ϕ has accuracy 3 if and only if $t \neq 0$, $\mu = 1/2$, and $\lambda = 1/4 + 2st$. Furthermore, ϕ has accuracy 4 if and only if $\lambda = -1/8$, $\mu = 1/2$, and $st = -3/16$.

2. **The Eigenvalue Condition.** In this section we show that if a vector of multiple refinable functions has accuracy k, then the corresponding subdivision operator has eigenvalues $1, 1/2, \ldots, (1/2)^{k-1}$.

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$. Suppose ϕ satisfies the refinement equation (1.4) with the mask a being a finitely supported sequence of $r \times r$ matrices. Let $K(\phi)$ be the linear space defined in (1.3).

Let S_a be the subdivision operator associated with a. For $b \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$, we have

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)^T \phi(\cdot - \alpha) = \sum_{\alpha\in\mathbb{Z}} \big(S_a b(\alpha)\big)^T \phi(2\cdot - \alpha). \tag{2.1}$$

Indeed, since ϕ satisfies the refinement equation (1.4), we have

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)^T \phi(\cdot - \alpha) = \sum_{\alpha\in\mathbb{Z}} b(\alpha)^T \sum_{\beta\in\mathbb{Z}} a(\beta)\phi(2\cdot - 2\alpha - \beta) = \sum_{\gamma\in\mathbb{Z}} c(\gamma)^T \phi(2\cdot - \gamma),$$

where

$$c(\gamma) = \sum_{\alpha\in\mathbb{Z}} a(\gamma - 2\alpha)^T b(\alpha), \quad \gamma \in \mathbb{Z}.$$

Hence $c = S_a b$. This verifies (2.1). It follows that $K(\phi)$ is invariant under S_a.

THEOREM 2.1. *Suppose ϕ is a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation in (1.4) with mask a. If ϕ has accuracy k, then $1, 1/2, \ldots, (1/2)^{k-1}$ are eigenvalues of the subdivision operator S_a. Moreover, if a is supported in $[N_1, N_2]$, where N_1 and N_2 are integers, then a nonzero complex number σ is an eigenvalue of S_a if and only if σ is an eigenvalue of the block matrix*

$$A_{[N_1,N_2]} := \big(a(-\alpha + 2\beta)^T\big)_{N_1\le\alpha,\beta\le N_2}.$$

PROOF. Let us prove the second statement first. For $u \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$, we have

$$S_a u(-\alpha) = \sum_{\beta\in\mathbb{Z}} a(-\alpha - 2\beta)^T u(\beta) = \sum_{\beta\in\mathbb{Z}} a(-\alpha + 2\beta)^T u(-\beta), \quad \alpha \in \mathbb{Z}. \tag{2.2}$$

For $\alpha \in [N_1, N_2]$ and $\beta \in \mathbb{Z} \setminus [N_1, N_2]$, we have $-\alpha + 2\beta \in \mathbb{Z} \setminus [N_1, N_2]$, for otherwise one would have $\beta = (\alpha - \alpha + 2\beta)/2 \in [N_1, N_2]$. Thus, for $\alpha \in [N_1, N_2]$, $a(-\alpha + 2\beta) \neq 0$ only if $\beta \in [N_1, N_2]$. Hence

$$S_a u(-\alpha) = \sum_{\beta=N_1}^{N_2} a(-\alpha + 2\beta)^T u(-\beta), \quad N_1 \le \alpha \le N_2. \tag{2.3}$$

Let P be the linear mapping defined by

$$Pu := \big[u(-N_1), u(-N_1 - 1), \ldots, u(-N_2)\big]^T, \quad u \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r).$$

It follows from (2.3) that

$$PS_a = A_{[N_1,N_2]} P. \tag{2.4}$$

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Suppose $\sigma \neq 0$ is an eigenvalue of the subdivision operator S_a. Then there exists a nonzero element $u \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ such that $S_a u = \sigma u$. It follows that $PS_a u = \sigma Pu$. This in connection with (2.4) gives

$$A_{[N_1,N_2]}(Pu) = \sigma(Pu).$$

But $Pu \neq 0$, for otherwise u would be 0 by (2.2). This shows that σ is an eigenvalue of the matrix $A_{[N_1,N_2]}$.

Conversely, suppose $[v(N_1), v(N_1+1), \ldots, v(N_2)]^T$ is an eigenvector of $A_{[N_1,N_2]}$ corresponding to an eigenvalue $\sigma \neq 0$. For $\alpha > N_2$, let $v(\alpha)$ be determined recursively by

$$v(\alpha) := \frac{1}{\sigma} \sum_{\beta=N_1}^{\alpha-1} a(-\alpha + 2\beta)^T v(\beta),$$

and, for $\alpha < N_1$, let

$$v(\alpha) := \frac{1}{\sigma} \sum_{\beta=\alpha+1}^{N_2} a(-\alpha + 2\beta)^T v(\beta).$$

Let u be the element in $\ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ given by $u(\alpha) = v(-\alpha)$, $\alpha \in \mathbb{Z}$. Then u is an eigenvector of the subdivision operator S_a corresponding to the eigenvalue σ.

