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NONEXPANSIVE PROJECTIONS ONTO TWO-DIMENSIONAL SUBSPACES OF BANACH SPACES

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We show that if a three dimensional normed space X has two linearly independent smooth points e and f such that every two-dimensional subspace containing e or f is the range of a nonexpansive projection then X is isometrically isomorphic to $\ell_p(3)$ for some p, $1 < p \leqslant \infty$. This leads to a characterisation of the Banach spaces c_0 and ℓ_p, $1 < p \leqslant \infty$, and a characterisation of real Hilbert spaces.

1 INTRODUCTION

In 1969, Ando [**1**] showed that a real three dimensional Banach lattice is isometrically isomorphic to $\ell_p(3)$ for some $p \in [1,\infty]$ if and only if all sublattices are ranges of positive nonexpansive projections. This and other results on characterising L_p spaces can be found in the books [**6**] and [**8**]. In recent work [**3**] and [**4**] we characterised $\ell_p(3)$ by only requiring sublattices through two of the coordinate axes to be ranges of nonexpansive projections. This allowed us to characterise the Banach lattices $\ell_p(n)$, c_0 and ℓ_p by requiring planes through $\mathbf{R}e_i$ to be ranges of nonexpansive projections for certain disjoint elements e_i.

In this work we show that those results generalise to Banach spaces which are not endowed with lattice structure and to e_i which are not necessarily orthogonal. In Theorem A we take two linearly independent smooth points e and f in a three-dimensional normed space X such that every two-dimensional subspace which intersects $\{e,f\}$ is the range of a nonexpansive projection and conclude that X is $\ell_p(3)$. If e and f are not orthogonal then we have $p = 2$.

We extend this result to higher dimensions in Theorems B and C. This yields a characterisation of ℓ_p and c_0 which requires only a small number of planes to be ranges of nonexpansive projections. In Theorem D we use this to characterise Hilbert spaces.

Let X be a Banach space. Recall that *duality mapping* J from X to subsets of $X^\star$ is defined by $x^\star \in Jx$ provided $x^\star(x) = \|x\|^2 = \|x^\star\|^2$. The norm is *smooth* at x, or x is a *smooth point*, provided Jx is a singleton. By *projection* we mean a linear map $P\colon X \to X$ such that $P^2 = P$. A point x is *orthogonal* to a point y provided $\|x+ty\| \geqslant \|x\|$ for all $t \in \mathbf{R}$.

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2 Three-dimensional normed spaces

The following result is basically by Blaschke [**2**] and appears in a form like this in Ando [**1**]. We give a different proof.

Lemma 1. *Let X be a real three-dimensional normed space with basis $\{e_1, e_2, e_3\}$ where e_1 is a unit vector. Suppose every two-dimensional subspace which contains e_1 is the range of a nonexpansive projection along a vector in* $\operatorname{span}\{e_2, e_3\}$. *Then there is a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\|x_1 e_1 + x_2 e_2 + x_3 e_3\| = F(x_1, \|x_2 e_2 + x_3 e_3\|) \qquad \textit{for all } x_i \in \mathbb{R}.$$

Proof: Let $y(t)$, $0 \leqslant t \leqslant T$, be a parametrization of the unit circle $\|y(t)\| = 1$ in $\operatorname{span}\{e_2, e_3\}$, such that

$$\lim_{h \to 0_+} \frac{y(t+h) - y(t)}{h} = p(t), \quad \|p(t)\| = 1,$$

and $y(0) = y(T) = \|e_2\|^{-1} e_2$. Then from the existance of a nonexpansive projection onto $\operatorname{span}\{e_1, y(t+h)\}$ along a vector $u(t+h)$ in $\operatorname{span}\{e_2, e_3\}$ for each t we see by taking the limit as $h \to 0_+$ that the projection along $p(t)$ is nonexpansive, so that $\|x_1 e_1 + y(t) + sp(t)\| \geqslant \|x_1 e_1 + y(t)\|$ for all x_1, s and t. Now for $h > 0$,

$$\begin{aligned} \|x_1 e_1 + y(t+h)\| &= \|x_1 e_1 + y(t) + y(t+h) - y(t)\| \\ &\geqslant \|x_1 e_1 + y(t) + hp(t)\| - \|y(t+h) - y(t) - hp(t)\| \\ &\geqslant \|x_1 e_1 + y(t)\| - \|y(t+h) - y(t) - hp(t)\| \end{aligned}$$

so that the right-hand derivative of $\|x_1 e_1 + y(t)\|$,

$$\lim_{h \to 0_+} \frac{\|x_1 e_1 + y(t+h)\| - \|x_1 e_1 + y(t)\|}{h} \geqslant 0.$$

Since $y(0) = y(T)$ we see that $\|x_1 e_1 + y(t)\| = F(x_1, 1)$ does not depend on t. The result follows by homogeneity of the norm. ∎

Now we introduce some standing assumptions for this Section.