Now suppose ϕ is a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation in (1.4) with mask a. If ϕ has accuracy k, then $\mathbb{S}(\phi)$ contains the monomials $1, x, \ldots, x^{k-1}$. Let p be the monomial $x \longmapsto x^j$, where $j \in \{0, 1, \ldots, k-1\}$. Then there exists a nonzero vector b in $\ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ such that

$$p = \sum_{\alpha\in\mathbb{Z}} b(\alpha)^T \phi(\cdot - \alpha). \tag{2.5}$$

By (2.1), it follows that

$$p(\cdot/2) = \sum_{\alpha\in\mathbb{Z}} \big(S_a b(\alpha)\big)^T \phi(\cdot - \alpha).$$

Note that $p(x/2) = (1/2)^j p(x)$, $x \in \mathbb{R}$. We deduce from the above two equations that

$$\sum_{\alpha\in\mathbb{Z}} \Big[S_a b(\alpha) - (1/2)^j b(\alpha)\Big]^T \phi(\cdot - \alpha) = 0.$$

Consequently,

$$S_a b - (1/2)^j\, b \in K(\phi). \tag{2.6}$$

Applying the linear operator P to $S_a b - (1/2)^j\, b$ and taking (2.4) into account, we obtain

$$A_{[N_1,N_2]}(Pb) - (1/2)^j (Pb) \in P\left(K(\phi)\right). \tag{2.7}$$

We claim $Pb \notin P\big(K(\phi)\big)$. Indeed, if $Pb \in P\big(K(\phi)\big)$, then there exists some $c \in K(\phi)$ such that $Pb = Pc$, *i.e.*, $b(-\alpha) = c(-\alpha)$ for $N_1 \leq \alpha \leq N_2$. Since ϕ is supported in $[N_1, N_2]$, $\phi(\cdot + \alpha)$ vanishes on $(-1, 1)$ for $\alpha < N_1$ or $\alpha > N_2$. Consequently,

$$\begin{aligned} &\sum_{\alpha\in\mathbb{Z}} b(\alpha)^T \phi(\cdot - \alpha)|_{(-1,1)} = \sum_{\alpha\in\mathbb{Z}} b(-\alpha)^T \phi(\cdot + \alpha)|_{(-1,1)} \\ &= \sum_{\alpha\in\mathbb{Z}} c(-\alpha)^T \phi(\cdot + \alpha)|_{(-1,1)} = \sum_{\alpha\in\mathbb{Z}} c(\alpha)^T \phi(\cdot - \alpha)|_{(-1,1)} = 0, \end{aligned}$$

which contradicts (2.5). This verifies our claim that $Pb \notin P\big(K(\phi)\big)$. Thus, (2.7) tells us that $(1/2)^j$ is an eigenvalue of $A_{[N_1,N_2]}$. This is true for $j = 0, 1, \ldots, k-1$. Therefore, we conclude that S_a has eigenvalues $1, 1/2, \ldots, (1/2)^{k-1}$, provided ϕ has accuracy k. ∎

The following example demonstrates that the mere existence of the eigenvalues $1, 1/2, \ldots, 1/2^{k-1}$ of the corresponding subdivision operator is not sufficient to ensure that ϕ has accuracy k even when the shifts of ϕ are linearly independent.

EXAMPLE 2.2. Let a be the sequence on $\mathbb{Z}$ given by

$$a(0) = 1/2,\ a(1) = 3/4,\ a(2) = 1/2,\ a(3) = 1/4, \text{ and}$$
$$a(\alpha) = 0 \text{ for } \alpha \in \mathbb{Z} \setminus \{0, 1, 2, 3\}.$$

Then the subdivision operator S_a has eigenvalues $1, 1/2, 1/4$. Let ϕ be the normalized solution of the refinement equation $\phi = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2\cdot -\alpha)$. Then ϕ is a compactly supported continuous function with linearly independent shifts. But ϕ does not have accuracy 2.

PROOF. First, ϕ is a compactly supported continuous function. This can be proved by using the results in [15] and [5]. The reader is also referred to [9, Theorems 3.3 and 4.1]. Indeed, we observe that the symbol of a can be factorized as $\tilde{a}(z) = (1+z)\tilde{b}(z)$, where

$$\tilde{b}(z) := (2 + z + z^2)/4.$$

Thus, $b(0) = 1/2, b(1) = 1/4, b(2) = 1/4$, and $b(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus [0, 2]$. We have

$$B_0 := \big(b(2j-1-k)\big)_{1\le j,k\le 2} = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/4 \end{bmatrix}$$

and

$$B_1 := \big(b(2j-k)\big)_{1\le j,k\le 2} = \begin{bmatrix} 1/4 & 1/2 \\ 0 & 1/4 \end{bmatrix}.$$

The maximum row sum norms of B_0 and B_1 are less than 1. Therefore, the uniform joint spectral radius of B_0 and B_1 is less than 1, and ϕ is continuous.

Second, the shifts of ϕ are linearly independent. Indeed, $\tilde{a}(z)$ does not have symmetric zeros. Moreover, if $m > 1$ is an odd integer and ω is an mth root of unity, then $\tilde{a}(\omega) \neq 0$. Therefore, by [13, Theorem 2], the shifts of ϕ are linearly independent.

Third, since $\tilde{a}(z)$ is divisible by $1+z$ but not by $(1+z)^2$, and since the shifts of ϕ are linearly independent, we conclude that $\Pi_0 \subset \mathbb{S}(\phi)$ but $\Pi_1 \not\subset \mathbb{S}(\phi)$ (see [4] and [5]).

Finally, by Theorem 2.1, S_a and the matrix $A_{[0,3]} := \big(a(-\alpha+2\beta)\big)_{0\le\alpha,\beta\le 3}$ have the same nonzero eigenvalues. But the eigenvalues of the matrix

$$A_{[0,3]} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix}$$

are $1, 1/2, 1/4$, and $1/4$. Hence the subdivision operator S_a has eigenvalues $1, 1/2, 1/4$.

To summarize, all the statements in Example 2.2 have been verified. ∎

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3. **The Eigenvector Condition.** In this section we give a method to test the accuracy of a vector of multiple refinable functions in terms of eigenvectors of the corresponding subdivision operator.

For a function f on $\mathbb{R}$, we use Df to denote its derivative. For $h > 0$, $\nabla_h f$ is defined by $\nabla_h f := f - f(\cdot - h)$. In particular, we write ∇ for ∇_1. For a sequence b on $\mathbb{Z}$, we define $\nabla b := b - b(\cdot - 1)$. For $n = 2, 3, \ldots$, define $\nabla^n := \nabla(\nabla^{n-1})$.