Standing assumptions. Let X be a real three-dimensional normed space with two linearly independent smooth points of norm 1, e and f, such that every two-dimensional subspace which intersects $\{e, f\}$ is the range of a nonexpansive projection. Let $e_1 = e$, $f_1 = f$, choose unit vectors e_2 and f_2 in $\operatorname{span}\{e, f\}$ such that $Je(e_2) = 0 = Jf(f_2)$ and let e_3 be a unit vector such that $Je(e_3) = 0 = Jf(e_3)$, and $e_3 \notin \operatorname{span}\{e, f\}$.

PROPOSITION 2. *For all numbers x_1, x_2 and x_3 we have*

$$\left\| \sum_{i=1}^{3} x_i e_i \right\| = \left\| \sum_{i=1}^{3} |x_i| e_i \right\| \tag{1}$$

PROOF: For any two-dimensional subspace M containing e, the nonexpansive projection onto M is in a direction p tangent to e so that $Je(p) = 0$. Thus all such p are in $\operatorname{span}\{e_2, e_3\}$.

By Lemma 1 we have for all x_i,

$$\|x_1e_1 + x_2e_2 + x_3e_3\| = \|x_1e_1 \pm \|x_2e_2 + x_3e_3\|\, e_3\|. \tag{2}$$

Now to show (1) we only have to show

$$\|x_2e_2 + x_3e_3\| = \|x_2e_2 - x_3e_3\| \tag{3}$$

for all x_3 and x_2.

Considering f instead of e we have

$$\|y_1f_1 + y_2f_2 + y_3f_3\| = \|y_1f_1 \pm \|y_2f_2 + y_3f_3\|\, e_3\|, \tag{4}$$

for all y_1, y_2 and y_3.

Now let

$$f_1 = \alpha e_1 + \beta e_2, \text{ so } \beta \neq 0, \text{ and } f_2 = \gamma e_1 + \delta e_2. \tag{5}$$

Thus for any t,

$$\begin{aligned} \|f_1 + tf_1\| &= \|f_1 + te_3\| && \text{by (4)} \\ &= \|\alpha e_1 + \beta e_2 + te_3\| && \text{by (5)} \\ &= \|\alpha e_1 + \|\beta e_2 + te_3\|\, e_3\| && \text{by (2).} \end{aligned}$$

But

$$\begin{aligned} \|f_1 + tf_2\| &= \|f_1 - tf_2\| && \text{by (4)} \\ &= \|\alpha e_1 + \|\beta e_2 - te_3\|\, e_3\| && \text{as above.} \end{aligned}$$

Suppose for purposes of obtaining a contradiction that

$$\|\beta e_2 + te_3\| \neq \|\beta e_2 - te_3\| \quad \text{for some } t. \tag{6}$$

Then

$$\begin{aligned}\|\alpha e_1 + \|\beta e_2 + te_3\|\, e_3\| &= \|\alpha e_1 - \|\beta e_2 + te_3\|\, e_3\| \\ &= \|\alpha e_1 + \|\beta e_2 - te_3\|\, e_3\|\end{aligned}$$

and the convexity of the norm implies that $\|\alpha e_1 + se_3\|$ is constant, and hence equal to $\|\alpha e_1\| = |\alpha|$, for $|s| \leqslant \max\{\|\beta e_2 + te_3\|, \|\beta e_2 - te_3\|\} = r$. Thus $\|\alpha e_1 + x_2 e_2 + x_3 e_3\| = |\alpha|$ whenever $\|x_2 e_2 + x_3 e_3\| \leqslant r$.

Thus for $s \leqslant t$ we have

$$\|f_1 + se_3\| = \|\alpha e_1 + \beta e_2 + se_3\| = |\alpha| \tag{7}$$

and putting $s = 0$ we see that $|\alpha| = 1$. Now

$$\begin{aligned}\|f_1 + se_3\| &= \|f_1 + sf_2\| \qquad \text{(by (4))} \\ &= \|\alpha e_1 + \beta e_2 + s(\gamma e_1 + \delta e_2)\| \\ &= \|(\alpha + s\gamma)e_1 + (\beta + s\delta)e_2\| \\ &= |\alpha + s\gamma| \left\|\alpha e_1 + (\alpha + s\gamma)^{-1}(\beta + s\delta)e_3\right\| \\ &= |\alpha + s\gamma| \quad \text{whenever } |(\alpha + s\gamma)^{-1}(\beta + s\delta)| \leqslant r.\end{aligned}$$

The convexity of the norm and (6) show that at least one of $\|\beta e_2 + te_3\|$ and $\|\beta e_2 - te_3\|$ is greater than $\|\beta e_2\| = |\beta|$. Since $|(\alpha + s\gamma)^{-1}(\beta + s\delta)|$ is equal to $|\beta|$ when $s = 0$, by continuity $|(\alpha + s\gamma)^{-1}(\beta + s\delta)| < r$ for s near 0. Using (7) we see that $|\alpha + s\gamma| = |\alpha| = 1$ for s near 0, thus $\gamma = 0$.