THEOREM 3.1. *Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation (1.4) with mask a. Then ϕ has accuracy k, provided that there exist polynomial sequences $u_1, \ldots, u_r$ of degree at most $k-1$ satisfying the following two conditions:*

(a) $S_a u = (1/2)^{k-1} u$, where u is given by $u(\alpha) = \big(u_1(\alpha), \ldots, u_r(\alpha)\big)^T$, $\alpha \in \mathbb{Z}$, and

(b) $\sum_{j=1}^r \hat{\phi}_j(0) u_j$ has degree $k-1$.

If this is the case, then

$$cx^{k-1} = \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}} u_j(\alpha)\phi_j(x-\alpha), \quad x \in \mathbb{R}, \tag{3.1}$$

where $c = \sum_{j=1}^r \hat{\phi}_j(0)\nabla^{k-1} u_j(0) / (k-1)! \neq 0$.

PROOF. Suppose $u_1, \ldots, u_r$ are polynomial sequences of degree at most $k-1$ satisfying conditions (a) and (b). Set

$$p := \sum_{\alpha\in\mathbb{Z}} u(\alpha)^T \phi(\cdot - \alpha) = \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}} u_j(\alpha)\phi_j(\cdot - \alpha), \tag{3.2}$$

where $u \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ is given by $u(\alpha) = \big(u_1(\alpha), \ldots, u_r(\alpha)\big)^T$, $\alpha \in \mathbb{Z}$. Since ϕ satisfies the refinement equation (1.4), by (2.1) we have

$$p = \sum_{\alpha\in\mathbb{Z}} (S_a u(\alpha))^T \phi(2\cdot - \alpha) = (1/2)^{k-1} \sum_{\alpha\in\mathbb{Z}} u(\alpha)^T \phi(2\cdot - \alpha) = (1/2)^{k-1} p(2\,\cdot).$$

An induction argument gives

$$p = (1/2)^{n(k-1)} \sum_{\alpha\in\mathbb{Z}} u(\alpha)^T \phi(2^n \cdot - \alpha), \quad n = 1, 2, \ldots.$$

Let m be an integer greater than $k-1$. Since $u_1, \ldots, u_r$ are polynomial sequences of degree at most $k-1$, we have $\nabla^m u_j = 0$ for $j = 1, \ldots, r$. It follows that

$$\nabla^m_{1/2^n} p = (1/2)^{n(k-1)} \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}} \nabla^m u_j(\alpha)\phi_j(2^n \cdot - \alpha) = 0, \quad n = 1, 2, \ldots.$$

Thus, we have proved that $p = (1/2)^{k-1} p(2\,\cdot)$ and $\nabla^m_{1/2^n} p = 0$ for all positive integers n. We shall derive from these two facts that $p(x) = c\,x^{k-1}$ for some constant c.

For this purpose, we consider the convolution of p with a function $\psi \in C_c^\infty(\mathbb{R})$. Since p is a distribution on $\mathbb{R}$, we have $f := p*\psi \in C^\infty(\mathbb{R})$ (see [1, p. 97]). Moreover,

$$\nabla^m_{1/2^n} f = \big(\nabla^m_{1/2^n} p\big)*\psi = 0, \quad n = 1, 2, \ldots.$$

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Consequently,

$$D^m f = \lim_{n\to\infty} (2^n)^m \nabla^m_{1/2^n} f = 0.$$

It follows that $(D^m p)*\psi = 0$ for all $\psi \in C_c^\infty(\mathbb{R})$. Choose $\psi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)\,dx = 1$. For $n = 1, 2, \ldots$, let $\psi_n := \psi(\cdot/n)/n$. Then $(D^m p)*\psi_n$ converges to $D^m p$ as $n \longrightarrow \infty$ in the following sense:

$$\lim_{n\to\infty} \langle (D^m p)*\psi_n, g\rangle = \langle D^m p, g\rangle \quad \forall\, g \in C_c^\infty(\mathbb{R}).$$

But $(D^m p)*\psi_n = 0$ for $n = 1, 2, \ldots$. Therefore $D^m p = 0$, and so p is a polynomial of degree less than m (see [1, p. 68]). Suppose $p(x) = c_0 + c_1 x + \cdots + c_j x^j$ for $x \in \mathbb{R}$ with $c_j \neq 0$. Then we deduce from $p = (1/2)^{k-1} p(2\,\cdot)$ that

$$c_0 + c_1 x + \cdots + c_j x^j = \left(c_0 + 2c_1 x + \cdots + 2^j c_j x^j\right)/2^{k-1}, \quad x \in \mathbb{R}.$$

This happens only if $j = k-1$ and $c_0 = c_1 = \cdots = c_{j-1} = 0$. Therefore, $p(x) = c\,x^{k-1}$ for some constant c. This in connection with (3.2) yields (3.1).

It remains to determine c. We observe that, for each j, $\nabla^{k-1} u_j$ is a constant sequence. Let $\lambda_j := \nabla^{k-1} u_j(0)/(k-1)!$, $j = 1, \ldots, r$. It follows from (3.1) that

$$c = \nabla^{k-1} p/(k-1)! = \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}} \phi_j(\cdot - \alpha) \nabla^{k-1} u_j(\alpha)/(k-1)! = \sum_{\alpha\in\mathbb{Z}} \left(\sum_{j=1}^r \lambda_j \phi_j\right)(\cdot - \alpha).$$

By the Poisson summation formula we obtain

$$c = \left(\sum_{j=1}^r \lambda_j \phi_j\right)\hat{}(0) = \sum_{j=1}^r \hat\phi_j(0) \nabla^{k-1} u_j(0)/(k-1)! = \nabla^{k-1}\left(\sum_{j=1}^r \hat\phi_j(0) u_j\right)(0)/(k-1)!.$$

Since $\sum_{j=1}^r \hat\phi_j(0) u_j$ has degree $k-1$, c must be nonzero.

We have proved that $\mathbb{S}(\phi)$ contains the monomial x^{k-1}. Since $\mathbb{S}(\phi)$ is shift-invariant, it contains $1, x, \ldots, x^{k-1}$. Therefore, we conclude that ϕ has accuracy k. ∎

Theorem 3.1 suggests the following algorithm to test the accuracy of a vector of multiple refinable functions.

ALGORITHM. Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation (1.4) with mask a supported in $[0, N_0]$, where N_0 is a positive integer. Let k be a positive integer and $N := \max\{N_0, 2k-1\}$.

Step 1. Find an eigenvector $[v(0), v(1), \ldots, v(N)]^T$ of the matrix $\left(a(-\alpha + 2\beta)^T\right)_{0\le\alpha,\beta\le N}$ corresponding to the eigenvalue $(1/2)^{k-1}$.

Step 2. Suppose $v(\alpha) = \left(v_1(\alpha), \ldots, v_r(\alpha)\right)^T$ for $0 \le \alpha \le N$. Find the Lagrange interpolating polynomials $u_1, \ldots, u_r$ of degree at most $k-1$ such that $u_j(-\alpha) = v_j(\alpha)$ for $0 \le \alpha \le k-1$ and $j = 1, \ldots, r$.

Step 3. Check whether $u_j(-\alpha) = v_j(\alpha)$ for all $0 \le \alpha \le N$ and $j = 1, \ldots, r$, and check whether $\sum_{j=1}^r \hat\phi_j(0) u_j$ has degree $k-1$. If the answer is yes, then ϕ has accuracy k.

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Let us justify our algorithm. Write σ for $(1/2)^{k-1}$. For $0 \le \alpha \le N$, $a(-\alpha+2\beta) \ne 0$ only if $0 \le \beta \le N$. Hence we have

$$\sigma v(\alpha) = \sum_{\beta=0}^{N} a(-\alpha+2\beta)^T v(\beta) = \sum_{\beta\in\mathbb{Z}} a(-\alpha+2\beta)^T v(\beta), \quad 0 \le \alpha \le N.$$

It follows that, for $-N \le \alpha \le 0$,

$$\sigma u(\alpha) = \sigma v(-\alpha) = \sum_{\beta\in\mathbb{Z}} a(\alpha+2\beta)^T v(\beta) = \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)^T u(\beta) = S_a u(\alpha).$$

Suppose $S_a u(\alpha) = \big(w_1(\alpha),\ldots,w_r(\alpha)\big)^T$ for $\alpha \in \mathbb{Z}$. We observe that

$$S_a u(2\alpha) = \sum_{\beta\in\mathbb{Z}} a(2\alpha-2\beta)^T u(\beta) = \sum_{\beta\in\mathbb{Z}} a(2\beta)^T u(\alpha-\beta), \quad \alpha \in \mathbb{Z}.$$

This shows that $\big(w_j(2\alpha)\big)_{\alpha\in\mathbb{Z}}$ $(j=1,\ldots,r)$ are polynomial sequences of degree at most $k-1$. But $S_a u(2\alpha) = \sigma u(2\alpha)$ for $-(k-1) \le \alpha \le 0$. Therefore, $S_a u(2\alpha) = \sigma u(2\alpha)$ for all $\alpha \in \mathbb{Z}$. Similarly, $S_a u(2\alpha-1) = \sigma u(2\alpha-1)$ for all $\alpha \in \mathbb{Z}$. In other words, $S_a u = \sigma u = (1/2)^{k-1} u$. By Theorem 3.1, we conclude that ϕ has accuracy k.

Let us apply our theory to the example mentioned in the introduction. Let a be the sequence of 2×2 matrices given by $a(\alpha) = 0$ for $\alpha \in \mathbb{Z}\setminus\{0,1,2\}$ and

$$a(0) = \begin{bmatrix} 1/2 & s/2 \\ t & \lambda \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad a(2) = \begin{bmatrix} 1/2 & -s/2 \\ -t & \lambda \end{bmatrix}, \tag{3.3}$$

where λ, μ, s, and t are real numbers.

EXAMPLE 3.2. Let a be the mask given by (3.3), and let $\phi = (\phi_1,\phi_2)^T$ be a vector of compactly supported distributions that satisfies the refinement equation

$$\phi = \sum_{\alpha=0}^{2} a(\alpha)\phi(2\cdot-\alpha) \tag{3.4}$$

subject to the condition $\hat\phi_1(0) = 1$ and $\hat\phi_2(0) = 0$. Suppose $st \ne 0$. Then

(a) ϕ has accuracy 3 if and only if $\mu = 1/2$ and $\lambda = 1/4 + 2st$, and

(b) ϕ has accuracy 4 if and only if $\lambda = -1/8$, $\mu = 1/2$, and $st = -3/16$.

PROOF. The matrix $A_{[0,2]} := \big(a(-\alpha+2\beta)^T\big)_{0\le\alpha,\beta\le 2}$ has the form

$$\begin{bmatrix} 1/2 & t & 1/2 & -t & & \\ s/2 & \lambda & -s/2 & \lambda & & \\ & & 1 & 0 & & \\ & & 0 & \mu & & \\ & & 1/2 & t & 1/2 & -t \\ & & s/2 & \lambda & -s/2 & \lambda \end{bmatrix}. \tag{3.5}$$

Since $st \ne 0$, $A_{[0,2]}$ has eigenvalues $1, 1/2, 1/4$ if and only if $\mu = 1/2$ and $\lambda = 1/4+2st$. Moreover, $A_{[0,2]}$ has eigenvalues $1, 1/2, 1/4, 1/8$ if and only if $\lambda = -1/8$, $\mu = 1/2$, and $st = -3/16$. In this case, $1, 1/2, 1/4, 1/4, 1/8, 1/8$ are all the eigenvalues of $A_{[0,2]}$. Thus, by Theorem 2.1, ϕ does not have accuracy 5 for any choice of the parameters.