If necessary taking f_1 to be $-f$ instead of f we have $f_1 = e_1 + \beta e_2$ and $f_2 = \delta e_2 = e_2$ without loss of generality. Thus $\|f_1 - \beta e_2\| = \|e_1\| = 1$ so that $\|f_1 + \beta e_2\| = 1$ by (4). This means $\|e_1 + 2\beta e_2\| = 1$, so $\|e_1 - 2\beta e_2\| = 1$ by (2), which means $\|f_1 - 3\beta e_2\| = 1$. By induction $\|e_1 + n\beta e_2\| = 1$ for all $n \in \mathbb{N}$, giving $\beta = 0$, so that e and f are not linearly independent. This contradiction shows that (6) is false and hence $\|\beta e_2 + te_3\| = \|\beta e_2 - te_3\|$ for all t. This yields (3) and completes the proof. ∎

We will need the following result from [**3**] or [**4**].

PROPOSITION 3. *Let X be a real three-dimensional Banach lattice with unit basis $\{e_1, e_2, e_3\}$ such that $e_i \wedge e_j = 0$ if $i \neq j$. Suppose every subspace which intersects $\{e_1, e_2\}$ is the range of a nonexpansive projection on X. Then X is isometrically isomorphic to $\ell_p(3)$ for some $p \in [1, \infty]$.*

PROPOSITION 4. *Under our standing assumptions, either X is isometrically isomorphic to $\ell_p(3)$ for some $p \in [1, \infty]$ or there is an isometry $R\colon X \to X$ such that*

$Re_3 = e_3$ and for some $t > 0$ and some odd integer $m > 2$, letting $\theta = \pi m^{-1}$ we have $Re_1 = \cos\theta e_1 + t^{-1}\sin\theta e_2$ and $Re_2 = \cos\theta e_2 - t\sin\theta e_1$.

PROOF: If $f_1 = \pm e_2$ then by Proposition 2, if we order X by the cone generated by $\{e_1, e_2, e_3\}$ then the hypotheses of Proposition 3 hold and X is isometrically isomorphic to $\ell_p(3)$ for some $p \in [1,\infty]$. Thus we assume that $f_1 = \alpha e_1 + \beta e_2$; $\alpha, \beta > 0$ without loss of generality (replacing e_1 or e_2 by its negative if necessary). Recall that $f_2 = \gamma e_1 + \delta e_2$. If $\delta = 0$ then $f_2 = \pm e_1$ and as above the required conclusion holds by Propositions 2 and 3, so we take $\delta > 0$ without loss of generality, changing the sign of f_2 if necessary. ∎

Now suppose $\gamma = 0$. Then $f_2 = e_2$ so that for all t, $\|f_1 + te_2\| = \|f_1 - te_2\|$. Thus $\|\alpha e_1 + (\beta + t)e_2\| = \|\alpha e_1 + (\beta - t)e_2\|$ and taking $t = n\beta$ we have $\|\alpha e_1 + (n+1)\beta e_2\| = \|\alpha e_1 + (n-1)\beta e_2\|$ and for even integers m we get $\|\alpha e_1 + m\beta e_2\| = \alpha$, so $\beta = 0$ giving a contradiction which shows that $\gamma \neq 0$. It will be seen later that $\gamma < 0$.

Since $\|y_1 f_1 + y_2 f_2 + x_3 e_3\| = \|y_1 f_1 - y_2 f_2 + x_3 e_3\|$ for all y_1, y_2 and x_3, we have $\|(y_1\alpha + y_2\gamma)e_1 + (y_1\beta + y_2\delta)e_2 + x_3 e_3\| = \|(y_1\alpha - y_2\gamma)e_1 + (y_1\beta - y_2\delta)e_2 + x_3 e_3\|$. Let $x_1 = y_1\alpha + y_2\gamma$ and $x_2 = y_1\beta + y_2\delta$ so we have $y_1 = (\alpha\delta - \beta\gamma)^{-1}(\delta x_1 - \gamma x_2)$ and $y_2 = (\alpha\delta - \beta\gamma)^{-1}(\alpha x_2 - \beta x_1)$; $\alpha\delta - \beta\gamma \neq 0$ since f_1 and f_2 are independent. Thus

$$\begin{aligned}\|x_1e_1 + x_2e_2 + x_3e_3\| &= \|(\alpha\delta x_1 - \alpha\gamma x_2 - \alpha\gamma x_2 + \gamma\beta x_1)(\alpha\delta - \beta\gamma)^{-1}e_1 \\ &\quad + (\beta\delta x_1 - \beta\gamma x_2 - \alpha\delta x_2 + \beta\delta x_1)(\alpha\delta - \beta\gamma)^{-1}e_2 + x_3e_3\| \\ &= \|(\alpha\delta + \beta\gamma)(\alpha\delta - \beta\gamma)^{-1}x_1e_1 - 2\alpha\gamma(\alpha\delta - \beta\gamma)^{-1}x_2e_1 \\ &\quad + 2\beta\delta(\alpha\delta - \beta\gamma)^{-1}x_1e_2 - (\alpha\delta + \beta\gamma)(\alpha\delta - \beta\gamma)^{-1}x_2e_2 \\ &\quad + x_3e_3\|.\end{aligned}$$

That means the reflection whose matrix with respect to $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix} \frac{\alpha\delta+\beta\gamma}{\alpha\delta-\beta\gamma} & \frac{-2\alpha\gamma}{\alpha\delta-\beta\gamma} & 0 \\ \frac{2\beta\delta}{\alpha\delta-\beta\gamma} & \frac{-\alpha\delta-\beta\gamma}{\alpha\delta-\beta\gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an isometry. Also by Proposition 2 the reflection whose matrix with respect to the basis $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an isometry and thus the composition of these reflections yields an isometry R whose matrix with respect to $\{e_1, e_2, e_3\}$ is

$$\begin{bmatrix} \frac{\alpha\delta+\beta\gamma}{\alpha\delta-\beta\gamma} & \frac{2\alpha\gamma}{\alpha\delta-\beta\gamma} & 0 \\ \frac{2\beta\delta}{\alpha\delta-\beta\gamma} & \frac{\alpha\delta+\beta\gamma}{\alpha\delta-\beta\gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To show that $\gamma < 0$ note that $\|\gamma\alpha^{-1}f_1 + f_2\| = \|\gamma\alpha^{-1}f_1 - f_2\|$ so that

$$\|\gamma\alpha^{-1}(\alpha e_1 + \beta e_2) + \gamma e_1 + \delta e_2\| = \|\gamma\alpha^{-1}(\alpha e_1 + \beta e_2) - \gamma e_1 - \delta e_2\|,$$

and hence

$$\|2\gamma e_1 + (\gamma\beta\alpha^{-1} + \delta)e_2\| = \|(\gamma\beta\alpha^{-1} - \delta)e_2\| = |\gamma\beta\alpha^{-1} - \delta|.$$

Now $\|2\gamma e_1 + (\gamma\beta\alpha^{-1} + \delta)e_2\| \geqslant |\gamma\beta\alpha^{-1} + \delta|$ so that δ has opposite sign to $\gamma\beta\alpha^{-1}$ and hence to γ. Thus $\gamma < 0$ as claimed, and $-1 < (\alpha\delta + \beta\gamma)(\alpha\delta - \beta\gamma)^{-1} < 1$. Let $\theta = \cos^{-1}\left((\alpha\delta + \beta\gamma)(\alpha\delta - \beta\gamma)^{-1}\right)$ and define t by $t\sin\theta = -2\alpha\gamma(\alpha\delta - \beta\gamma)^{-1}$; since $0 < \theta < \pi$ we have $t > 0$. The matrix for the isometry R is

$$(8) \qquad \begin{bmatrix} \cos\theta & -t\sin\theta & 0 \\ t^{-1}\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and by an easy induction, for all integers n, the matrix for R^n is

$$\begin{bmatrix} \cos(n\theta) & -t\sin(n\theta) & 0 \\ t^{-1}\sin(n\theta) & \cos(n\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now if $\theta\pi^{-1}$ is irrational then there is a sequence (n_j) of integers such that $\cos(n_j\theta) \to 0$ and $\sin(n_j\theta) \to 1$. Then we have the matrices for R^{n_j} converging to

$$(9) \qquad \begin{bmatrix} 0 & -t & 0 \\ t^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This limit isometry takes e_1 to $t^{-1}e_2$ and hence to e_2 (and $t = 1$) so we have nonexpansive projections onto every two-dimensional subspace containing e_2 and by Proposition 3 the required conclusion holds.