Let us show that ϕ has accuracy 3 if $\mu = 1/2$ and $\lambda = 1/4+2st$. In this case, we have $N_0 = 2$, $k = 3$, and $N = \max\{N_0, 2k-1\} = 5$. We find an eigenvector $[v(0), v(1), v(2)]^T$ of $A_{[0,2]}$ corresponding to the eigenvalue $\sigma := 1/4$ as follows:

$$v(0) = \begin{bmatrix} t \\ -1/4 \end{bmatrix}, \quad v(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v(2) = \begin{bmatrix} t \\ 1/4 \end{bmatrix}.$$

To find $v(\alpha)$ for $\alpha > 2$, we may use the formula

$$v(\alpha) = \frac{1}{\sigma} \sum_{\beta=0}^{\alpha-1} a(-\alpha + 2\beta)^T v(\beta). \tag{3.6}$$

In this way we obtain

$$v(3) = \begin{bmatrix} 4t \\ 1/2 \end{bmatrix}, \quad v(4) = \begin{bmatrix} 9t \\ 3/4 \end{bmatrix}, \quad v(5) = \begin{bmatrix} 16t \\ 1 \end{bmatrix}.$$

Choose $u_1(x) := t(x+1)^2$ and $u_2(x) = -(x+1)/4$ for $x \in \mathbb{R}$ and set $u(\alpha) := [u_1(\alpha), u_2(\alpha)]^T$ for $\alpha \in \mathbb{Z}$. Then $u(-\alpha) = v(\alpha)$ for $0 \le \alpha \le 5$. Moreover,

$$\sum_{j=1}^{2} \hat{\phi}_j(0) \nabla^2 u_j(0)/2! = t \neq 0.$$

Therefore, by Theorem 3.1, ϕ has accuracy 3 and

$$x^2 = \sum_{\alpha \in \mathbb{Z}} \Big[(\alpha+1)^2 \phi_1(x-\alpha) - \frac{\alpha+1}{4t} \phi_2(x-\alpha) \Big], \quad x \in \mathbb{R}. \tag{3.7}$$

It is proved in [12] that ϕ is continuous if $\mu = 1/2$, $\lambda = 1/4 + 2st$, and $|2\lambda + \mu| < 2$. Thus, ϕ provides approximation order 3.

Now consider the case $\lambda = -1/8$, $\mu = 1/2$, and $st = -3/16$. In this case, we have $N_0 = 2$, $k = 4$, and $N = \max\{N_0, 2k-1\} = 7$. We find an eigenvector $[v(0), v(1), v(2)]^T$ of $A_{[0,2]}$ corresponding to the eigenvalue $\sigma := 1/8$ as follows:

$$v(0) = \begin{bmatrix} 1 \\ 2s \end{bmatrix}, \quad v(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad v(2) = \begin{bmatrix} -1 \\ 2s \end{bmatrix}.$$

Using formula (3.6) to find $v(\alpha)$ for $\alpha > 2$, we obtain

$$v(\alpha) = \begin{bmatrix} -(\alpha-1)^3 \\ 2s(\alpha-1)^2 \end{bmatrix}, \quad \text{for } 3 \le \alpha \le 7.$$

Choose $u_1(x) := (x+1)^3$ and $u_2(x) = 2s(x+1)^2$ for $x \in \mathbb{R}$ and set $u(\alpha) := [u_1(\alpha), u_2(\alpha)]^T$ for $\alpha \in \mathbb{Z}$. Then $u(-\alpha) = v(\alpha)$ for $0 \le \alpha \le 7$. Moreover,

$$\sum_{j=1}^{2} \hat{\phi}_j(0) \nabla^3 u_j(0)/3! = 1.$$

Therefore, by Theorem 3.1, ϕ provides approximation order 4 and

$$x^3 = \sum_{\alpha\in\mathbb{Z}} \Big[(\alpha+1)^3\phi_1(x-\alpha) + 2s(\alpha+1)^2\phi_2(x-\alpha)\Big], \quad x \in \mathbb{R}.$$

In fact, in this case, $\phi = (\phi_1, \phi_2)^T$ can be solved explicitly:

$$\phi_1(x) = \begin{cases} x^2(-2x+3) & \text{for } 0 \le x \le 1, \\ (2-x)^2(2x-1) & \text{for } 1 \le x \le 2, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$\phi_2(x) = \begin{cases} x^2(x-1)3/(2s) & \text{for } 0 \le x \le 1, \\ (2-x)^2(x-1)3/(2s) & \text{for } 1 \le x \le 2, \\ 0 & \text{elsewhere.} \end{cases}$$

The special case $\lambda = -1/8$, $\mu = 1/2$, $s = 3/2$, and $t = -1/8$ was discussed in [7]. ∎

4. **A Characterization of Accuracy.** In this section we give a complete characterization for the accuracy of a vector of multiple refinable functions in terms of the refinement mask. We also complete our study of the example discussed in the previous section.

THEOREM 4.1. *Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be a vector of compactly supported distributions on $\mathbb{R}$ satisfying the refinement equation (1.4). Then ϕ has accuracy k if and only if there exist polynomial sequences $u_1, \ldots, u_r$ on $\mathbb{Z}$ such that the element $u \in \ell(\mathbb{Z} \longrightarrow \mathbb{C}^r)$ given by $u(\alpha) = (u_1(\alpha), \ldots, u_r(\alpha))^T$, $\alpha \in \mathbb{Z}$, satisfies*

$$u \notin K(\phi) \quad \text{and} \quad S_a u - (1/2)^{k-1} u \in K(\phi). \tag{4.1}$$

Consequently, if the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent, then ϕ has accuracy k if and only if there exist polynomial sequences $u_1, \ldots, u_r$ of degree at most $k-1$ such that the vector $u: \alpha \longmapsto \big(u_1(\alpha), \ldots, u_r(\alpha)\big)^T$ ($\alpha \in \mathbb{Z}$) is an eigenvector for S_a corresponding to the eigenvalue $(1/2)^{k-1}$.