Otherwise there are co-prime integers k and m so that $m\theta = k\pi$. We take integers i and j such that $ik + jm = 1$ so that the isometry $(-1)^j R^i$ has matrix (8) with $\theta = \pi m^{-1}$. We replace R by this isometry and we may assume that m is odd, for otherwise $m = 2b$ and R^b has matrix (9) and as above we can apply Proposition 3.

The next stage is to show that the existence of such an R leads to a Euclidean norm.

PROPOSITION 5. *Suppose under our standing assumptions that there is an isometry* $R\colon X \to X$ *such that* $Re_3 = e_3$ *and for some* $t > 0$ *and some odd integer* $m > 2$, *letting* $\theta = \pi m^{-1}$ *we have* $Re_1 = \cos\theta e_1 + t^{-1}\sin\theta e_2$ *and* $Re_2 = \cos\theta e_2 - t\sin\theta e_1$. *Then* $t = 1$ *and* X *is isometrically isomorphic to* $\ell_2(3)$.

PROOF: For each ν there is a nonexpansive projection onto $\operatorname{span}\{e_1, e_3 + \nu e_2\}$ along a vector $p(\nu)$ in $\operatorname{span}\{e_2, e_3\}$. We can take $p(\nu) = e_2 - g(\nu)e_3$ unless $p(\nu)$ is in $\mathbb{R}e_3$ in which case take $p(\nu) = e_3$, in fact we will see this case cannot occur.

For each α, ν and s we have

$$\alpha Re_1 + \nu Re_2 + e_3 = (\alpha\cos\theta - \nu t\sin\theta)e_1 + \big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)e_2 + e_3$$

so that if $g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)$ exists we have

$$\|\alpha Re_1 + \nu Re_2 + e_3\| \leqslant \Big\|\alpha Re_1 + \nu Re_2 + e_3 + s\big(e_2 - g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)e_3\big)\Big\|$$

and applying the isometry R^{-1} we get

$$\begin{aligned}\|\alpha e_1 + \nu e_2 + e_3\| &\leqslant \Big\|\alpha e_1 + \nu e_2 + e_3 + s\big(R^{-1}e_2 - g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)e_3\big)\Big\|\\ &= \Big\|\alpha e_1 + \nu e_2 + e_3 + s\big(\cos\theta e_2 + t\sin\theta e_1 - g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)e_3\big)\Big\|\end{aligned}$$

and using R^{-1} instead of R, we have

$$\begin{aligned}&\|\alpha e_1 + \nu e_2 + e_3\|\\ &\qquad\leqslant \Big\|\alpha e_1 + \nu e_2 + e_3 + s\big(\cos\theta e_2 - t\sin\theta e_1 - g\big(\nu\cos\theta - \alpha t^{-1}\sin\theta\big)e_3\big)\Big\|\end{aligned}$$

provided $g\big(\nu\cos\theta - \alpha t^{-1}\sin\theta\big)$ exists.

Now if $x = \alpha e_1 + \nu e_2 + e_3$ is a smooth point then $Jx(p(\nu)) = 0$ and $Jx\big(\cos\theta e_2 + t\sin\theta e_1 - g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)e_3\big) = 0$ and hence we have that $Jx\big(\cos\theta e_2 - t\sin\theta e_1 - g\big(\nu\cos\theta - \alpha t^{-1}\sin\theta\big)e_3\big) = 0$ so that the vectors $p(\nu)$, $\cos\theta e_2 + t\sin\theta e_1 - g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)e_3$ and $\cos\theta e_2 - t\sin\theta e_1 - g\big(\nu\cos\theta - \alpha t^{-1}\sin\theta\big)e_3$ are linearly dependent. So if $g(\nu)$ exists then

$$\begin{vmatrix} 0 & 1 & g(\nu)\\ t\sin\theta & \cos\theta & g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)\\ -t\sin\theta & \cos\theta & g\big(\nu\cos\theta - \alpha t^{-1}\sin\theta\big)\end{vmatrix} = 0.$$

That means

$$2\cos\theta\, g(\nu) = g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big) + g\big(\nu\cos\theta - \alpha t^{-1}\sin\theta\big). \tag{10}$$

On the other hand if $g(\nu)$ ever fails to exist then since $0 < \cos\theta < 1$ we can choose a smooth point $\alpha e_1 + \nu e_2 + e_3$ so that $p(\nu) = e_3$ while $g\big(\nu\cos\theta + \alpha t^{-1}\sin\theta\big)$

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and $g(\nu\cos\theta - \alpha t^{-1}\sin\theta)$ both exist. But the vectors e_3, $\cos\theta e_2 + t\sin\theta e_1 - g(\nu\cos\theta + \alpha t^{-1}\sin\theta)e_3$ and $\cos\theta e_2 - t\sin\theta e_1 - g(\nu\cos\theta - \alpha t^{-1}\sin\theta)e_3$ are linearly independent, contradicting the linear dependence noted above. Thus $g(\nu)$ exists for all ν and (10) holds for almost all (α,ν) in $\mathbf{R}^2$ since almost all points in X are smooth.