PROOF. First observe that if the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent, then the first condition in (4.1) reduces to $u \neq 0$ because $u \neq 0$ implies $u \notin K(\phi) = \{0\}$. Suppose u satisfies the conditions in (4.1). Set

$$p = \sum_{\alpha\in\mathbb{Z}} u(\alpha)^T \phi(\cdot - \alpha) = \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}} u_j(\alpha)\phi_j(\cdot - \alpha). \tag{4.2}$$

Evidently, $u \notin K(\phi)$ implies $p \neq 0$. Since $u_1, \ldots, u_r$ are polynomial sequences, there exists a positive integer m such that $\nabla^m u_j = 0$ for $j = 1, \ldots, r$. Applying ∇^m to both sides of (4.2), we obtain

$$\nabla^m p = \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}} \nabla^m u_j(\alpha)\phi_j(\cdot - \alpha) = 0. \tag{4.3}$$

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Since ϕ satisfies the refinement equation (1.4), by (2.1) and (4.1) we have

$$(4.4)\quad p = \sum_{\alpha\in\mathbb{Z}} (S_a u(\alpha))^T \phi(2\cdot -\alpha) = (1/2)^{k-1} \sum_{\alpha\in\mathbb{Z}} u(\alpha)^T \phi(2\cdot -\alpha) = (1/2)^{k-1} p(2\,\cdot).$$

From these two facts we deduce that $p(x) = cx^{k-1}$ for some constant c (see the proof of Theorem 3.1). Thus, $\mathbb{S}(\phi)$ contains the monomial x^{k-1}, and so ϕ has accuracy k. This establishes the sufficiency part of the theorem.

If ϕ has accuracy k, then $\mathbb{S}(\phi)$ contains the monomial $p\colon x \longmapsto x^{k-1}$, $x \in \mathbb{R}$. There exist sequences $u_1, \ldots, u_r$ on $\mathbb{Z}$ such that (4.2) holds true. Obviously, $u \notin K(\phi)$. Also, p satisfies (4.4). It follows that

$$\sum_{\alpha\in\mathbb{Z}} \Big[S_a u(\alpha) - (1/2)^{k-1} u(\alpha) \Big]^T \phi(2\cdot -\alpha) = 0.$$

Hence $S_a u - (1/2)^{k-1} u \in K(\phi)$. Thus, in order to prove the necessity part of the theorem, it suffices to show that there exist *polynomial* sequences $u_1, \ldots, u_r$ that satisfy (4.2). Choosing m to be k in (4.3), we obtain $\nabla^k p = 0$. If the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent, then it follows that $\nabla^k u_j = 0$ for $j = 1, \ldots, r$. Hence each u_j is a polynomial sequence of degree at most $k-1$. In general, this will be proved in the following lemma. ∎

LEMMA 4.2. *Let $\Phi = \{\phi_1, \ldots, \phi_r\}$ be a finite collection of compactly supported distributions on $\mathbb{R}$. If p is a polynomial in $\mathbb{S}(\Phi)$, then there exist polynomial sequences $q_1, \ldots, q_r$ such that*

$$(4.5)\qquad p = \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}} q_j(\alpha) \phi_j(\cdot - \alpha).$$

PROOF. Since p lies in $\mathbb{S}(\Phi)$, there exist sequences $q_1, \ldots, q_r$ on $\mathbb{Z}$ such that (4.5) holds true. If the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent, then each q_j is a polynomial sequence, as was proved above.

In general, we shall prove the lemma by induction on the length of Φ. Let ϕ be a nonzero compactly supported distribution on $\mathbb{R}$. Let $[s_\phi, t_\phi]$ be the smallest integer-bounded interval containing the support of ϕ. The *length* of ϕ is defined to be $t_\phi - s_\phi$, and denoted by $l(\phi)$. The *length* of Φ is defined by $l(\Phi) := \sum_{\phi\in\Phi} l(\phi)$. For each ϕ_j, let $s_j := s_{\phi_j}$ and $t_j := t_{\phi_j}$. After shifting ϕ_j appropriately, we may assume that all $s_j = 0$.

If $l(\Phi) = 0$, then $\phi_1, \ldots, \phi_r$ are all supported at 0; hence the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent if and only if $\phi_1, \ldots, \phi_r$ are linearly independent. Choose a linearly independent spanning subset Ψ of Φ. Then $\mathbb{S}(\Phi) = \mathbb{S}(\Psi)$ and the shifts of the elements in Ψ are linearly independent. Note further that the elements of $\mathbb{S}(\Phi)$ are supported only on the integers so that $\mathbb{S}(\Phi)$ cannot contain a non-zero polynomial. Therefore, in what follows we may assume without loss of any generality that the set $\{\phi \in \Phi : l(\phi) = 0\}$ is linearly independent.

Suppose $l(\Phi) \geq 1$. If the shifts of $\phi_1, \ldots, \phi_r$ are linearly dependent, then we can find some $\theta \in \mathbb{C} \setminus \{0\}$ and $(c_1, \ldots, c_r) \in \mathbb{C}^r \setminus \{0\}$ such that

$$\left(c_1\theta^0, \ldots, c_r\theta^0 \right)^T \in K(\Phi),$$

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where θ^0 denotes the sequence $k \longmapsto \theta^k$, $k \in \mathbb{Z}$ (see [11, Theorem 3.3]). In other words,

$$\sum_{j=1}^{r} \sum_{k=-\infty}^{\infty} c_j \theta^k \phi_j(\cdot - k) = 0. \tag{4.6}$$

Let $l := \max\{l(\phi_j) : c_j \neq 0\}$. Since the set $\{\phi \in \Phi : l(\phi) = 0\}$ is linearly independent, we have $l \geq 1$. For simplicity, we assume that $c_1 \neq 0$ and $l(\phi_1) = l$. Let

$$\rho := \sum_{j=1}^{r} c_j \phi_j \quad \text{and} \quad \psi := \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k).$$