Since g is monotone increasing we may assume that g is continuous from the right and it then follows that (10) holds for all α and ν. Putting $\alpha = 0$ gives $\cos\theta\, g(\nu) = g(\cos\theta\,\nu)$ so we have for all ν, y in $\mathbf{R}$,

(11) $$2g(\nu) = g(\nu + y) + g(\nu - y).$$

It is known (see [9], §72) that the only monotone solutions of (11) are the affine functions $g(\nu) = k\nu + r$ and $r = 0$ since $\cos\theta\, g(\nu) = g(\cos\theta\,\nu)$. Thus $g(\nu) = k\nu$ for all ν, so for each ν there is a nonexpansive projection onto $\operatorname{span}\{e_1, e_3 + \nu e_2\}$ along the vector $e_2 - k\nu e_3$. It follows that the convex function $N(x,y) = \|xe_2 + ye_3\|$ has $\nabla N(x,y) \perp (y, -kx)$ almost everywhere and the solutions of the differential equation

$$\frac{dy}{dx} = -k\frac{x}{y}$$

give curves with $N(x,y)$ constant. Thus $y^2 + kx^2 = c$ are curves with $N(x,y)$ constant and evaluating at $(1,0)$ and $(0,1)$ we find that if $y^2 + x^2 = 1$ then $N(x,y) = 1$. Thus $\|xe_2 + ye_3\| = (y^2 + x^2)^{\frac{1}{2}}$ and so

$$\begin{aligned}
\|xe_1 + ye_2 + ze_3\| &= \left\|xe_1 + (y^2 + z^2)^{\frac{1}{2}} e_3\right\| = \left\|xRe_1 + (y^2 + z^2)^{\frac{1}{2}} e_3\right\| \\
&= \left\|x(\cos\theta e_1 + t^{-1}\sin\theta e_2) + (y^2 + z^2)^{\frac{1}{2}} e_3\right\| \\
&= \left\|x\cos\theta e_1 + (t^{-2}\sin^2\theta\, x^2 + y^2 + z^2)^{\frac{1}{2}} e_3\right\| \\
&= \Big\|x\cos^2\theta e_1 \\
&\qquad + \left((1 + \cos^2\theta)t^{-2}\sin^2\theta\, x^2 + y^2 + z^2\right)^{\frac{1}{2}} e_3\Big\| \\
&= \Big\|x\cos^n\theta e_1 \\
&\qquad + \left((1 + \cos^2\theta + \cdots + \cos^{2n-2}\theta)t^{-2}\sin^2\theta\, x^2 + y^2 + z^2\right)^{\frac{1}{2}} e_3\Big\|
\end{aligned}$$

by an easy induction, so that letting $n \to \infty$ we therefore have $\|xe_1 + ye_2 + ze_3\| = \left\|(t^{-2}x^2 + y^2 + z^2)^{\frac{1}{2}} e_3\right\|$. Evaluating at e_1 gives $t = 1$ and $\|xe_1 + ye_2 + ze_3\| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ as required. ∎

Putting these propositions together we have proved the following result.

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Theorem A. *Let X be a 3-dimensional normed space with linearly independent smooth points e and f such that every 2-dimensional subspace which intersects $\{e, f\}$ is the range of a nonexpansive projection. Then there is $p \in (1, \infty]$ such that X is isometrically isomorphic to $\ell_p(3)$.*

Proof: Using Propositions 4 and 5 we see that X is isomorphic to $\ell_p(3)$ for some $p \in [1, \infty]$. But $\ell_1(3)$ does not have any smooth points e such that every 2-dimensional subspace containing e is the range of a nonexpansive projection. ∎

Corollary. *Let X be a 3-dimensional normed space with basis $\{e_1, e_2, e_3\}$ of smooth points such that every 2-dimensional subspace which intersects $\{e_1, e_2, e_2\}$ is the range of a nonexpansive projection. Then there is $p \in (1, \infty]$ such that X is isometrically isomorphic to $\ell_p(3)$.*

3 Spaces of higher dimension

We first need to record which points in $\ell_p(n)$ have the property we are interested in.