By our choice of ρ, we deduce from (4.6) that

$$\sum_{k=-\infty}^{\infty} \theta^k \rho(\cdot - k) = 0.$$

Since $\rho(\cdot - k)|_{(l-1,\infty)} = 0$ for $k < 0$, it follows that

$$\psi|_{(l-1,\infty)} = \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k)|_{(l-1,\infty)} = \sum_{k=-\infty}^{\infty} \theta^k \rho(\cdot - k)|_{(l-1,\infty)} = 0.$$

Also, $\psi|_{(-\infty,0)} = 0$. Consequently, ψ is supported on $[0, l-1]$. Moreover,

$$\psi - \theta\psi(\cdot - 1) = \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k) - \sum_{k=0}^{\infty} \theta^{k+1} \rho(\cdot - k - 1) = \rho.$$

Let $\Psi := \{\psi, \phi_2, \ldots, \phi_r\}$. Clearly, $\mathbb{S}(\Phi) \subseteq \mathbb{S}(\Psi)$ and $l(\Psi) < l(\Phi)$.

Suppose p is a nonzero polynomial in $\mathbb{S}(\Phi)$. If $l(\Phi) = 1$, then the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. For otherwise, $l(\Psi) = 0$ and $p \in \mathbb{S}(\Psi)$, which is a contradiction.

Now suppose $l(\Phi) > 1$. We have verified the lemma if the shifts of $\phi_1, \ldots, \phi_r$ are linearly independent. Otherwise, we can find $\Psi = \{\psi, \phi_2, \ldots, \phi_r\}$ with $l(\Psi) < l(\Phi)$ and all the properties stated in the above. By the induction hypothesis, there exist polynomials $q_1, q_2, \ldots, q_r$ such that

$$p = \sum_{\alpha \in \mathbb{Z}} q_1(\alpha)\psi(\cdot - \alpha) + \sum_{j=2}^{r} \sum_{\alpha \in \mathbb{Z}} q_j(\alpha)\phi_j(\cdot - \alpha).$$

If we can find a polynomial q such that

$$p = \sum_{\alpha \in \mathbb{Z}} q(\alpha)\rho(\cdot - \alpha) + \sum_{j=2}^{r} \sum_{\alpha \in \mathbb{Z}} q_j(\alpha)\phi_j(\cdot - \alpha), \tag{4.7}$$

then the induction procedure will be complete, because ρ is a linear combination of $\phi_1, \ldots, \phi_r$. But $\rho = \psi - \theta\psi(\cdot - 1)$. Hence we have

$$\sum_{\alpha \in \mathbb{Z}} q(\alpha)\rho(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}} \big[q(\alpha) - \theta q(\alpha - 1)\big] \psi(\cdot - \alpha).$$

It is easily seen that there exists a polynomial q such that $q - \theta q(\cdot - 1) = q_1$. For this q, (4.7) holds true. The proof of the lemma is complete. ∎

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Now we are in a position to discuss the exceptional case $st = 0$ in Example 3.2.

EXAMPLE 4.3. Let $a\colon \mathbb{Z} \longrightarrow \mathbb{R}^{2\times 2}$ be the mask given in (3.3). Assume that $|2\lambda+\mu| < 2$. Let $\phi = (\phi_1, \phi_2)^T$ be the normalized solution of the refinement equation (3.4). Suppose $st = 0$. For any choice of the parameters s, t, λ, and μ (subject to the condition $st = 0$), ϕ has accuracy 2 but does not have accuracy 4. Moreover, ϕ has accuracy 3 if and only if $t \neq 0$, $\lambda = 1/4$ and $\mu = 1/2$.

PROOF. The case $t = 0$ is trivial. Indeed, in this case, $\phi_2 = 0$ and

$$\phi_1(x) = \begin{cases} x & \text{for } 0 \leq x < 1, \\ 2 - x & \text{for } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

So ϕ has accuracy 2 but does not have accuracy 3.

In what follows, we assume that $s = 0$ and $t \neq 0$. In this case, ϕ_1 is the function given in (4.8). Thus, ϕ has accuracy at least 2.

Let us first discuss the case where the shifts of ϕ_1 and ϕ_2 are linearly *dependent*. We observe that the shifts of ϕ_1 are linearly independent. Let $\Phi := \{\phi_1, \phi_2\}$. If the shifts of ϕ_1, ϕ_2 are linearly dependent, then from the proof of Lemma 4.2 we see that there exists a compactly supported distribution $\psi \in \mathbb{S}(\Phi)$ such that $\Psi := \{\phi_1, \psi\}$ satisfies $l(\Psi) < l(\Phi) \leq 4$ and $\mathbb{S}(\Psi) = \mathbb{S}(\Phi)$. Thus, $\mathbb{S}(\Psi)|_{(0,1)}$ has dimension at most 3. Hence $\mathbb{S}(\Phi) = \mathbb{S}(\Psi)$ does not contain Π_3. This shows that ϕ does not have accuracy 4. If ϕ has accuracy 3, then $\mathbb{S}(\Psi) = \mathbb{S}(\Phi) \supseteq \Pi_2$. But the dimension of $\Pi_2|_{(0,1)}$ is 3. Hence $S(\Phi)|_{(0,1)} = S(\Psi)|_{(0,1)} = \Pi_2|_{(0,1)}$. This shows that $\phi_2|_{(0,1)}$ is a quadratic polynomial. Suppose

$$\phi_2(x) = c_0x^2 + c_1x + c_2, \quad \text{for } 0 < x < 1,$$

where the leading coefficient $c_0 \neq 0$. Since ϕ_2 is anti-symmetric about 1, we have

$$\phi_2(x) = -c_0(2-x)^2 - c_1(2-x) - c_2, \quad \text{for } 1 < x < 2.$$

For $0 < x < 1/2$, the refinement equation (3.4) reads as $\phi(x) = a(0)\phi(2x)$, that is,

$$\begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ t & \lambda \end{bmatrix} \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \end{bmatrix}.$$