Proposition 6. *If $e \in \ell_p(n)$, $n > 2$, $p \neq 2$ is a smooth point such that every two-dimensional subspace containing e is the range of a nonexpansive projection then e has exactly one nonzero coordinate.*

Proof: Let $e = (1, \alpha_2, \alpha_3, \dots, \alpha_n)$ where $\alpha_2 \neq 0$ and $|\alpha_i| \leqslant 1$ for each i. Let $x = (0, 1, -1, 0, \dots, 0)$ if $\alpha_2.\alpha_3 > 0$ and $x = (0, 1, 1, 0, \dots, 0)$ otherwise. Then $\text{span}\{e, x\}$ is not the range of a nonexpansive projection. For p finite this is a consequence of [7], Theorem 2.a.4. If $p = \infty$ then smoothness at e implies that $|\alpha_i| \neq 1$; then suppose that P is a nonexpansive projection onto $\text{span}\{e, x\}$. Since, for $j = 1, 2$ and 3, $\text{span}\{e, x\}$ intersects the interior of the face of the unit sphere in $\ell_\infty(n)$ on which the j^{th} coordinate is 1, we have $\dim P^{-1}(0) \leqslant n - 3$. But that means the range of P is at least 3-dimensional, giving a contradiction.

The smoothness at e is needed in the case of $\ell_\infty(n)$ because for example every 2-dimensional subspace containing $e = (1, 1, \dots, 1)$ is the range of a nonexpansive projection. ∎

Theorem B. *Let X be a n-dimensional normed space and let $\{e_2, e_3, \dots, e_n\}$ be a linearly independent set of smooth points in X such that every 2-dimensional subspace intersecting $\{e_2, e_3, \dots, e_n\}$ is the range of a nonexpansive projection. Then X is isometrically isomorphic to $\ell_p(n)$ for some $p \in (1, \infty]$.*

Proof: This is true for $n = 3$ by Theorem A. Let $k > 3$ and assume that it is true for $n = k - 1$. Suppose $\{e_2, e_3, \dots, e_k\}$ is a linearly independent set of smooth points in X such that every 2-dimensional subspace intersecting $\{e_2, e_3, \dots, e_k\}$ is a the range of a nonexpansive projection. We choose e_1 such that $e_1 \notin \operatorname{span}\{e_2, e_3, \ldots, e_k\}$ and $Je_i(e_1) = 0$ for $2 \leqslant i \leqslant k$ and we also suppose that $\|e_i\| = 1$ for $1 \leqslant i \leqslant k$.

Let x_i be scalars, $1 \leqslant i \leqslant k$, with $x_1 \neq 0$. By our inductive hypothesis there are p, q and r in $(1, \infty]$ such that

$$\begin{aligned} &\operatorname{span}\{e_2, e_3, \ldots, e_k\} \text{ is isometrically isomorphic to } \ell_p(k-1), \\ &\operatorname{span}\{e_1, e_2, \ldots, e_{k-1}\} \text{ is isometrically isomorphic to } \ell_q(k-1), \text{ and} \\ &\operatorname{span}\{x_1 e_1 + x_2 e_2, e_3, \ldots, e_k\} \text{ is isometrically isomorphic to } \ell_r(k-1). \end{aligned}$$

Now $\operatorname{span}\{e_2, e_3, \ldots, e_{k-1}\}$ is isometrically isomorphic to $\ell_p(k-2)$ by Proposition 6 if $p \neq 2$ and by subspaces of Euclidean spaces being isometrically isomorphic to Euclidean spaces if $p = 2$. Similarly $\operatorname{span}\{e_2, e_3, \ldots, e_{k-1}\}$ is isometrically isomorphic to $\ell_q(k-2)$, so we have $p = q$. Considering $\operatorname{span}\{e_3, \ldots, e_k\}$ we see similarly that $p = r$.

If $p \neq 2$ then Proposition 6 and our choice of e_1 show that e_i is orthogonal to $x_1 e_1 + x_2 e_2$ for $3 \leqslant i \leqslant k$, giving

$$\left\| \sum_{i=1}^{k} x_i e_i \right\| = \left\| \left(\|x_1 e_1 + x_2 e_2\|, x_3, \ldots, x_k \right) \right\|_p$$

and similarly $\|x_1 e_1 + x_2 e_2\| = (x_1, x_2)_p$ so that, as required

$$\left\| \sum_{i=1}^{k} x_i e_i \right\| = \|(x_1, x_2, \ldots, x_k)\|_p .$$

If $p = 2$ then $Je_2(e_1) = 0$ implies that $\|x_1 e_1 + x_2 e_2\|^2 = x_1^2 + x_2^2$. Since $\operatorname{span}\{x_1 e_1 + x_2 e_2, e_3, \ldots, e_k\}$ is isometrically isomorphic to $\ell_2(k-1)$ we have