It follows that

$$c_0x^2 + c_1x + c_2 = \lambda(4c_0x^2 + 2c_1x + c_2) + t(2x).$$

Comparing the corresponding coefficients of the two sides of this equation, we obtain $\lambda = 1/4$, $c_1 = 4t$, and $c_2 = 0$. For $1/2 < x < 1$, the refinement equation (3.4) reads as follows:

$$\phi(x) = a(0)\phi(2x) + a(1)\phi(2x-1),$$

that is,

$$\begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ t & \lambda \end{bmatrix} \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix}.$$

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It follows that

$$c_0x^2 + c_1x = t(2-2x) + \lambda\big[-c_0(2-2x)^2 - c_1(2-2x)\big] + \mu\big[c_0(2x-1)^2 + c_1(2x-1)\big].$$

Comparing the corresponding coefficients of the two sides of this equation, we obtain $\mu = 1/2$ and $c_0 = -4t$. This shows that ϕ has accuracy 3 only if $\lambda = 1/4$ and $\mu = 1/2$. If this is the case, then the proof of Example 3.2 shows that ϕ has accuracy 3 and (3.7) holds true. In addition,

$$\phi_2(x) = \begin{cases} 4tx(1-x) & \text{for } 0 \le x < 1, \\ -4t(2-x)(x-1) & \text{for } 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since both ϕ_1 and ϕ_2 are continuous, we conclude that ϕ provides approximation order 3.

We claim that the shifts of ϕ_2 are linearly dependent if $\mu = 2\lambda$. Indeed, it follows from the refinement equation (3.4) that

$$\phi_2 = \lambda\phi_2(2\,\cdot) + \mu\phi_2(2\cdot -1) + \lambda\phi_2(2\cdot -2) + t\phi_1(2\,\cdot) - t\phi_1(2\cdot -2).$$

Taking the Fourier transform of both sides of the above equation, we obtain

$$\hat{\phi}_2(\xi) = (\lambda + \mu e^{-i\xi/2} + \lambda e^{-i\xi})\hat{\phi}_2(\xi/2)/2 + (t - te^{-i\xi})\hat{\phi}_1(\xi/2)/2 \quad \forall\, \xi \in \mathbb{R}.$$

For $k \in \mathbb{Z}$, setting $\xi = 2k\pi$ in the above equation gives $\hat{\phi}_2(2k\pi) = 0$, provided $\mu = 2\lambda$. This verifies our claim.

It remains to deal with the case where the shifts of ϕ_1 and ϕ_2 are linearly independent. In this case, if ϕ has accuracy 3, then Theorem 4.1 tells us that there exist polynomials u_1 and u_2 of degree at most 2 such that $u\colon \alpha \longmapsto \big(u_1(\alpha), u_2(\alpha)\big)^T$, $\alpha \in \mathbb{Z}$, satisfies $u \ne 0$ and $S_a u = (1/4)u$. It follows that

$$u(\alpha) = 4\sum_{\beta\in\mathbb{Z}} a(\alpha - 2\beta)^T u(\beta) \quad \forall\, \alpha \in \mathbb{Z}. \tag{4.9}$$

Suppose $u(1) = [b_1, b_2]^T$. From (4.9) we deduce that

$$u(2^{n+1}-1) = 4a(1)^T u(2^n - 1) = \begin{bmatrix} 4 & 0 \\ 0 & 4\mu \end{bmatrix} u(2^n - 1), \quad \text{for } n = 1, 2, \ldots.$$

An induction argument gives

$$u(2^{n+1}-1) = \begin{bmatrix} 4^n b_1 \\ (4\mu)^n b_2 \end{bmatrix}, \quad \text{for } n = 0, 1, \ldots. \tag{4.10}$$

It follows from (4.10) that $u_1(2^{n+1}-1) = 4^n b_1$ for $n = 0, 1, \ldots$. Since u_1 is a polynomial of degree at most 2, we have

$$u_1(x) = b_1(x+1)^2/4 \quad \forall\, x \in \mathbb{R}.$$

Likewise, since u_2 is a polynomial of degree at most 2, (4.10) holds true only if $\mu = 1$, $1/2$, or $1/4$:

$$u_2(x) = \begin{cases} b_2(x+1)^2/4, & \text{if } \mu = 1, \\ b_2(x+1)/2, & \text{if } \mu = 1/2, \\ b_2, & \text{if } \mu = 1/4. \end{cases} \tag{4.11}$$

Setting $\alpha = 2$ in (4.9), we obtain

$$u(2) = 4a(0)^T u(1) + 4a(2)^T u(0),$$

or,

$$\begin{bmatrix} 9b_1/4 \\ u_2(2) \end{bmatrix} = 4 \begin{bmatrix} 1/2 & t \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + 4 \begin{bmatrix} 1/2 & -t \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} b_1/4 \\ u_2(0) \end{bmatrix}. \tag{4.12}$$

If $\mu = 1/4$ and $u_2(x) = b_2$, then the equation for the second component in (4.12) implies that either $\lambda = 1/8$ or $b_2 = u_2(x) = 0$. In the first case, $\mu = 2\lambda$ and the shifts of ϕ_2 are linearly dependent; a contradiction. In the second case, the equation for the first component of (4.12) yields $9b_1/4 = 5b_1/2$ which implies $b_1 = 0$. But then $u_1 = 0$, in contradiction to the fact that $u \neq 0$.

For the remaining cases in (4.11), we observe that, for the case $s = 0$, the eigenvalues of the matrix $A_{[0,2]}$ given in (3.5) are $1, 1/2, 1/2, \lambda, \lambda, \mu$. Thus, if $\mu \neq 1/4$, then ϕ has accuracy 3 implies $\lambda = 1/4$, by Theorem 2.1. Hence, when $\mu = 1$ or $\mu = 1/2$, the equation for the second component of (4.12) becomes $u_2(2) = b_2 + u_2(0)$ which implies $b_2 = 0$ for the u_2 given in (4.11), and this would lead to a contradiction as before. ∎

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