$$\begin{aligned} \left\| \sum_{i=1}^{k} x_i e_i \right\|^2 &= \left\langle \sum_{i=1}^{k} x_i e_i, \sum_{i=1}^{k} x_i e_i \right\rangle \\ &= \|x_1 e_1 + x_2 e_2\|^2 + \left\langle \sum_{i=3}^{k} x_i e_i, \; 2x_1 e_1 + 2x_2 e_2 + \sum_{i=3}^{k} x_i e_i \right\rangle \\ &= \sum_{i=1}^{k} x_i^2 + 2 \sum_{i=3}^{k} x_i Je_i(x_1 e_1 + x_2 e_2) + 2 \sum_{i=3}^{k-1} \sum_{j=i+1}^{k} x_i x_j Je_j(e_i) \\ &= \sum_{i=1}^{k} x_i^2 + 2 \sum_{j=2}^{k-1} \sum_{i=j+1}^{k} Je_i(e_j) x_i x_j \end{aligned}$$

which shows that the norm of $\text{span}\{e_1, e_2, \ldots, e_k\}$ is Euclidean, as required to complete the proof by induction. ∎

Note that the corollary to Theorem A does not suffice to prove the corresponding weaker form of Theorem B with $\{e_2, e_3, \ldots, e_n\}$ replaced by $\{e_1, e_2, e_3, \ldots, e_n\}$. This theorem extends to infinite dimensional spaces as follows.

THEOREM C. *Let E be a Banach space of dimension at least 3 over $\mathbb{R}$ and let $\{e_i : i \in I\}$ be a linearly independent set of smooth points with $\text{span}\{e_i : i \in I\}$ dense in E. Suppose that every two-dimensional subspace intersecting $\{e_i : i \in I\}$ is the range of a nonexpansive projection. Then either*

(a) *E is isometrically isomorphic to $c_0(I)$ or to $\ell_p(I)$ for some $p \neq 2$, in such a way that each e_i corresponds to an element of the canonical basis or*

(b) *E is isometrically isomorphic to a Hilbert space.*

PROOF: If I is finite then this follows from Theorem B and Proposition 6. Otherwise there is p such that for each finite subset F of I, $\text{span}\{e_i : i \in F\}$ is isometrically isomorphic to $\ell_p(F)$, with each e_i, $i \in F$, corresponding to an element of the canonical basis in $\ell_p(F)$ if $p \neq 2$. For $p = \infty$ it follows that E is isometrically isomorphic to $c_0(I)$, while for finite $p \neq 2$ it follows that E is isometrically isomorphic to $\ell_p(I)$, in both cases the elements e_i corresponding to canonical basis elements.

In the case $p = 2$ let $x, y \in E$ and let x_n, y_n be elements of $\text{span}\{e_i : i \in I\}$ such that $\|x - x_n\|$ and $\|y - y_n\|$ are less than n^{-1}. Then

$$\langle x_n, y_n \rangle = 4^{-1}\left(\|x_n + y_n\|^2 - \|x_n - y_n\|^2\right) \to 4^{-1}\left(\|x + y\|^2 - \|x - y\|^2\right)$$

so we define $\langle x, y \rangle$ to be

$$4^{-1}\left(\|x + y\|^2 - \|x - y\|^2\right) = \lim \langle x_n, y_n \rangle .$$

Then $\langle \ , \ \rangle$ is a bilinear form such that $\|x\|^2 = \langle x, x \rangle$ and E is a Hilbert space. ∎

Remark. The difference between this result and our previous work [3], [4] and [5] is that the Banach space does not need to be a lattice and the points e_i do not need to be orthogonal. We do require the norm to be smooth at the points e_i; without this some other spaces satisfy our standard hypotheses.

The condition that $\text{span}\{e_i : i \in I\}$ is dense in E can be weakened to $\text{span}\{e_i : i \in I\}$ being dense in some hyperplane in E. A similar modification is possible in our final result which is a characterisation of real Hilbert spaces.

THEOREM D. *Let E be a real Banach space of dimension at least 3. Then E is a Hilbert space if and only if there is a linearly independent set A of smooth points in E such that the linear span of A is dense in E, every 2-dimensional subspace intersecting A is the range of a nonexpansive projection and $\|x + ty\| < \|x\|$ for some distinct $x, y \in A$ and $t \in \mathbf{R}$.*

PROOF: In c_0 and ℓ_p, $p \neq 2$, the elements of A must be mutually orthogonal. ∎

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